Abstract fractional inequalities over a line segment of a Banach space

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Abstract

Here we present first a complete fractional calculus between a pair of Banach spaces. That is regular and sequential fractionality. Based on these we give a collection of left and right related fractional inequalities relied on a line segment of a Banach space. We include also the case of connected line segments. We treat as well the case of sequential inequalities. Our results include Ostrowski type inequalities, Poincaré and Sobolev type inequalities, Opial type inequalities and Hilbert-Pachpatte type inequalities.

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1 Introduction

We are motivated greatly by the following basic result:

Theorem 1 *(1938. Ostrowski [3]) Let $f : [a,b] \to \mathbb{R}$ be continuous on $[a,b]$ and differentiable on (a,b) whose derivative $f' : (a,b) \to \mathbb{R}$ is bounded on (a,b), i.e., $\|f'\|_{\infty}^{\sup} := \sup_{t \in (a,b)} |f'(t)| < +\infty$. Then*

$$\left| \frac{1}{b-a} \int_a^b f(t)\,dt - f(x) \right| \leq \left[\frac{1}{4} + \frac{\left(x - \frac{a+b}{2}\right)^2}{(b-a)^2} \right] (b-a) \|f'\|_{\infty}^{\sup}, \tag{1}$$

for any $x \in [a,b]$. The constant $\frac{1}{4}$ is the best possible.

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The problem of estimating the difference of a value of a function from its average is a top one. The answer to it are the Ostrowski type inequalities, see (1). Ostrowski type inequalities are very useful among others in Numerical Analysis for approximating integrals.

In this article we present a full array of abstract left and right fractional inequalities, of regular and sequential types.

Our studied functions here are between Banach spaces, and we develop first the related abstract regular and sequential fractional calculi which are based on a Banach space segment. Great sources to support our goal are the books [1], [4].

2 Complete Fractionality between a pair of Banach spaces

2.1 Regular Fractionality

We make

Remark 2 *Throughout this article let* $(X, \|\cdot\|_1)$ *and* $(Y, \|\cdot\|_2)$ *be Banach spaces. Here* X^j *denotes the* j*-fold product space* $\underbrace{X \times X \times ... \times X}_{j}$ *endowed with the max-norm* $\|x\|_{X^j} := \max_{1 \leq \lambda \leq j} \|x_\lambda\|_1$, *where* $x := (x_1, ..., x_j) \in X^j$.

Let the space of $L_j := L_j\left(X^j, Y\right)$ *of all* j*-multilinear continuous maps* $h : X^j \to Y$, $j = 1, ..., m$, *which is a Banach space with norm*

$$\|h\| = \|h\|_{L_j} := \sup_{\left(\|x\|_{X^j}=1\right)} \|h(x)\|_2 = \sup \frac{\|h(x)\|_2}{\|x_1\|_1 ... \|x_j\|_1}. \tag{2}$$

Let M *be a non-empty convex and compact set of* X *and* $x_0 \in M$ *is fixed.*

Let O *be an open subset of* $X : M \subset O$.

Let $f : O \to Y$ *be a continuous function, whose Fréchet derivatives ([4], pp. 87-127)* $f^{(j)} : O \to L_j = L_j\left(X^j, Y\right)$ *exist and are continuous for* $1 \leq j \leq m$, $m \in \mathbb{N}$.

Call $(x - x_0)^j := (x - x_0, ..., x - x_0) \in X^j$, $x \in M$.

We will work with $f|_M$.

Here we set $f^{(0)}(x_0)(x - x_0)^0 = f(x_0)$.

We obtain

$$\left\|\left(f^{(m)}(x_0 + u(x - x_0))\right)(x - x_0)^m\right\|_2 \leq \left\|f^{(m)}(x_0 + u(x - x_0))\right\| \|x - x_0\|_1^m. \tag{3}$$

Above $(f|_M)^{(j)}$, $j = 0, 1, ..., m$, *are norm bounded by continuity, and thus they are integrable.*

Here

$$L(x_0, x_1) := \{\overline{x} | \overline{x} = \theta x_1 + (1-\theta) x_0,\ 0 \leq \theta \leq 1\} \tag{4}$$

is the line segment joining the points x_0 *and* x_1 *(notice that the last* $\overline{x} = x_0 + \theta(x_1 - x_0)$*).*

Denote also $L(x_0, x_1) = \overline{x_0 x_1}$*.*

Here $(\cdot - x_0)^j$ *maps* M *into* X^j *and it is continuous, also* $f^{(j)}(x_0)$ *maps* X^j *into* Y *and it is continuous. Hence their composition* $f^{(j)}(x_0)(\cdot - x_0)^j$ *is continuous from* M *into* Y*.*

Let us restrict f *on the line segment* $\overline{x_0 x_1}$*. Then for*

$$\overline{x}(u) = u x_1 + (1-u) x_0 = x_0 + u(x_1 - x_0),\ \ 0 \leq u \leq 1,$$

the abstract function

$$f(u) = f(\overline{x}(u)) = f(x_0 + u(x_1 - x_0))$$

will map $[0,1]$ *into an abstract arc in* Y*, which starts at* $y_0 = f(x_0)$ *and ends at* $y_1 = f(x_1)$*.*

By [4], p. 124, we have that

$$f^{(k)}(u) = f^{(k)}(x_0 + u(x_1 - x_0))(x_1 - x_0)^k, \tag{5}$$

for $k = 1, 2, ..., m;\ u \in [0,1]$*.*

We need

Definition 3 *All as in Remark 2. We define the vector left Caputo-Fréchet fractional derivative of order* $\alpha > 0$, $m = \lceil \alpha \rceil$ *(*$\lceil \cdot \rceil$ *ceiling of the number), by*

$$D_{*0}^{\alpha}(f(x_0 + u(x_1 - x_0))) := J_0^{m-\alpha}\left(\left(f^{(m)}(x_0 + u(x_1 - x_0))\right)(x_1 - x_0)^m\right)$$

$$:= \frac{1}{\Gamma(m-\alpha)} \int_0^u (u-t)^{m-\alpha-1} \left(f^{(m)}(x_0 + t(x_1 - x_0))\right)(x_1 - x_0)^m \, dt, \tag{6}$$

all $0 \leq u \leq 1$,

defined via the vector left Riemann-Liouville fractional integral J_0 *([1], p. 2).*

Then, we observe that

$$J_0^{\alpha} D_{*0}^{\alpha}(f(x_0 + u(x_1 - x_0))) = J_0^{\alpha} J_0^{m-\alpha}\left(f^{(m)}(x_0 + u(x_1 - x_0))\right)(x_1 - x_0)^m =$$

(by [1], p. 6)

$$J_0^{m}\left(f^{(m)}(x_0 + u(x_1 - x_0))\right)(x_1 - x_0)^m = \tag{7}$$

$$\frac{1}{(m-1)!}\int_0^u (u-t)^{m-1}\left(f^{(m)}\left(x_0+t\left(x_1-x_0\right)\right)\right)\left(x_1-x_0\right)^m dt,$$

true for $0 \le u \le 1$.

So, we have proved that

$$\frac{1}{\Gamma(\alpha)}\int_0^u (u-t)^{\alpha-1} D_{*0}^{\alpha}\left(f\left(x_0+t\left(x_1-x_0\right)\right)\right) dt =$$

$$\frac{1}{(m-1)!}\int_0^u (u-t)^{m-1}\left(f^{(m)}\left(x_0+t\left(x_1-x_0\right)\right)\right)\left(x_1-x_0\right)^m dt, \tag{8}$$

for all $0 \le u \le 1$.

Consequently (by [1], p. 12) it holds the new left fractional-Fréchet Taylor formula on $\overline{x_0x_1}$:

Theorem 4 *All as above. Then*

$$f\left(x_0+u\left(x_1-x_0\right)\right)=\sum_{k=0}^{m-1}\frac{f^{(k)}\left(x_0\right)\left(x_1-x_0\right)^k}{k!}u^k+$$

$$\frac{1}{\Gamma(\alpha)}\int_0^u (u-t)^{\alpha-1} D_{*0}^{\alpha} f\left(x_0+t\left(x_1-x_0\right)\right) dt, \tag{9}$$

for all $0 \le u \le 1$.

So, we have the particular vector left hand side Caputo-Fréchet fractional Taylor's formula ($u=1$).

Corollary 5 *All as above. Then*

$$f\left(x_1\right)=\sum_{j=0}^{m-1}\frac{f^{(j)}\left(x_0\right)\left(x_1-x_0\right)^j}{j!}+ \tag{10}$$

$$\frac{1}{\Gamma(\alpha)}\int_0^1 (1-w)^{\alpha-1} D_{*0}^{\alpha}\left(f\left(x_0+w\left(x_1-x_0\right)\right)\right) dw,$$

for all $x_0, x_1 \in M$.

We make

Remark 6 *We are again working on the segment* $\overline{x_0x_1}$, *where* $x_0, x_1 \in M$; $0 \le u \le 1$, *with* $\overline{x} = \overline{x}(u) := x_0 + u\left(x_1-x_0\right)$, *and* $f(u) = f(\overline{x}(u)) = f\left(x_0+u\left(x_1-x_0\right)\right)$, *i.e.* $f:[0,1]\to Y$.

When $u=0$, $f(0)=f\left(x_0\right)$, *and when* $u=1$, $f(1)=f\left(x_1\right)$.

We have (by [1], pp. 121-122) the abstract Riemann integral:

$$\int_{x_0}^{x_1} f(\overline{x})\, d\overline{x} = \int_0^1 f(u)\, du. \tag{11}$$

We also have that ([1], p. 122, and p. 124)

$$f'(u) = f'(x_0 + u(x_1 - x_0))(x_1 - x_0),$$

and

$$f^{(k)}(u) = f^{(k)}(x_0 + u(x_1 - x_0))(x_1 - x_0)^k, \tag{12}$$

$k = 1, 2, ..., m.$

Let $\alpha > 0$, such that $\lceil \alpha \rceil = m$.

We consider the vector valued right hand side Riemann-Liouville fractional integral ([1], p. 34),

$$J_{1-}^{\alpha} f(u) = \frac{1}{\Gamma(\alpha)} \int_u^1 (J - u)^{\alpha - 1} f(J) \, dJ, \tag{13}$$

and the vector valued right hand side Caputo fractional derivative of order $\alpha > 0$ ([1], p. 42), by

$$D_{1-}^{\alpha} f(u) := (-1)^m J_{1-}^{m-\alpha} f^{(m)}(x) = \frac{(-1)^m}{\Gamma(m-\alpha)} \int_u^1 (J - u)^{m-\alpha-1} f^{(m)}(J) \, dJ. \tag{14}$$

(we have that $J_{1-}^{\alpha} J_{-1}^{\beta} f = J_{-1}^{\alpha+\beta} f = J_{-1}^{\beta} J_{-1}^{\alpha} f$, when $f \in C([0,1], Y)$ or $\alpha + \beta \geq 1$, see [1], p. 39).

We will use the right hand side Taylor's fractional formula.

Theorem 7 *([1], p. 44, by Theorem 2.16) Let $f \in C^m([0,1], Y)$, $u \in [0,1]$, $\alpha > 0$, $m = \lceil \alpha \rceil$. Then*

$$f(u) = \sum_{k=0}^{m-1} \frac{f^{(k)}(1)}{k!} (u - 1)^k + \frac{1}{\Gamma(\alpha)} \int_u^1 (J - u)^{\alpha - 1} D_{1-}^{\alpha} f(J) \, dJ. \tag{15}$$

Equation (15) implies the vector right hand side corresponding fractional Taylor's formula:

Theorem 8 *All as in Remarks 2, 6. Then*

$$f(x_0 + u(x_1 - x_0)) = \sum_{k=0}^{m-1} \frac{f^{(k)}(x_1)(x_1 - x_0)^k}{k!} (u - 1)^k + \tag{16}$$

$$\frac{1}{\Gamma(\alpha)} \int_u^1 (J - u)^{\alpha - 1} D_{1-}^{\alpha} f(J) \, dJ,$$

all $0 \leq u \leq 1$,

where the vector right Caputo-Fréchet fractional derivative of order $\alpha > 0$ is given by,

$$D_{1-}^{\alpha} f(J) = D_{1-}^{\alpha} f(x_0 + J(x_1 - x_0)) = \frac{(-1)^m}{\Gamma(m-\alpha)} \int_J^1 (t-J)^{m-\alpha-1} f^{(m)}(t)\, dt$$

$$= \frac{(-1)^m}{\Gamma(m-\alpha)} \int_J^1 (t-J)^{m-\alpha-1} f^{(m)}(x_0 + t(x_1 - x_0))(x_1 - x_0)^m\, dt. \tag{17}$$

When $u = 0$ we obtain

Corollary 9 *All as in Remarks 2, 6. Then*

$$f(x_0) = \sum_{k=0}^{m-1} \frac{f^{(k)}(x_1)(x_1 - x_0)^k}{k!} (-1)^k +$$

$$\frac{1}{\Gamma(\alpha)} \int_0^1 J^{\alpha-1} D_{1-}^{\alpha} f(x_0 + J(x_1 - x_0))\, dJ, \tag{18}$$

for all $x_0, x_1 \in M$.

2.2 Sequential Fractionality

We need

Definition 10 *In particular when $0 < \alpha < 1$ we have*

$$D_{*0}^{\alpha}(f(x_0 + u(x_1 - x_0))) =$$

$$\frac{1}{\Gamma(1-\alpha)} \int_0^u (u-t)^{-\alpha} (f'(x_0 + t(x_1 - x_0)))(x_1 - x_0)\, dt, \tag{19}$$

all $0 \leq u \leq 1$.

If $\alpha \in \mathbb{N}$, we set $D_{*0}^{\alpha} f := f^{(\alpha)}$ the ordinary Y-valued derivative, and also set $D_{*0}^{0} f := f$.

Definition 11 *In particular when $0 < \alpha < 1$ we have*

$$D_{1-}^{\alpha}(f(x_0 + u(x_1 - x_0))) =$$

$$\frac{-1}{\Gamma(1-\alpha)} \int_u^1 (t-u)^{-\alpha} f'(x_0 + t(x_1 - x_0))(x_1 - x_0)\, dt, \tag{20}$$

all $0 \leq u \leq 1$.

We set $D_{1-}^{m} f = (-1)^m f^{(m)}$, for $m \in \mathbb{N}$, and $D_{1-}^{0} f = f$.

Denote the sequential Caputo-Bochner left and right fractional derivatives ($\alpha > 0$)

$$D_{*a}^{n\alpha} := D_{*a}^{\alpha} D_{*a}^{\alpha} ... D_{*a}^{\alpha} \quad (n\text{-times}),\ n \in \mathbb{N} \tag{21}$$

and

$$D_{b-}^{n\alpha} := D_{b-}^{\alpha} D_{b-}^{\alpha} ... D_{b}^{\alpha} \quad (n\text{-times}). \tag{22}$$

For the detailed definitions of D_{*a}^{α}, D_{b-}^{α} see [1], pp. 128-129. In this article we consider $[a, b] = [0, 1]$.

We mention the following left alternative fractional Taylor's formula

Theorem 12 *([1], p. 129, Theorem 4.43) Let Y a Banach space and $0 < \alpha \leq 1$, $n \in \mathbb{N}$, $f \in C^1([a, b], Y)$.*

For $k = 1, ..., n$, we assume that $D_{*a}^{k\alpha} f \in C^1([a, b], Y)$ and $D_{*a}^{(n+1)\alpha} f \in C([a, b], Y)$. Then

$$f(x) = \sum_{i=0}^{n} \frac{(x - a)^{i\alpha}}{\Gamma(i\alpha + 1)} \left(D_{*a}^{i\alpha} f\right)(a) + \tag{23}$$

$$\frac{1}{\Gamma((n+1)\alpha)} \int_a^x (x - t)^{(n+1)\alpha - 1} \left(D_{*a}^{(n+1)\alpha} f\right)(t)\, dt,$$

$\forall\ x \in [a, b]$.

We also mention the following right alternative fractional Taylor's formula.

Theorem 13 *([1], p. 129, Theorem 4.44) Let Y a Banach space and $0 < \alpha \leq 1$, $n \in \mathbb{N}$, $f \in C^1([a, b], Y)$. For $k = 1, ..., n$, we assume that $D_{b-}^{k\alpha} f \in C^1([a, b], Y)$ and $D_{b-}^{(n+1)\alpha} f \in C([a, b], Y)$. Then*

$$f(x) = \sum_{i=0}^{n} \frac{(b - x)^{i\alpha}}{\Gamma(i\alpha + 1)} \left(D_{b-}^{i\alpha} f\right)(b) + \tag{24}$$

$$\frac{1}{\Gamma((n+1)\alpha)} \int_x^b (t - x)^{(n+1)\alpha - 1} \left(D_{b-}^{(n+1)\alpha} f\right)(t)\, dt,$$

$\forall\ x \in [a, b]$.

We give the corresponding left and right fractional alternative Taylor's formulae on the segment $\overline{x_0 x_1}$.

Theorem 14 *Let Y a Banach space and $0 < \alpha \leq 1$, $n \in \mathbb{N}$, $f(x_0 + u(x_1 - x_0)) \in C^1([0, 1], Y)$.*

For $k = 1, ..., n$, we assume that $D_{*0}^{k\alpha} f(x_0 + u(x_1 - x_0)) \in C^1([0, 1], Y)$ and $D_{*0}^{(n+1)\alpha} f(x_0 + u(x_1 - x_0)) \in C([0, 1], Y)$. Then

$$f(x_0 + u(x_1 - x_0)) = \sum_{i=0}^{n} \frac{u^{i\alpha}}{\Gamma(i\alpha + 1)} \left(D_{*0}^{i\alpha}(f(x_0 + u(x_1 - x_0)))\right)(0) + \tag{25}$$

$$\frac{1}{\Gamma((n+1)\alpha)}\int_0^u (u-t)^{(n+1)\alpha-1}\left(D_{*0}^{(n+1)\alpha}(f(x_0+u(x_1-x_0)))\right)(t)\,dt,$$

$\forall\ u\in[0,1]$.

Proof. By Theorem 12. ■

Theorem 15 *Let Y a Banach space and $0<\alpha\leq 1$, $n\in\mathbb{N}$, $f(x_0+u(x_1-x_0))\in C^1([0,1],Y)$.*

For $k=1,...,n$, we assume that $D_{1-}^{k\alpha}f(x_0+u(x_1-x_0))\in C^1([0,1],Y)$ and $D_{1-}^{(n+1)\alpha}f(x_0+u(x_1-x_0))\in C([0,1],Y)$. Then

$$f(x_0+u(x_1-x_0))=\sum_{i=0}^{n}\frac{(1-u)^{i\alpha}}{\Gamma(i\alpha+1)}\left(D_{1-}^{i\alpha}f(x_0+u(x_1-x_0))\right)(1)+ \tag{26}$$

$$\frac{1}{\Gamma((n+1)\alpha)}\int_u^1 (t-u)^{(n+1)\alpha-1}\left(D_{1-}^{(n+1)\alpha}f(x_0+u(x_1-x_0))\right)(t)\,dt,$$

$\forall\ u\in[0,1]$.

Proof. By Theorem 13. ■

3 Main Results

We present

Theorem 16 *All as in Remarks 2, 6. Assume $f^{(k)}(x_0)=f^{(k)}(x_1)=0$, $k=1,...,m-1$.*

Then

$$E(f,x_0,x_1):=\int_{x_0}^{x_1}f(x)\,dx-\left(\frac{f(x_0)+f(x_1)}{2}\right)= \tag{27}$$

$$\frac{1}{2\Gamma(\alpha)}\left[\int_0^1\left(\int_0^u (u-t)^{\alpha-1}D_{*0}^{\alpha}(f(x_0+t(x_1-x_0)))\,dt\right)du+\right.$$

$$\left.\int_0^1\left(\int_u^1 (J-u)^{\alpha-1}D_{1-}^{\alpha}(f(x_0+J(x_1-x_0)))\,dJ\right)du\right].$$

Proof. By (11) and Theorem 4 we have

$$\int_{x_0}^{x_1}f(x)\,dx=\int_0^1 f(x_0+u(x_1-x_0))\,du=\sum_{k=0}^{m-1}\frac{f^{(k)}(x_0)(x_1-x_0)^k}{k!}\int_0^1 u^k du+$$

$$\frac{1}{\Gamma(\alpha)}\int_0^1\left(\int_0^u (u-t)^{\alpha-1}D_{*0}^{\alpha}f(x_0+t(x_1-x_0))\,dt\right)du=$$

$$\sum_{k=0}^{m-1} \frac{f^{(k)}(x_0)(x_1-x_0)^k}{(k+1)!} + \frac{1}{\Gamma(\alpha)} \int_0^1 \left(\int_0^u (u-t)^{\alpha-1} D_{*0}^{\alpha} f(x_0 + t(x_1-x_0))\, dt \right) du. \tag{28}$$

Also by Theorem 8 we have

$$\int_{x_0}^{x_1} f(x)\, dx = \int_0^1 f(x_0 + u(x_1-x_0))\, du =$$

$$\sum_{k=0}^{m-1} \frac{f^{(k)}(x_1)(x_1-x_0)^k}{k!} \int_0^1 (u-1)^k\, du +$$

$$\frac{1}{\Gamma(\alpha)} \int_0^1 \left(\int_u^1 (J-u)^{\alpha-1} D_{1-}^{\alpha} f(J)\, dJ \right) du =$$

$$\sum_{k=0}^{m-1} \frac{f^{(k)}(x_1)(x_1-x_0)^k}{(k+1)!} (-1)^k + \frac{1}{\Gamma(\alpha)} \int_0^1 \left(\int_u^1 (J-u)^{\alpha-1} D_{1-}^{\alpha} f(J)\, dJ \right) du. \tag{29}$$

Assume $f^{(k)}(x_0) = 0,\ f^{(k)}(x_1) = 0,\ k = 1, ..., m-1$. Then

$$\int_{x_0}^{x_1} f(x)\, dx - f(x_0) = \frac{1}{\Gamma(\alpha)} \int_0^1 \left(\int_0^u (u-t)^{\alpha-1} D_{*0}^{\alpha} f(x_0 + t(x_1-x_0))\, dt \right) du, \tag{30}$$

and

$$\int_{x_0}^{x_1} f(x)\, dx - f(x_1) = \frac{1}{\Gamma(\alpha)} \int_0^1 \left(\int_u^1 (J-u)^{\alpha-1} D_{1-}^{\alpha} f(J)\, dJ \right) du. \tag{31}$$

Adding (30) and (31) and divide by 2 we derive

$$\int_{x_0}^{x_1} f(x)\, dx - \left(\frac{f(x_0) + f(x_1)}{2} \right) = \tag{32}$$

$$\frac{1}{2\Gamma(\alpha)} \left[\int_0^1 \left(\int_0^u (u-t)^{\alpha-1} D_{*0}^{\alpha} f(x_0 + t(x_1-x_0))\, dt \right) du + \int_0^1 \left(\int_u^1 (J-u)^{\alpha-1} D_{1-}^{\alpha} f(x_0 + J(x_1-x_0))\, dJ \right) du \right].$$

■

We also give:

Theorem 17 *All as in Theorems 14, 15. Additionally assume that* $D_{*0}^{i\alpha}(f(x_0 + u(x_1-x_0)))(0) = D_{1-}^{i\alpha}(f(x_0 + u(x_1-x_0)))(1) = 0$, $i = 1, ..., n$. *Then*

$$E(f, x_0, x_1) := \int_{x_0}^{x_1} f(x)\, dx - \left(\frac{f(x_0) + f(x_1)}{2} \right) = \tag{33}$$

$$\frac{1}{2\Gamma\left(\left(n+1\right)\alpha\right)}\left[\int_{0}^{1}\left(\int_{0}^{u}\left(u-t\right)^{\left(n+1\right)\alpha-1}D_{*0}^{\left(n+1\right)\alpha}\left(f\left(x_{0}+t\left(x_{1}-x_{0}\right)\right)\right)\left(t\right)dt\right)du+\right.$$

$$\left.\int_{0}^{1}\left(\int_{u}^{1}\left(t-u\right)^{\left(n+1\right)\alpha-1}D_{1-}^{\left(n+1\right)\alpha}\left(f\left(x_{0}+t\left(x_{1}-x_{0}\right)\right)\right)\left(t\right)dt\right)du\right].$$

Proof. By Theorem 14 we have

$$\int_{x_0}^{x_1}f\left(x\right)dx=\int_{0}^{1}f\left(x_{0}+u\left(x_{1}-x_{0}\right)\right)du=$$

$$\sum_{i=0}^{n}\frac{\left(\int_{0}^{1}u^{i\alpha}du\right)}{\Gamma\left(i\alpha+1\right)}D_{*0}^{i\alpha}\left(f\left(x_{0}+u\left(x_{1}-x_{0}\right)\right)\right)\left(0\right)+$$

$$\frac{1}{\Gamma\left(\left(n+1\right)\alpha\right)}\int_{0}^{1}\left(\int_{0}^{u}\left(u-t\right)^{\left(n+1\right)\alpha-1}D_{*0}^{\left(n+1\right)\alpha}\left(f\left(x_{0}+t\left(x_{1}-x_{0}\right)\right)\right)\left(t\right)dt\right)du=$$

$$\sum_{i=0}^{n}\frac{1}{\Gamma\left(i\alpha+2\right)}D_{*0}^{i\alpha}\left(f\left(x_{0}+u\left(x_{1}-x_{0}\right)\right)\right)\left(0\right)+\tag{34}$$

$$\frac{1}{\Gamma\left(\left(n+1\right)\alpha\right)}\int_{0}^{1}\left(\int_{0}^{u}\left(u-t\right)^{\left(n+1\right)\alpha-1}D_{*0}^{\left(n+1\right)\alpha}\left(f\left(x_{0}+t\left(x_{1}-x_{0}\right)\right)\right)\left(t\right)dt\right)du,$$

all $0\leq u\leq 1$.

By Theorem 15, we have

$$\int_{x_0}^{x_1}f\left(x\right)dx=\sum_{i=0}^{n}\frac{1}{\Gamma\left(i\alpha+2\right)}D_{1-}^{i\alpha}\left(f\left(x_{0}+u\left(x_{1}-x_{0}\right)\right)\right)\left(1\right)+\tag{35}$$

$$\frac{1}{\Gamma\left(\left(n+1\right)\alpha\right)}\int_{0}^{1}\left(\int_{u}^{1}\left(t-u\right)^{\left(n+1\right)\alpha-1}D_{1-}^{\left(n+1\right)\alpha}\left(f\left(x_{0}+t\left(x_{1}-x_{0}\right)\right)\right)\left(t\right)dt\right)du.$$

We assume that $D_{*0}^{i\alpha}\left(f\left(x_{0}+u\left(x_{1}-x_{0}\right)\right)\right)\left(0\right)=D_{1-}^{i\alpha}\left(f\left(x_{0}+u\left(x_{1}-x_{0}\right)\right)\right)\left(1\right)=0$, $i=1,...,n$. Then

$$\int_{x_0}^{x_1}f\left(x\right)dx-f\left(x_{0}\right)=$$

$$\frac{1}{\Gamma\left(\left(n+1\right)\alpha\right)}\int_{0}^{1}\left(\int_{0}^{u}\left(u-t\right)^{\left(n+1\right)\alpha-1}D_{*0}^{\left(n+1\right)\alpha}\left(f\left(x_{0}+t\left(x_{1}-x_{0}\right)\right)\right)\left(t\right)dt\right)du,\tag{36}$$

and

$$\int_{x_0}^{x_1}f\left(x\right)dx-f\left(x_{1}\right)=$$

$$\frac{1}{\Gamma\left(\left(n+1\right)\alpha\right)}\int_{0}^{1}\left(\int_{u}^{1}\left(t-u\right)^{\left(n+1\right)\alpha-1}D_{1-}^{\left(n+1\right)\alpha}\left(f\left(x_{0}+t\left(x_{1}-x_{0}\right)\right)\right)\left(t\right)dt\right)du.\tag{37}$$

Therefore it holds

$$\int_{x_0}^{x_1} f(x)\,dx - \left(\frac{f(x_0)+f(x_1)}{2}\right) = \tag{38}$$

$$\frac{1}{2\Gamma((n+1)\alpha)}\left[\int_0^1\left(\int_0^u (u-t)^{(n+1)\alpha-1} D_{*0}^{(n+1)\alpha}(f(x_0+t(x_1-x_0)))(t)\,dt\right)du+\right.$$

$$\left.\int_0^1\left(\int_u^1 (t-u)^{(n+1)\alpha-1} D_{1-}^{(n+1)\alpha}(f(x_0+t(x_1-x_0)))(t)\,dt\right)du\right].$$

■

We present the following basic abstract fractional Ostrowski type inequalities:

Theorem 18 *All as in Theorem 16. Then*

i)

$$\|E(f,x_0,x_1)\|_2 \leq \frac{1}{2\Gamma(\alpha+2)}\left\{\left\|\left\|D_{*0}^{\alpha}(f(x_0+t(x_1-x_0)))\right\|_2\right\|_{t,\infty,[0,1]}\right. \tag{39}$$

$$\left.+\left\|\left\|D_{1-}^{\alpha}(f(x_0+J(x_1-x_0)))\right\|_2\right\|_{J,\infty,[0,1]}\right\},$$

ii) when $\alpha \geq 1$, we get

$$\|E(f,x_0,x_1)\|_2 \leq \frac{1}{2\Gamma(\alpha+1)}\left\{\left\|\left\|D_{*0}^{\alpha}(f(x_0+t(x_1-x_0)))\right\|_2\right\|_{t,L_1([0,1])}\right. \tag{40}$$

$$\left.+\left\|\left\|D_{1-}^{\alpha}(f(x_0+J(x_1-x_0)))\right\|_2\right\|_{J,L_1([0,1])}\right\},$$

iii) when $p,q>1: \frac{1}{p}+\frac{1}{q}=1$, with $\alpha > \frac{1}{q}$, we find that

$$\|E(f,x_0,x_1)\|_2 \leq \frac{1}{2\Gamma(\alpha)(p(\alpha-1)+1)^{\frac{1}{p}}\left(\alpha+\frac{1}{p}\right)} \tag{41}$$

$$\left[\left\|\left\|D_{*0}^{\alpha}(f(x_0+t(x_1-x_0)))\right\|_2\right\|_{t,L_q([0,1])}+\left\|\left\|D_{1-}^{\alpha}(f(x_0+J(x_1-x_0)))\right\|_2\right\|_{J,L_q([0,1])}\right].$$

Proof. We have that

$$\|E(f,x_0,x_1)\|_2 \overset{(27)}{\leq} \frac{1}{2\Gamma(\alpha)}$$

$$\left[\left\|\int_0^1\left(\int_0^u (u-t)^{\alpha-1} D_{*0}^{\alpha}(f(x_0+t(x_1-x_0)))\,dt\right)du\right\|_2+\right.$$

$$\left.\left\|\int_0^1\left(\int_u^1 (J-u)^{\alpha-1} D_{1-}^{\alpha}f(x_0+J(x_1-x_0))\,dJ\right)du\right\|_2\right] \leq$$

$$\frac{1}{2\Gamma(\alpha)}\left[\int_0^1\left(\int_0^u (u-t)^{\alpha-1}\left\|D_{*0}^{\alpha}f(x_0+t(x_1-x_0))\right\|_2 dt\right)du+\right. \tag{42}$$

$$\left.\int_0^1\left(\int_u^1 (J-u)^{\alpha-1}\left\|D_{1-}^{\alpha}f(x_0+J(x_1-x_0))\right\|_2 dJ\right)du\right]=:(\psi).$$

i) We have that

$$(\psi)\leq\frac{1}{2\Gamma(\alpha+1)}\left[\left(\int_0^1 u^{\alpha}du\right)\left\|\left\|D_{*0}^{\alpha}\left(f(x_0+t(x_1-x_0))\right)\right\|_2\right\|_{t,\infty,[0,1]}+\right.$$

$$\left.\left(\int_0^1(1-u)^{\alpha}du\right)\left\|\left\|D_{1-}^{\alpha}\left(f(x_0+J(x_1-x_0))\right)\right\|_2\right\|_{J,\infty,[0,1]}\right]=$$

$$\frac{1}{2\Gamma(\alpha+2)}\left\{\left\|\left\|D_{*0}^{\alpha}f(x_0+t(x_1-x_0))\right\|_2\right\|_{t,\infty,[0,1]}\right.$$

$$\left.+\left\|\left\|D_{1-}^{\alpha}\left(f(x_0+J(x_1-x_0))\right)\right\|_2\right\|_{J,\infty,[0,1]}\right\}, \tag{43}$$

proving (i).

ii) If $\alpha\geq 1$, then

$$(\psi)\leq\frac{1}{2\Gamma(\alpha)}\left[\left(\int_0^1 u^{\alpha-1}du\right)\left\|\left\|D_{*0}^{\alpha}\left(f(x_0+t(x_1-x_0))\right)\right\|_2\right\|_{t,L_1([0,1])}\right. \tag{44}$$

$$\left.+\left(\int_0^1(1-u)^{\alpha-1}du\right)\left\|\left\|D_{1-}^{\alpha}\left(f(x_0+J(x_1-x_0))\right)\right\|_2\right\|_{J,L_1([0,1])}\right]=$$

$$\frac{1}{2\Gamma(\alpha+1)}\left[\left\|\left\|D_{*0}^{\alpha}\left(f(x_0+t(x_1-x_0))\right)\right\|_2\right\|_{t,L_1([0,1])}\right.$$

$$\left.+\left\|\left\|D_{1-}^{\alpha}\left(f(x_0+J(x_1-x_0))\right)\right\|_2\right\|_{J,L_1([0,1])}\right],$$

proving (ii).

iii) Let $p,q>1:\frac{1}{p}+\frac{1}{q}=1$, with $\alpha>\frac{1}{q}$. Then (by applying Hölder's inequality) we obtain:

$$(\psi)\leq\frac{1}{2\Gamma(\alpha)}\left[\int_0^1\frac{u^{\alpha-1+\frac{1}{p}}}{(p(\alpha-1)+1)^{\frac{1}{p}}}\left\|\left\|D_{*0}^{\alpha}\left(f(x_0+t(x_1-x_0))\right)\right\|_2\right\|_{t,L_q([0,1])}du\right. \tag{45}$$

$$\left.+\int_0^1\frac{(1-u)^{\alpha-1+\frac{1}{p}}}{(p(\alpha-1)+1)^{\frac{1}{p}}}\left\|\left\|D_{1-}^{\alpha}\left(f(x_0+J(x_1-x_0))\right)\right\|_2\right\|_{J,L_q([0,1])}du\right]=$$

$$\frac{1}{2\Gamma(\alpha)(p(\alpha-1)+1)^{\frac{1}{p}}\left(\alpha+\frac{1}{p}\right)}\left[\left\|\left\|D_{*0}^{\alpha}\left(f(x_0+t(x_1-x_0))\right)\right\|_2\right\|_{t,L_q([0,1])}\right.$$

$$\left.+\left\|\left\|D_{1-}^{\alpha}\left(f(x_0+J(x_1-x_0))\right)\right\|_2\right\|_{J,L_q([0,1])}\right],$$

proving (iii). ■

We continue with sequential fractional Ostrowski inequalities.

Theorem 19 *All as in Theorem 17. Then*

i)

$$\|E(f,x_0,x_1)\|_2 \le \frac{1}{2\Gamma((n+1)\alpha+2)} \left\{ \left\| \left\| D_{*0}^{(n+1)\alpha} (f(x_0+t(x_1-x_0))) \right\|_2 \right\|_{t,\infty,[0,1]} \right. \tag{46}$$

$$\left. + \left\| \left\| D_{1-}^{(n+1)\alpha} (f(x_0+J(x_1-x_0))) \right\|_2 \right\|_{J,\infty,[0,1]} \right\},$$

ii) when $\alpha \ge \frac{1}{(n+1)}$, *we get*

$$\|E(f,x_0,x_1)\|_2 \le \frac{1}{2\Gamma((n+1)\alpha+1)} \left\{ \left\| \left\| D_{*0}^{(n+1)\alpha} (f(x_0+t(x_1-x_0))) \right\|_2 \right\|_{t,L_1([0,1])} \right. \tag{47}$$

$$\left. + \left\| \left\| D_{1-}^{(n+1)\alpha} (f(x_0+J(x_1-x_0))) \right\|_2 \right\|_{J,L_1([0,1])} \right\},$$

iii) when $p,q>1 : \frac{1}{p}+\frac{1}{q}=1$, *with* $\alpha > \frac{1}{q(n+1)}$, *we derive*

$$\|E(f,x_0,x_1)\|_2 \le \frac{1}{2\Gamma((n+1)\alpha)(p((n+1)\alpha-1)+1)^{\frac{1}{p}}\left((n+1)\alpha+\frac{1}{p}\right)} \tag{48}$$

$$\left[\left\| \left\| D_{*0}^{(n+1)\alpha} (f(x_0+t(x_1-x_0))) \right\|_2 \right\|_{t,L_q([0,1])} + \right.$$

$$\left. \left\| \left\| D_{1-}^{(n+1)\alpha} (f(x_0+J(x_1-x_0))) \right\|_2 \right\|_{J,L_q([0,1])} \right].$$

Proof. As similar to Theorem 18 is omitted, use of (33). ■

We proceed with a Poincaré like left fractional inequality:

Theorem 20 *Let* $p,q>1 : \frac{1}{p}+\frac{1}{q}=1$, *and* $\nu > \frac{1}{q}$, $m=\lceil \nu \rceil$. *Here all as in Remark 2 and Definition 3. Assume further that* $f^{(k)}(x_0)=0$, $k=0,1,...,m-1$. *Then*

$$\|\|f(x_0+u(x_1-x_0))\|_2\|_{u,L_q([0,1])} \le \frac{\|\|D_{*0}^{\nu} f(x_0+t(x_1-x_0))\|_2\|_{t,L_q([0,1])}}{\Gamma(\nu)(p(\nu-1)+1)^{\frac{1}{p}}(p\nu)^{\frac{1}{q}}}. \tag{49}$$

Proof. Direct application of Theorem 1.38, p. 25, along with Theorem 1.5, p.3 and Definition 1.13, pp. 10-11, all from [2]. We use also Theorem 4 from here. ■

It follows a Sobolev like left fractional inequality:

Theorem 21 *All as in Theorem 20 and* $r>0$. *Then*

$$\|\|f(x_0+u(x_1-x_0))\|_2\|_{u,L_r([0,1])} \le \frac{\|\|D_{*0}^{\nu} f(x_0+t(x_1-x_0))\|_2\|_{t,L_q([0,1])}}{\Gamma(\nu)(p(\nu-1)+1)^{\frac{1}{p}}\left(r\left(\nu-\frac{1}{q}\right)+1\right)^{\frac{1}{r}}}. \tag{50}$$

Proof. As in the proof of Theorem 1.39, p. 26 of [2]. ■

Next comes an Opial type left fractional inequality:

Theorem 22 *All as in Theorem 20. Then*

$$\int_0^z \|f(x_0+w(x_1-x_0))\|_2 \|D_{*0}^{\nu} f(x_0+w(x_1-x_0))\|_2\, dw \leq$$

$$\frac{z^{\nu-1+\frac{2}{p}}}{2^{\frac{1}{q}}\Gamma(\nu)\left((p(\nu-1)+1)(p(\nu-1)+2)\right)^{\frac{1}{p}}}\left(\int_0^z \|D_{*0}^{\nu} f(x_0+w(x_1-x_0))\|_2^q\, dw\right)^{\frac{2}{q}}, \tag{51}$$

$\forall\ z \in [0,1]$.

Proof. By Theorem 1.40, p. 27 of [2]. ■

We continue with a Hilbert-Pachpatte type left fractional inequality:

Theorem 23 *Let $p,q>1 : \frac{1}{p}+\frac{1}{q}=1$, and $\nu_1 > \frac{1}{q}$, $\nu_2 > \frac{1}{p}$, $m_i := \lceil \nu_i \rceil$, $i=1,2$. Let f_i, $i=1,2$, as is f in Remark 2 and Definition 3.*

Assume further that $f_i^{(k_i)}(x_0)=0$, $k_i=0,1,...,m_i-1$; $i=1,2$. Then

$$\int_0^1\int_0^1 \frac{\|f_1(x_0+t_1(x_1-x_0))\|_2 \|f_2(x_0+t_2(x_1-x_0))\|_2\, dt_1 dt_2}{\left(\frac{t_1^{p(\nu_1-1)+1}}{p(p(\nu_1-1)+1)}+\frac{t_2^{q(\nu_2-1)+1}}{q(q(\nu_2-1)+1)}\right)} \leq \tag{52}$$

$$\frac{\left\|\|D_{*0}^{\nu_1} f_1(x_0+t_1(x_1-x_0))\|_2\right\|_{t_1,L_q([0,1])}\left\|\|D_{*0}^{\nu_2} f_2(x_0+t_2(x_1-x_0))\|_2\right\|_{t_2,L_p([0,1])}}{\Gamma(\nu_1)\Gamma(\nu_2)}.$$

Proof. By Theorem 1.41, p. 29 of [2] and Theorem 4 here. ■

We proceed with a Poincaré like right fractional inequality:

Theorem 24 *Let $p,q>1 : \frac{1}{p}+\frac{1}{q}=1$, and $\alpha > \frac{1}{q}$, $m=\lceil \alpha \rceil$. Here all as in Theorem 8. Assume further that $f^{(k)}(x_1)=0$, $k=0,1,...,m-1$. Then*

$$\left\|\|f(x_0+u(x_1-x_0))\|_2\right\|_{u,L_q([0,1])} \leq \frac{\left\|\left\|D_{1-}^{\alpha} f(x_0+t(x_1-x_0))\right\|_2\right\|_{t,L_q([0,1])}}{\Gamma(\alpha)(p(\alpha-1)+1)^{\frac{1}{p}}(p\alpha)^{\frac{1}{q}}}. \tag{53}$$

Proof. By Theorem 2.24, p. 49, along with Theorem 2.5, p. 37, Definition 2.13, pp. 42-43, all from [2]. We use also Theorem 8 from here. ■

It follows a Sobolev like right fractional inequality:

Theorem 25 *All as in Theorem 24 and $r>0$. Then*

$$\left\|\|f(x_0+u(x_1-x_0))\|_2\right\|_{u,L_r([0,1])} \leq \frac{\left\|\left\|D_{1-}^{\alpha} f(x_0+t(x_1-x_0))\right\|_2\right\|_{t,L_q([0,1])}}{\Gamma(\alpha)(p(\alpha-1)+1)^{\frac{1}{p}}\left(r\left(\alpha-\frac{1}{q}\right)+1\right)^{\frac{1}{r}}}. \tag{54}$$

Proof. As in the proof of Theorem 2.25, p. 50 of [2]. ■

Next comes an Opial type right fractional inequality:

Theorem 26 *All as in Theorem 24. Then*

$$\int_z^1 \|f(x_0+w(x_1-x_0))\|_2 \left\|D_{1-}^{\alpha} f(x_0+w(x_1-x_0))\right\|_2 dw \leq$$

$$\frac{(1-z)^{\alpha-1+\frac{2}{p}}}{2^{\frac{1}{q}}\Gamma(\alpha)\left((p(\alpha-1)+1)(p(\alpha-1)+2)\right)^{\frac{1}{p}}} \left(\int_z^1 \left\|D_{1-}^{\alpha} f(x_0+w(x_1-x_0))\right\|_2^q dw\right)^{\frac{2}{q}}, \tag{55}$$

$\forall\ z \in [0,1]$.

Proof. By Theorem 2.26, p. 51 of [2]. ■

We continue with a Hilbert-Pachpatte type right fractional inequality:

Theorem 27 *Let* $p,q>1:\frac{1}{p}+\frac{1}{q}=1$, *and* $\alpha_1>\frac{1}{q}$, $\alpha_2>\frac{1}{p}$, $m_i:=\lceil\alpha_i\rceil$, $i=1,2$. *Let* f_i, $i=1,2$, *as is* f *in Theorem 8.*

Assume further that $f_i^{(k_i)}(x_1)=0$, $k_i=0,1,...,m_i-1$; $i=1,2$. *Then*

$$\int_0^1\int_0^1 \frac{\|f_1(x_0+t_1(x_1-x_0))\|_2\|f_2(x_0+t_2(x_1-x_0))\|_2\,dt_1dt_2}{\left(\frac{(1-t_1)^{p(\alpha_1-1)+1}}{p(p(\alpha_1-1)+1)}+\frac{(1-t_2)^{q(\alpha_2-1)+1}}{q(q(\alpha_2-1)+1)}\right)} \leq \tag{56}$$

$$\frac{\left\|\left\|D_{1-}^{\alpha_1}f_1(x_0+t_1(x_1-x_0))\right\|_2\right\|_{t_1,L_q([0,1])}\left\|\left\|D_{1-}^{\alpha_2}f_2(x_0+t_2(x_1-x_0))\right\|_2\right\|_{t_2,L_p([0,1])}}{\Gamma(\alpha_1)\Gamma(\alpha_2)}.$$

Proof. By Theorem 2.27, p. 53 of [2] and Theorem 8 here. ■

We continue with a sequential left fractional Poincaré type inequality:

Theorem 28 *Let* $\gamma>0$ *with* $\lceil\gamma\rceil=m$; $n,k\in\mathbb{N}$; $p,q>1:\frac{1}{p}+\frac{1}{q}=1$; $0<\alpha<1$ *be such that* $\max\left\{\frac{m+(k-1)\gamma}{n+1},\frac{k\gamma q+1}{(n+1)q}\right\}<\alpha<1$. *Here* $(Y,\|\cdot\|_2)$ *is a Banach space, with* $f(x_0+u(x_1-x_0))\in C^1([0,1],Y)$ *and* $D_{*0}^{k\alpha}f(x_0+u(x_1-x_0))\in C^1([0,1],Y)$, $k=1,...,n$ *and* $\left(D_{*0}^{i\alpha}f(x_0+u(x_1-x_0))\right)(0)=0$, $i=0,2,3,...,n$. *Then*

$$\left\|\left\|D_{*0}^{k\gamma}f(x_0+u(x_1-x_0))\right\|_2\right\|_{u,L_q([0,1])} \leq \tag{57}$$

$$\frac{\left\|\left\|D_{*0}^{(n+1)\alpha}f(x_0+u(x_1-x_0))\right\|_2\right\|_{u,L_q([0,1])}}{\Gamma((n+1)\alpha-k\gamma)\left(p((n+1)\alpha-k\gamma-1)+1\right)^{\frac{1}{p}}}.$$

Proof. By Theorem 10.19, p. 212 of [2] for $g=id$ map. ■

Next comes the sequential right fractional Poincaré type inequality:

Theorem 29 *Let $\gamma > 0$ with $\lceil \gamma \rceil = m$; $n, k \in \mathbb{N}$; $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$; $0 < \alpha < 1$ be such that $\max \left\{ \frac{m+(k-1)\gamma}{n+1}, \frac{k\gamma q+1}{(n+1)q} \right\} < \alpha < 1$. Here $(Y, \|\cdot\|_2)$ is a Banach space, with $f(x_0 + u(x_1 - x_0)) \in C^1([0,1], Y)$ and $D_{1-}^{k\alpha} f(x_0 + u(x_1 - x_0)) \in C^1([0,1], Y)$, $k = 1, ..., n$ and $\left(D_{1-}^{i\alpha} f(x_0 + u(x_1 - x_0))\right)(1) = 0$, $i = 0, 2, 3, ..., n$. Then*

$$\left\| \left\| D_{1-}^{k\gamma} f(x_0 + u(x_1 - x_0)) \right\|_2 \right\|_{u, L_q([0,1])} \leq \tag{58}$$

$$\frac{\left\| \left\| D_{1-}^{(n+1)\alpha} f(x_0 + u(x_1 - x_0)) \right\|_2 \right\|_{u, L_q([0,1])}}{\Gamma((n+1)\alpha - k\gamma)(p((n+1)\alpha - k\gamma - 1) + 1)^{\frac{1}{p}}}.$$

Proof. By Theorem 10.20, p. 213 of [2] for $g = id$ map. ■

It follows a sequential left fractional Opial type inequality:

Theorem 30 *All as in Theorem 28. Then*

$$\int_0^y \left\| D_{*0}^{k\gamma} f(x_0 + u(x_1 - x_0)) \right\|_2 \left\| D_{*0}^{(n+1)\alpha} f(x_0 + u(x_1 - x_0)) \right\|_2 du \leq$$

$$\frac{y^{((n+1)\alpha - k\gamma - 1) + \frac{2}{p}}}{2^{\frac{1}{q}} \Gamma((n+1)\alpha - k\gamma) \left[(p((n+1)\alpha - k\gamma - 1) + 1)(p((n+1)\alpha - k\gamma - 1) + 2) \right]^{\frac{1}{p}}}$$

$$\left(\int_0^y \left\| D_{*0}^{(n+1)\alpha} f(x_0 + u(x_1 - x_0)) \right\|_2^q du \right)^{\frac{2}{q}}, \quad \forall\, y \in [0,1]. \tag{59}$$

Proof. By Theorem 10.21, p. 214 of [2] for $g = id$ map. ■

We continue with a sequential right fractional Opial type inequality:

Theorem 31 *All as in Theorem 29. Then*

$$\int_y^1 \left\| D_{1-}^{k\gamma} f(x_0 + u(x_1 - x_0)) \right\|_2 \left\| D_{1-}^{(n+1)\alpha} f(x_0 + u(x_1 - x_0)) \right\|_2 du \leq$$

$$\frac{(1-y)^{((n+1)\alpha - k\gamma - 1) + \frac{2}{p}}}{2^{\frac{1}{q}} \Gamma((n+1)\alpha - k\gamma) \left[(p((n+1)\alpha - k\gamma - 1) + 1)(p((n+1)\alpha - k\gamma - 1) + 2) \right]^{\frac{1}{p}}}$$

$$\left(\int_y^1 \left\| D_{1-}^{(n+1)\alpha} f(x_0 + u(x_1 - x_0)) \right\|_2^q du \right)^{\frac{2}{q}}, \quad \forall\, y \in [0,1]. \tag{60}$$

Proof. By Theorem 10.22, pp. 216-217 of [2] for $g = id$ map ■

Next comes a sequential Hilbert-Pachpatte type left ftactional inequality:

Theorem 32 *Let $i = 1, 2$; $0 < \alpha_i < 1$, as in (61), (62), $f_i(x_0 + u(x_1 - x_0)) \in C^1([0,1], Y)$. Assume that $D_{*0}^{k_i\alpha_i} f_i(x_0 + u(x_1 - x_0)) \in C^1([0,1], Y)$, $k_i = 1, ..., n_i$, and $\left(D_{*0}^{\lambda_i\alpha_i} f_i(x_0 + u(x_1 - x_0))\right)(0) = 0$, $\lambda_i = 0, 2, 3, ..., n_i$, where $n_i \in \mathbb{N}$. Let $\gamma_i > 0$ and $\lceil \gamma_i \rceil = m_i$, $k_i \in \mathbb{N}$, $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$. We further assume:*

$$1 > \alpha_1 > \max\left(\frac{m_1 + (k_1 - 1)\gamma_1}{n_1 + 1}, \frac{k_1\gamma_1 q + 1}{(n_1 + 1) q}\right), \tag{61}$$

and

$$1 > \alpha_2 > \max\left(\frac{m_2 + (k_2 - 1)\gamma_2}{n_2 + 1}, \frac{k_2\gamma_2 p + 1}{(n_2 + 1) p}\right), \tag{62}$$

Then

$$\int_0^1 \int_0^1 \frac{\left\|D_{*0}^{k_1\gamma_1} f_1(x_0 + u_1(x_1 - x_0))\right\|_2 \left\|D_{*0}^{k_2\gamma_2} f_2(x_0 + u_2(x_1 - x_0))\right\|_2 du_1 du_2}{\left(\frac{u_1^{p((n+1)\alpha_1 - k_1\gamma_1 - 1)+1}}{p(p((n_1+1)\alpha_1 - k_1\gamma_1 - 1)+1)} + \frac{u_2^{q((n_2+1)\alpha_2 - k_2\gamma_2 - 1)+1}}{q(q((n_2+1)\alpha_2 - k_2\gamma_2 - 1)+1)}\right)} \leq \tag{63}$$

$$\frac{1}{\Gamma((n_1 + 1)\alpha_1 - k_1\gamma_1)\Gamma((n_2 + 1)\alpha_2 - k_2\gamma_2)}$$

$$\left\|\left\|D_{*0}^{(n_1+1)\alpha_1} f_1(x_0 + u(x_1 - x_0))\right\|_2\right\|_{u, L_q([0,1])}$$

$$\left\|\left\|D_{*0}^{(n_2+1)\alpha_2} f_2(x_0 + u(x_1 - x_0))\right\|_2\right\|_{u, L_p([0,1])}.$$

Proof. By Theorem 10.23, p. 217 of [2] for $g_1 = g_2 = id$ map. ■

We finish this article with a sequential Hilbert-Pachpatte type right ftactional inequality:

Theorem 33 *Let $i = 1, 2$; $0 < \alpha_i < 1$, as in (64), (65), $f_i(x_0 + u(x_1 - x_0)) \in C^1([0,1], Y)$. Assume that $D_{1-}^{k_i\alpha_i} f_i(x_0 + u(x_1 - x_0)) \in C^1([0,1], Y)$, $k_i = 1, ..., n_i$, and $\left(D_{1-}^{\lambda_i\alpha_i} f_i(x_0 + u(x_1 - x_0))\right)(1) = 0$, $\lambda_i = 0, 2, 3, ..., n_i$, where $n_i \in \mathbb{N}$. Let $\gamma_i > 0$ and $\lceil \gamma_i \rceil = m_i$, $k_i \in \mathbb{N}$, $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$. We further assume:*

$$1 > \alpha_1 > \max\left(\frac{m_1 + (k_1 - 1)\gamma_1}{n_1 + 1}, \frac{k_1\gamma_1 q + 1}{(n_1 + 1) q}\right), \tag{64}$$

and

$$1 > \alpha_2 > \max\left(\frac{m_2 + (k_2 - 1)\gamma_2}{n_2 + 1}, \frac{k_2\gamma_2 p + 1}{(n_2 + 1) p}\right), \tag{65}$$

Then

$$\int_0^1 \int_0^1 \frac{\left\|D_{1-}^{k_1\gamma_1} f_1(x_0 + u_1(x_1 - x_0))\right\|_2 \left\|D_{1-}^{k_2\gamma_2} f_2(x_0 + u_2(x_1 - x_0))\right\|_2 du_1 du_2}{\left(\frac{(1-u_1)^{p((n_1+1)\alpha_1 - k_1\gamma_1 - 1)+1}}{p(p((n_1+1)\alpha_1 - k_1\gamma_1 - 1)+1)} + \frac{(1-u_2)^{q((n_2+1)\alpha_2 - k_2\gamma_2 - 1)+1}}{q(q((n_2+1)\alpha_2 - k_2\gamma_2 - 1)+1)}\right)} \leq \tag{66}$$

$$\frac{1}{\Gamma\left(\left(n_1+1\right)\alpha_1-k_1\gamma_1\right)\Gamma\left(\left(n_2+1\right)\alpha_2-k_2\gamma_2\right)}$$
$$\left\|\left\|D_{1-}^{(n_1+1)\alpha_1}f_1\left(x_0+u\left(x_1-x_0\right)\right)\right\|_2\right\|_{u,L_q([0,1])}$$
$$\left\|\left\|D_{1-}^{(n_2+1)\alpha_2}f_2\left(x_0+u\left(x_1-x_0\right)\right)\right\|_2\right\|_{u,L_p([0,1])}.$$

Proof. By Theorem 10.24, p. 220 of [2] for $g_1 = g_2 = id$ map. ■

References

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