SOME PROPERTIES OF TENSORIAL PERSPECTIVE FOR CONVEX FUNCTIONS OF SELFADJOINT OPERATORS IN HILBERT SPACES

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Abstract. Let H be a Hilbert space. Assume that $f : [0, \infty) \to \mathbb{R}$ is continuous and A, $B > 0$. We define the *tensorial perspective* for the function f and the pair of operators (A, B) by

$$\mathcal{P}_{f,\otimes}(A, B) := (1 \otimes B) f\left(A \otimes B^{-1}\right).$$

In this paper we show among others that, if f is differentiable convex, then

$$\mathcal{P}_{f,\otimes}(A, B) \geq \left[f(u) - f'(u) u\right](1 \otimes B) + f'(u)(A \otimes 1),$$

for A, $B > 0$ and $u > 0$. Moreover, if $\operatorname{Sp}(A) \subset I$, $\operatorname{Sp}(B) \subset J$ and such that $0 < \gamma \leq \frac{t}{s} \leq \Gamma$ for $t \in I$ and $s \in J$, then

$$\begin{aligned}\mathcal{P}_{f,\otimes}(A, B) &\leq \left[f(u) - f'(u) u\right](1 \otimes B) + f'(u)(A \otimes 1)\\ &\quad + \left[f'_-(\Gamma) - f'_+(\gamma)\right]\left|A \otimes 1 - u(1 \otimes B)\right|\end{aligned}$$

for $u \in [\gamma, \Gamma]$.

1. Introduction

Let $I_1, ..., I_k$ be intervals from $\mathbb{R}$ and let $f : I_1 \times ... \times I_k \to \mathbb{R}$ be an essentially bounded real function defined on the product of the intervals. Let $A = (A_1, ..., A_n)$ be a k-tuple of bounded selfadjoint operators on Hilbert spaces $H_1, ..., H_k$ such that the spectrum of A_i is contained in I_i for $i = 1, ..., k$. We say that such a k-tuple is in the domain of f. If

$$A_i = \int_{I_i} \lambda_i dE_i(\lambda_i)$$

is the spectral resolution of A_i for $i = 1, ..., k$; by following [2], we define

$$f(A_1, ..., A_k) := \int_{I_1} ... \int_{I_k} f(\lambda_1, ..., \lambda_1)\, dE_1(\lambda_1) \otimes ... \otimes dE_k(\lambda_k) \tag{1.1}$$

as a bounded selfadjoint operator on the tensorial product $H_1 \otimes ... \otimes H_k$.

If the Hilbert spaces are of finite dimension, then the above integrals become finite sums, and we may consider the functional calculus for arbitrary real functions. This construction [2] extends the definition of Korányi [7] for functions of two variables and have the property that

$$f(A_1, ..., A_k) = f_1(A_1) \otimes ... \otimes f_k(A_k),$$

whenever f can be separated as a product $f(t_1, ..., t_k) = f_1(t_1)...f_k(t_k)$ of k functions each depending on only one variable.

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It is know that, *if f is super-multiplicative (sub-multiplicative)* on $[0,\infty)$, namely

$$f(st) \geq (\leq) f(s) f(t) \text{ for all } s,t \in [0,\infty)$$

and if f is continuous on $[0,\infty)$, then [10, p. 173]

$$f(A \otimes B) \geq (\leq) f(A) \otimes f(B) \text{ for all } A,\ B \geq 0. \tag{1.2}$$

This follows by observing that, if

$$A = \int_{[0,\infty)} t dE(t) \text{ and } B = \int_{[0,\infty)} s dF(s)$$

are the spectral resolutions of A and B, then

$$f(A \otimes B) = \int_{[0,\infty)} \int_{[0,\infty)} f(st)\, dE(t) \otimes dF(s) \tag{1.3}$$

for the continuous function f on $[0,\infty)$.

Recall the *geometric operator mean* for the positive operators $A,\ B > 0$

$$A\#_t B := A^{1/2}(A^{-1/2}BA^{-1/2})^t A^{1/2},$$

where $t \in [0,1]$ and

$$A\# B := A^{1/2}(A^{-1/2}BA^{-1/2})^{1/2} A^{1/2}.$$

By the definitions of $\#$ and $\otimes$ we have

$$A\#B = B\#A \text{ and } (A\#B) \otimes (B\#A) = (A \otimes B) \# (B \otimes A).$$

In 2007, S. Wada [13] obtained the following *Callebaut type inequalities* for tensorial product

$$\begin{aligned}(A\#B) \otimes (A\#B) &\leq \frac{1}{2}\left[(A\#_\alpha B) \otimes (A\#_{1-\alpha} B) + (A\#_{1-\alpha} B) \otimes (A\#_\alpha B)\right] \\ &\leq \frac{1}{2}(A \otimes B + B \otimes A)\end{aligned} \tag{1.4}$$

for $A,\ B > 0$ and $\alpha \in [0,1]$.

Assume that $f : [0,\infty) \to \mathbb{R}$ is continuous and $A,\ B > 0$. We define the *tensorial perspective* for the function f and the pair of operators (A,B)

$$\mathcal{P}_{f,\otimes}(A,B) := (1 \otimes B) f\left(A \otimes B^{-1}\right).$$

Motivated by the above results, in this paper we show among others that, if f is differentiable convex, then

$$\mathcal{P}_{f,\otimes}(A,B) \geq [f(u) - f'(u) u](1 \otimes B) + f'(u)(A \otimes 1),$$

for $A,\ B > 0$ and $u > 0$. Moreover, if $\mathrm{Sp}(A) \subset I$, $\mathrm{Sp}(B) \subset J$ and such that $0 < \gamma \leq \frac{t}{s} \leq \Gamma$ for $t \in I$ and $s \in J$, then

$$\begin{aligned}\mathcal{P}_{f,\otimes}(A,B) &\leq [f(u) - f'(u) u](1 \otimes B) + f'(u)(A \otimes 1) \\ &\quad + \left[f'_-(\Gamma) - f'_+(\gamma)\right] |A \otimes 1 - u(1 \otimes B)|\end{aligned}$$

for $u \in [\gamma,\Gamma]$.

2. Some Preliminary Facts

Recall the following property of the tensorial product

$$(AC) \otimes (BD) = (A \otimes B)(C \otimes D) \tag{2.1}$$

that holds for any $A, B, C, D \in B(H)$.

If we take $C = A$ and $D = B$, then we get

$$A^2 \otimes B^2 = (A \otimes B)^2 .$$

By induction and using (2.1) we derive that

$$A^n \otimes B^n = (A \otimes B)^n \text{ for natural } n \geq 0. \tag{2.2}$$

In particular

$$A^n \otimes 1 = (A \otimes 1)^n \text{ and } 1 \otimes B^n = (1 \otimes B)^n \tag{2.3}$$

for all $n \geq 0$.

We also observe that, by (2.1), the operators $A \otimes 1$ and $1 \otimes B$ are commutative and

$$(A \otimes 1)(1 \otimes B) = (1 \otimes B)(A \otimes 1) = A \otimes B. \tag{2.4}$$

Moreover, for two natural numbers m, n we have

$$(A \otimes 1)^m (1 \otimes B)^n = (1 \otimes B)^n (A \otimes 1)^m = A^m \otimes B^n. \tag{2.5}$$

According with the properties of tensorial products and functional calculus for continuous functions of selfadjoint operators, we have

$$\begin{aligned}
\mathcal{P}_{f,\otimes}(A, B) &= (1 \otimes B) f\left((A \otimes 1)(1 \otimes B)^{-1}\right) \\
&= f\left((A \otimes 1)(1 \otimes B)^{-1}\right)(1 \otimes B) \\
&= f\left((1 \otimes B)^{-1}(A \otimes 1)\right)(1 \otimes B),
\end{aligned}$$

due to the commutativity of $A \otimes 1$ and $1 \otimes B$.

In the following, we consider the spectral resolutions of A and B given by

$$A = \int_{[0,\infty)} t dE(t) \text{ and } B = \int_{[0,\infty)} s dF(s). \tag{2.6}$$

We have the following representation result for continuous functions:

Lemma 1. *Assume that $f : [0, \infty) \to \mathbb{R}$ is continuous and A, $B > 0$, then*

$$\mathcal{P}_{f,\otimes}(A, B) = \int_{[0,\infty)} \int_{[0,\infty)} s f\left(\frac{t}{s}\right) dE(t) \otimes dF(s). \tag{2.7}$$

Proof. By Stone-Weierstrass theorem, any continuous function can be approximated by a sequence of polynomials, therefore it suffices to prove the equality for the power function $\varphi(t) = t^n$ with n any natural number.

We have that

$$\begin{aligned}
&\int_{[0,\infty)}\int_{[0,\infty)} s\varphi\left(\frac{t}{s}\right) dE(t)\otimes dF(s)\\
&=\int_{[0,\infty)}\int_{[0,\infty)} s\left(\frac{t}{s}\right)^{n} dE(t)\otimes dF(s)\\
&=\int_{[0,\infty)}\int_{[0,\infty)} t^{n}s^{1-n} dE(t)\otimes dF(s)\\
&=A^{n}\otimes B^{1-n}=A^{n}\otimes BB^{-n}=(1\otimes B)\left(A^{n}\otimes B^{-n}\right)\\
&=(1\otimes B)\left(A\otimes B^{-1}\right)^{n}=\mathcal{P}_{\varphi,\otimes}(A,B),
\end{aligned}$$

which shows that (2.7) holds for the power function.

This proves the lemma. □

We assume in the following that $A,\, B>0$.

If we consider the function $\Pi_r(u)=u^r-1$, $u\geq 0$, $r>0$, then we have

$$\begin{aligned}
\mathcal{P}_{\Pi_r,\otimes}(A,B) &:= (1\otimes B)\,\Pi_r\left(A\otimes B^{-1}\right)\\
&=(1\otimes B)\left[\left(A\otimes B^{-1}\right)^{r}-1\right]\\
&=(A\otimes 1)^{r}(1\otimes B)^{1-r}-1\otimes B.
\end{aligned}$$

If we take $f=-\ln(\cdot)$, then we get

$$\begin{aligned}
\mathcal{P}_{-\ln(\cdot),\otimes}(A,B) &:= -(1\otimes B)\ln\left(A\otimes B^{-1}\right)\\
&=-\ln\left((1\otimes B)^{-1}(A\otimes 1)\right)(1\otimes B)\\
&=(1\otimes B)\left[\ln(1\otimes B)-\ln(A\otimes 1)\right].
\end{aligned}$$

If we take $f=(\cdot)\ln(\cdot)$, then we get

$$\begin{aligned}
\mathcal{P}_{(\cdot)\ln(\cdot),\otimes}(A,B) &:= (1\otimes B)\left(A\otimes B^{-1}\right)\ln\left(A\otimes B^{-1}\right)\\
&=(A\otimes 1)\left[\ln(A\otimes 1)-\ln(1\otimes B)\right].
\end{aligned}$$

If we take $f=|\cdot-\alpha|$, $\alpha\in\mathbb{R}$, then

$$\begin{aligned}
\mathcal{P}_{|\cdot-\alpha|,\otimes}(A,B) &= \int_{[0,\infty)}\int_{[0,\infty)} s\left|\frac{t}{s}-\alpha\right| dE(t)\otimes dF(s)\\
&=\int_{[0,\infty)}\int_{[0,\infty)} |t-\alpha s|\, dE(t)\otimes dF(s)\\
&=|A\otimes 1-\alpha 1\otimes B|,
\end{aligned}$$

where for the last equality we used the result obtained in [6],

$$\psi(h(A)\otimes 1+1\otimes k(B))=\int_I\int_J \psi(h(t)+k(s))\, dE(t)\otimes dF(s), \tag{2.8}$$

here A and B are selfadjoint operators with $\operatorname{Sp}(A)\subset I$ and $\operatorname{Sp}(B)\subset J$, h is continuous on I, k is continuous on J and ψ is continuous on an interval U that contains the sum of the intervals $h(I)+k(J)$, while A and B have the spectral resolutions

$$A=\int_I t dE(t) \text{ and } B=\int_J s dF(s).$$

For $f=|\cdot-1|$ we get

$$\mathcal{P}_{|\cdot-1|,\otimes}(A,B)=|A\otimes 1-1\otimes B|.$$

Consider the q-logarithm defined by

$$\ln_q u=\begin{cases}\frac{u^{1-q}-1}{1-q} \text{ if } q\neq 1,\\ \\ \ln u \text{ if } q=1.\end{cases}$$

For $q\neq 1$ we define

$$\begin{aligned}\mathcal{P}_{\ln_q,\otimes}(A,B)&:=(1\otimes B)\ln_q\left((A\otimes 1)(1\otimes B)^{-1}\right)\\ &=\frac{(A\otimes 1)^{1-q}(1\otimes B)^q-1\otimes B}{1-q}.\end{aligned}\tag{2.9}$$

Let f be a continuous function defined on the interval I of real numbers, B a self-adjoint operator on the Hilbert space H and A a positive invertible operator on H. Assume that the spectrum $\operatorname{Sp}\left(A^{-1/2}BA^{-1/2}\right)\subset\mathring{I}$. Then by using the continuous functional calculus, we can define the *perspective* $\mathcal{P}_f(B,A)$ by setting

$$\mathcal{P}_f(B,A):=A^{1/2}f\left(A^{-1/2}BA^{-1/2}\right)A^{1/2}.$$

If A and B are commutative, then

$$\mathcal{P}_f(B,A)=Af\left(BA^{-1}\right)$$

provided $\operatorname{Sp}\left(BA^{-1}\right)\subset\mathring{I}$.

It is well known that (see for instance [8]), if f is an *operator convex function* defined in the positive half-line, then the mapping

$$(B,A)\mapsto\mathcal{P}_f(B,A)$$

defined in pairs of positive definite operators, is operator convex.

The following inequality is also of interest, see [12]:

Theorem 1. *Assume that f is nonnegative and operator monotone on $[0,\infty)$. If $A\geq C>0$ and $B\geq D>0$, then*

$$\mathcal{P}_f(A,B)\geq\mathcal{P}_f(C,D).\tag{2.10}$$

We can state the following result for the tensorial perspective:

Theorem 2. *If f is an operator convex function defined in the positive half-line, then $\mathcal{P}_{f,\otimes}$ is operator convex in pairs of positive definite operators as well. If $A\geq C>0$ and $B\geq D>0$, then also*

$$\mathcal{P}_{f,\otimes}(A,B)\geq\mathcal{P}_{f,\otimes}(C,D).\tag{2.11}$$

Proof. Assume f is an *operator convex function* in the positive half-line. Since $A\otimes 1$ and $1\otimes B$ are commutative, hence

$$\mathcal{P}_{f,\otimes}(A,B)=(1\otimes B)f\left((A\otimes 1)(1\otimes B)^{-1}\right)=\mathcal{P}_f(A\otimes 1,1\otimes B)\tag{2.12}$$

for A, $B>0$.

If A, B, C, $D > 0$ and $\lambda \in [0,1]$, then we have

$$\begin{aligned}
&\mathcal{P}_{f,\otimes}((1-\lambda)(A,B)+\lambda(C,D)) \\
&= \mathcal{P}_{f,\otimes}(((1-\lambda)A+\lambda C,(1-\lambda)B+\lambda D)) \\
&= \mathcal{P}_{f}(((1-\lambda)A+\lambda C)\otimes 1, 1\otimes((1-\lambda)B+\lambda D)) \\
&= \mathcal{P}_{f}((1-\lambda)A\otimes 1+\lambda C\otimes 1,(1-\lambda)1\otimes B+\lambda 1\otimes D) \\
&= \mathcal{P}_{f}((1-\lambda)(A\otimes 1,1\otimes B)+\lambda(C\otimes 1,1\otimes D)) \\
&\leq (1-\lambda)\mathcal{P}_{f}(A\otimes 1,1\otimes B)+\lambda\mathcal{P}_{f}(C\otimes 1,1\otimes D) \\
&= (1-\lambda)\mathcal{P}_{f,\otimes}(A,B)+\lambda\mathcal{P}_{f,\otimes}(C,D),
\end{aligned}$$

which shows that $\mathcal{P}_{f,\otimes}$ is operator convex in pairs of positive definite operators.

If $A \geq C > 0$ and $B \geq D > 0$, then $A\otimes 1 \geq C\otimes 1 > 0$ and $1\otimes B \geq 1\otimes D > 0$. By utilizing Theorem 1 we derive that

$$\mathcal{P}_{f}(A\otimes 1,1\otimes B) \geq \mathcal{P}_{f}(C\otimes 1,1\otimes D).$$

By utilizing the representation (2.12) we derive the desired result (2.11). □

3. Main Results

Suppose that I is an interval of real numbers with interior $\mathring{I}$ and $f: I \to \mathbb{R}$ is a convex function on I. Then f is continuous on $\mathring{I}$ and has finite left and right derivatives at each point of $\mathring{I}$. Moreover, if $x,y \in \mathring{I}$ and $x < y$, then $f'_{-}(x) \leq f'_{+}(x) \leq f'_{-}(y) \leq f'_{+}(y)$ which shows that both f'_{-} and f'_{+} are nondecreasing function on $\mathring{I}$. It is also known that a convex function must be differentiable except for at most countably many points.

For a convex function $f: I \to \mathbb{R}$, the subdifferential of f denoted by ∂f is the set of all functions $\varphi: I \to [-\infty,\infty]$ such that $\varphi\left(\mathring{I}\right) \subset \mathbb{R}$ and

$$f(x) \geq f(a)+(x-a)\varphi(a) \text{ for any } x,a \in I. \tag{3.1}$$

It is also well known that if f is convex on I, then ∂f is nonempty, f'_{-}, $f'_{+} \in \partial f$ and if $\varphi \in \partial f$, then

$$f'_{-}(x) \leq \varphi(x) \leq f'_{+}(x) \text{ for any } x \in \mathring{I}.$$

In particular, φ is a nondecreasing function.

If f is differentiable and convex on $\mathring{I}$, then $\partial f = \{f'\}$.

Theorem 3. *Assume that f is convex on $(0,\infty)$, A, $B > 0$ and $u \in (0,\infty)$ while $\varphi \in \partial f$, then*

$$\mathcal{P}_{f,\otimes}(A,B) \geq [f(u)-\varphi(u)u](1\otimes B)+\varphi(u)(A\otimes 1). \tag{3.2}$$

Moreover, if f is differentiable, then

$$\mathcal{P}_{f,\otimes}(A,B) \geq [f(u)-f'(u)u](1\otimes B)+f'(u)(A\otimes 1), \tag{3.3}$$

for all A, $B > 0$ and $u \in (0,\infty)$.

Proof. By the gradient inequality we have

$$f(x) \geq f(u)+(x-u)\varphi(u) \tag{3.4}$$

for all x, $u \in (0,\infty)$.

If we take $x=\frac{t}{s}$ in (3.4), then we get

$$f\left(\frac{t}{s}\right)\geq f\left(u\right)+\left(\frac{t}{s}-u\right)\varphi\left(u\right) \tag{3.5}$$

for all $t,\ s>0$.

If we multiply (3.5) by $s>0$, then we get

$$sf\left(\frac{t}{s}\right)\geq sf\left(u\right)+\varphi\left(u\right)\left(t-us\right) \tag{3.6}$$

for all $t,\ s>0$.

We consider the spectral resolutions of A and B given by

$$A=\int_{[0,\infty)}tdE\left(t\right)\text{ and }B=\int_{[0,\infty)}sdF\left(s\right).$$

If we take in (3.6) the integral $\int_{[0,\infty)}\int_{[0,\infty)}$ over $dE\left(t\right)\otimes dF\left(s\right)$, then we get

$$\begin{aligned}&\int_{[0,\infty)}\int_{[0,\infty)}sf\left(\frac{t}{s}\right)dE\left(t\right)\otimes dF\left(s\right)\\&\geq\int_{[0,\infty)}\int_{[0,\infty)}\left[sf\left(u\right)+\varphi\left(u\right)\left(t-us\right)\right]dE\left(t\right)\otimes dF\left(s\right)\\&=f\left(u\right)\int_{[0,\infty)}\int_{[0,\infty)}sdE\left(t\right)\otimes dF\left(s\right)\\&+\varphi\left(u\right)\left[\int_{[0,\infty)}\int_{[0,\infty)}tdE\left(t\right)\otimes dF\left(s\right)-u\int_{[0,\infty)}\int_{[0,\infty)}sdE\left(t\right)\otimes dF\left(s\right)\right]\\&=f\left(u\right)\left(1\otimes B\right)+\varphi\left(u\right)\left(A\otimes 1-u1\otimes B\right)\end{aligned}$$

and by the representation (2.7) we get the desired inequality (3.2). □

Corollary 1. *With the assumptions of Theorem 3 and for $x,\ y\in H$ with $\|x\|=\|y\|=1$, we have*

$$\left\langle\mathcal{P}_{f,\otimes}\left(A,B\right)\left(x\otimes y\right),x\otimes y\right\rangle\geq\left[f\left(u\right)-\varphi\left(u\right)u\right]\left\langle By,y\right\rangle+\varphi\left(u\right)\left\langle Ax,x\right\rangle, \tag{3.7}$$

for all $u>0$.

If f is differentiable, then

$$\left\langle\mathcal{P}_{f,\otimes}\left(A,B\right)\left(x\otimes y\right),x\otimes y\right\rangle\geq\left[f\left(u\right)-f'\left(u\right)u\right]\left\langle By,y\right\rangle+f'\left(u\right)\left\langle Ax,x\right\rangle. \tag{3.8}$$

In particular, if we take $u=\frac{\langle Ax,x\rangle}{\langle By,y\rangle}$ in (3.7) then we get the Jensen's type inequality of interest

$$\frac{\left\langle\mathcal{P}_{f,\otimes}\left(A,B\right)\left(x\otimes y\right),x\otimes y\right\rangle}{\left\langle By,y\right\rangle}\geq f\left(\frac{\left\langle Ax,x\right\rangle}{\left\langle By,y\right\rangle}\right). \tag{3.9}$$

Proof. If we take the tensorial inner product over $x\otimes y$ in (3.2), then we get

$$\begin{aligned}&\left\langle\mathcal{P}_{f,\otimes}\left(A,B\right)\left(x\otimes y\right),x\otimes y\right\rangle\\&\geq f\left(u\right)\left\langle\left(1\otimes B\right)\left(x\otimes y\right),x\otimes y\right\rangle\\&+\varphi\left(u\right)\left\langle\left(A\otimes 1-u1\otimes B\right)\left(x\otimes y\right),x\otimes y\right\rangle\\&=f\left(u\right)\left\langle\left(1\otimes B\right)\left(x\otimes y\right),x\otimes y\right\rangle\\&+\varphi\left(u\right)\left[\left\langle\left(A\otimes 1\right)\left(x\otimes y\right),x\otimes y\right\rangle-u\left\langle 1\otimes B\left(x\otimes y\right),x\otimes y\right\rangle\right].\end{aligned} \tag{3.10}$$

Observe that for x, $y \in H$ with $\|x\| = \|y\| = 1$, we have

$$\begin{aligned}\left\langle \left(1 \otimes B\right)\left(x \otimes y\right), x \otimes y\right\rangle &= \left\langle \left(1x \otimes By\right), x \otimes y\right\rangle \\ &= \left\langle 1x, x\right\rangle \left\langle By, y\right\rangle = \|x\|^2 \left\langle By, y\right\rangle = \left\langle By, y\right\rangle\end{aligned}$$

and

$$\begin{aligned}\left\langle \left(A \otimes 1\right)\left(x \otimes y\right), x \otimes y\right\rangle &= \left\langle Ax \otimes 1y, x \otimes y\right\rangle \\ &= \left\langle Ax, x\right\rangle \left\langle 1y, y\right\rangle = \left\langle Ax, x\right\rangle \|y\|^2 = \left\langle Ax, x\right\rangle\end{aligned}$$

and by (3.10) we deduce (3.7).

If we take $u = \frac{\langle Ax,x\rangle}{\langle By,y\rangle}$ in (3.7), then we get

$$\begin{aligned}&\left\langle \mathcal{P}_{f,\otimes}\left(A, B\right)\left(x \otimes y\right), x \otimes y\right\rangle \\ &\geq \left[f\left(\frac{\langle Ax, x\rangle}{\langle By, y\rangle}\right) - \varphi\left(\frac{\langle Ax, x\rangle}{\langle By, y\rangle}\right)\frac{\langle Ax, x\rangle}{\langle By, y\rangle}\right]\langle By, y\rangle \\ &+ \varphi\left(\frac{\langle Ax, x\rangle}{\langle By, y\rangle}\right)\langle Ax, x\rangle \\ &= f\left(\frac{\langle Ax, x\rangle}{\langle By, y\rangle}\right)\langle By, y\rangle,\end{aligned}$$

which gives (3.9). □

Corollary 2. *Assume that f is convex on $(0, \infty)$, $0 < m \leq A$, $B \leq M$ for some constants m, M and $\varphi \in \partial f$, then*

$$\begin{aligned}\mathcal{P}_{f,\otimes}\left(A, B\right) \geq &\left[f\left(\frac{m+M}{2}\right) - \varphi\left(\frac{m+M}{2}\right)\frac{m+M}{2}\right]\left(1 \otimes B\right) \\ &+ \varphi\left(\frac{m+M}{2}\right)\left(A \otimes 1\right)\end{aligned} \tag{3.11}$$

and, if f is differentiable,

$$\begin{aligned}\mathcal{P}_{f,\otimes}\left(A, B\right) \geq &\left[f\left(\frac{m+M}{2}\right) - f'\left(\frac{m+M}{2}\right)\frac{m+M}{2}\right]\left(1 \otimes B\right) \\ &+ f'\left(\frac{m+M}{2}\right)\left(A \otimes 1\right).\end{aligned} \tag{3.12}$$

Also

$$\begin{aligned}\mathcal{P}_{f,\otimes}\left(A, B\right) \geq &\left(\frac{f\left(M\right) - f\left(m\right)}{M - m}\right)\left(A \otimes 1\right) \\ &+ \left(\frac{2}{M - m}\int_m^M f\left(u\right)du - \frac{Mf\left(M\right) - mf\left(m\right)}{M - m}\right)\left(1 \otimes B\right).\end{aligned} \tag{3.13}$$

Proof. If we take the integral mean in (3.2), then we get

$$\begin{aligned}\mathcal{P}_{f,\otimes}(A,B) \geq & \left(\frac{1}{M-m}\int_m^M f(u)\,du\right)(1\otimes B) \\ & + \left(\frac{1}{M-m}\int_m^M \varphi(u)\,du\right)(A\otimes 1) \\ & - \left(\frac{1}{M-m}\int_m^M \varphi(u)\,udu\right)(1\otimes B).\end{aligned} \tag{3.14}$$

Observe that, since $\varphi \in \partial\Phi$, hence

$$\frac{1}{M-m}\int_m^M \varphi(u)\,du = \frac{f(M)-f(m)}{M-m}$$

and

$$\begin{aligned}\frac{1}{M-m}\int_m^M u\varphi(u)\,du &= \frac{1}{M-m}\left[uf(u)|_m^M - \int_m^M f(u)\,du\right] \\ &= \frac{Mf(M)-mf(m)}{M-m} - \frac{1}{M-m}\int_m^M f(u)\,du.\end{aligned}$$

Therefore

$$\begin{aligned}&\left(\frac{1}{M-m}\int_m^M f(u)\,du\right)(1\otimes B) + \left(\frac{1}{M-m}\int_m^M \varphi(u)\,du\right)(A\otimes 1) \\ &- \left(\frac{1}{M-m}\int_m^M \varphi(u)\,udu\right)(1\otimes B). \\ &= \left(\frac{f(M)-f(m)}{M-m}\right)(A\otimes 1) \\ &+ \left(\frac{2}{M-m}\int_m^M f(u)\,du - \frac{Mf(M)-mf(m)}{M-m}\right)(1\otimes B)\end{aligned}$$

and by (3.14) we obtain (3.13). $\square$

Theorem 4. *Assume that f is continuously differentiable convex on $(0,\infty)$, A, $B>0$ and $u\in(0,\infty)$, then*

$$\mathcal{P}_{f,\otimes}(A,B) \leq f(u)(1\otimes B) + \mathcal{P}^{\dagger}_{f',\otimes}(A,B) - u\mathcal{P}_{f',\otimes}(A,B), \tag{3.15}$$

where for a continuous function g on $(0,\infty)$,

$$\begin{aligned}\mathcal{P}^{\dagger}_{g,\otimes}(A,B) &:= \int_{[0,\infty)}\int_{[0,\infty)} tg\left(\frac{t}{s}\right)dE(t)\otimes dF(s) \\ &= (A\otimes 1)\,g\left(A\otimes B^{-1}\right) \\ &= (A\otimes 1)\,g\left((A\otimes 1)(1\otimes B)^{-1}\right).\end{aligned} \tag{3.16}$$

Proof. By the gradient inequality we have

$$f(x) \leq f(u) + (x-u)f'(x) \tag{3.17}$$

for all x, $u\in(0,\infty)$.

If we take $x = \frac{t}{s}$ in (3.17) and multiply with s, then we get

$$sf\left(\frac{t}{s}\right) \leq sf(u) + tf'\left(\frac{t}{s}\right) - usf'\left(\frac{t}{s}\right) \tag{3.18}$$

for all $t,\, s \in (0, \infty)$.

We consider the spectral resolutions of A and B given by

$$A = \int_{[0,\infty)} t dE(t) \text{ and } B = \int_{[0,\infty)} s dF(s).$$

If we take in (3.18) the integral $\int_{[0,\infty)} \int_{[0,\infty)}$ over $dE(t) \otimes dF(s)$, then we get

$$\begin{aligned}
&\int_{[0,\infty)} \int_{[0,\infty)} sf\left(\frac{t}{s}\right) dE(t) \otimes dF(s) \\
&\leq f(u) \int_{[0,\infty)} \int_{[0,\infty)} s dE(t) \otimes dF(s) \\
&+ \int_{[0,\infty)} \int_{[0,\infty)} tf'\left(\frac{t}{s}\right) dE(t) \otimes dF(s) \\
&- u \int_{[0,\infty)} \int_{[0,\infty)} sf'\left(\frac{t}{s}\right) dE(t) \otimes dF(s),
\end{aligned} \tag{3.19}$$

which gives the desired inequality (3.15). □

Corollary 3. *With the assumptions of Theorem 4 and for $x,\, y \in H$ with $\|x\| = \|y\| = 1$, we have*

$$\begin{aligned}
&\left\langle \mathcal{P}_{f,\otimes}(A, B)(x \otimes y), x \otimes y \right\rangle \\
&\leq f(u) \langle By, y \rangle + \left\langle \mathcal{P}^{\dagger}_{f',\otimes}(A, B)(x \otimes y), x \otimes y \right\rangle \\
&- u \left\langle \mathcal{P}_{f',\otimes}(A, B)(x \otimes y), x \otimes y \right\rangle,
\end{aligned} \tag{3.20}$$

for all $u > 0$.

In particular, if we take $u = \frac{\langle Ax, x\rangle}{\langle By, y\rangle}$ in (3.7) then we get the Jensen's type inequality of interest

$$\begin{aligned}
0 &\leq \frac{\left\langle \mathcal{P}_{f,\otimes}(A, B)(x \otimes y), x \otimes y \right\rangle}{\langle By, y\rangle} - f\left(\frac{\langle Ax, x\rangle}{\langle By, y\rangle}\right) \\
&\leq \frac{\left\langle \mathcal{P}^{\dagger}_{f',\otimes}(A, B)(x \otimes y), x \otimes y \right\rangle}{\langle By, y\rangle} \\
&- \frac{\langle Ax, x\rangle}{\langle By, y\rangle^2} \left\langle \mathcal{P}_{f',\otimes}(A, B)(x \otimes y), x \otimes y \right\rangle.
\end{aligned} \tag{3.21}$$

Corollary 4. *With the assumptions of Theorem 4 and if $0 < m \leq A,\, B \leq M$ for some constants m, M, then*

$$\begin{aligned}
\mathcal{P}_{f,\otimes}(A, B) &\leq f\left(\frac{m+M}{2}\right)(1 \otimes B) + \mathcal{P}^{\dagger}_{f',\otimes}(A, B) \\
&- \frac{m+M}{2} \mathcal{P}_{f',\otimes}(A, B)
\end{aligned} \tag{3.22}$$

and

$$\mathcal{P}_{f,\otimes}(A,B) \le \left(\frac{1}{M-m}\int_m^M f(u)\,du\right)(1\otimes B) + \mathcal{P}^{\dagger}_{f',\otimes}(A,B) - \frac{m+M}{2}\mathcal{P}_{f',\otimes}(A,B). \tag{3.23}$$

We also have:

Theorem 5. *Assume that f is convex on $(0,\infty)$, A, $B>0$ with spectra $\operatorname{Sp}(A)\subset I$, $\operatorname{Sp}(B)\subset J$ and such that $0<\gamma\le\frac{t}{s}\le\Gamma$ for $t\in I$ and $s\in J$, then*

$$\begin{aligned}\mathcal{P}_{f,\otimes}(A,B) &\le [f(u)-u\varphi(u)](1\otimes B)+\varphi(u)(A\otimes 1)\\ &\quad+\left[f'_-(\Gamma)-f'_+(\gamma)\right]|A\otimes 1-u(1\otimes B)|\end{aligned} \tag{3.24}$$

for $u\in[\gamma,\Gamma]$ and $\varphi\in\partial f$.

Proof. Observe that, by the gradient inequality we have

$$\begin{aligned}f(x) &\le f(u)+(x-u)\varphi(x)\\ &= f(u)+(x-u)\varphi(u)+(x-u)[\varphi(x)-\varphi(u)]\end{aligned} \tag{3.25}$$

for x, $u>0$ and $\varphi\in\partial f$.

Since φ is monotonic nondrecreasing, then

$$\begin{aligned}0 &\le (f'(x)-f'(u))(x-u) = |(f'(x)-f'(u))(x-u)|\\ &= |f'(x)-f'(u)|\,|x-u| \le \left[f'_-(\Gamma)-f'_+(\gamma)\right]|x-u|,\end{aligned}$$

for x, $u\in[\gamma,\Gamma]$ and by (3.25)

$$f(x)\le f(u)+(x-u)\varphi(u)+\left[f'_-(\Gamma)-f'_+(\gamma)\right]|x-u| \tag{3.26}$$

for x, $u\in[\gamma,\Gamma]$.

If we take in (3.26) $x=\frac{t}{s}$ and multiply with s, then we get

$$sf\left(\frac{t}{s}\right)\le sf(u)+(t-us)\varphi(u)+\left[f'_-(\Gamma)-f'_+(\gamma)\right]|t-us| \tag{3.27}$$

for t, $s>0$ with $\frac{t}{s}$, $u\in[\gamma,\Gamma]$.

We consider the spectral resolutions of A and B given by

$$A=\int_I t\,dE(t) \text{ and } B=\int_J s\,dF(s).$$

If we take in (3.18) the integral $\int_I\int_J$ over $dE(t)\otimes dF(s)$, then we get

$$\begin{aligned}&\int_I\int_J sf\left(\frac{t}{s}\right)dE(t)\otimes dF(s)\\ &\le f(u)\int_I\int_J s\,dE(t)\otimes dF(s)+\varphi(u)\int_I\int_J(t-us)\,dE(t)\otimes dF(s)\\ &+\left[f'_-(\Gamma)-f'_+(\gamma)\right]\int_I\int_J|t-us|\,dE(t)\otimes dF(s),\end{aligned}$$

which, as above, gives the desired result (3.24). □

Corollary 5. *With the assumptions of Theorem 5 and for x, $y \in H$ with $\|x\| = \|y\| = 1$, we have*

$$\begin{aligned} &\langle \mathcal{P}_{f,\otimes}(A,B)(x\otimes y), x\otimes y\rangle \\ &\leq [f(u) - u\varphi(u)]\langle By, y\rangle + \langle Ax, x\rangle \varphi(u) \\ &+ \left[f'_-(\Gamma) - f'_+(\gamma)\right]\langle |A\otimes 1 - u(1\otimes B)|(x\otimes y), x\otimes y\rangle \end{aligned} \tag{3.28}$$

for all $u \in [\gamma, \Gamma]$.

In particular, if we take $u = \frac{\langle Ax,x\rangle}{\langle By,y\rangle} \in [\gamma,\Gamma]$ in (3.28) then we get the reverse of Jensen's inequality

$$\begin{aligned} &0 \leq \frac{\langle \mathcal{P}_{f,\otimes}(A,B)(x\otimes y), x\otimes y\rangle}{\langle By, y\rangle} - f\left(\frac{\langle Ax,x\rangle}{\langle By,y\rangle}\right) \\ &\leq \left[f'_-(\Gamma) - f'_+(\gamma)\right] \\ &\times \left\langle \frac{1}{\langle By,y\rangle}\left|A\otimes 1 - \frac{\langle Ax,x\rangle}{\langle By,y\rangle}(1\otimes B)\right|(x\otimes y), x\otimes y\right\rangle. \end{aligned} \tag{3.29}$$

4. Some Examples

Consider the function $\Pi_r(u) = u^r - 1$, $u \geq 0$, $r \geq 1$, then by (3.3) we get

$$\mathcal{P}_{\Pi_r,\otimes}(A,B) \geq ru^{r-1}(A\otimes 1) - [(r-1)u^r + 1](1\otimes B), \tag{4.1}$$

for A, $B > 0$ and $u > 0$.

If there exist the constants m_1, M_1, m_2 and M_2 with

$$0 < m_1 \leq A \leq M_1, m_2 \leq B \leq M_2, \tag{4.2}$$

then we can take in Theorem 5 $\gamma = \frac{m_1}{M_2}$ and $\Gamma = \frac{M_1}{m_2}$ and from (3.24) we derive

$$\begin{aligned} \mathcal{P}_{\Pi_r,\otimes}(A,B) &\leq A\otimes 1 - [(r-1)u^r + 1](1\otimes B) \\ &+ r\left(\left(\frac{M_1}{m_2}\right)^{r-1} - \left(\frac{m_1}{M_2}\right)^{r-1}\right)|A\otimes 1 - u(1\otimes B)|. \end{aligned} \tag{4.3}$$

For x, $y \in H$ with $\|x\| = \|y\| = 1$, we have by (3.9) that

$$\frac{\langle \mathcal{P}_{\Pi_r,\otimes}(A,B)(x\otimes y), x\otimes y\rangle}{\langle By,y\rangle} \geq \left(\frac{\langle Ax,x\rangle}{\langle By,y\rangle}\right)^r - 1 \tag{4.4}$$

for A, $B > 0$.

If the condition (4.2) is satisfied, then by (3.29) we get

$$\begin{aligned} &0 \leq \frac{\langle \mathcal{P}_{f,\otimes}(A,B)(x\otimes y), x\otimes y\rangle}{\langle By,y\rangle} - \left(\frac{\langle Ax,x\rangle}{\langle By,y\rangle}\right)^r + 1 \\ &\leq r\left(\left(\frac{M_1}{m_2}\right)^{r-1} - \left(\frac{m_1}{M_2}\right)^{r-1}\right) \\ &\times \left\langle \frac{1}{\langle By,y\rangle}\left|A\otimes 1 - \frac{\langle Ax,x\rangle}{\langle By,y\rangle}(1\otimes B)\right|(x\otimes y), x\otimes y\right\rangle \end{aligned} \tag{4.5}$$

for x, $y \in H$ with $\|x\| = \|y\| = 1$.

If we take the convex function $f = (\cdot)\ln(\cdot)$, then we get by (3.3) that

$$\mathcal{P}_{(\cdot)\ln(\cdot),\otimes}(A,B) \geq (\ln u + 1)(A\otimes 1) - u(1\otimes B), \tag{4.6}$$

for A, $B > 0$ and $u > 0$.

By (3.9) we obtain

$$\frac{\left\langle \mathcal{P}_{(\cdot)\ln(\cdot),\otimes}\left(A,B\right)\left(x\otimes y\right),x\otimes y\right\rangle}{\langle Ax,x\rangle} \geq \ln\left(\frac{\langle Ax,x\rangle}{\langle By,y\rangle}\right) \tag{4.7}$$

for $x,\, y \in H$ with $\|x\| = \|y\| = 1$.

If the condition (4.2) is satisfied, then by (3.24) we obtain

$$\begin{aligned} \mathcal{P}_{(\cdot)\ln(\cdot),\otimes}\left(A,B\right) &\leq \left(\ln u+1\right)\left(A\otimes 1\right) - u\left(1\otimes B\right) \\ &\quad + \ln\left(\frac{M_1M_2}{m_2m_1}\right)\left|A\otimes 1 - u\left(1\otimes B\right)\right| \end{aligned} \tag{4.8}$$

for $u \in \left[\frac{m_1}{M_2}, \frac{M_1}{m_2}\right]$.

From (3.29) we also derive

$$\begin{aligned} 0 &\leq \frac{\left\langle \mathcal{P}_{(\cdot)\ln(\cdot),\otimes}\left(A,B\right)\left(x\otimes y\right),x\otimes y\right\rangle}{\langle Ax,x\rangle} - \ln\left(\frac{\langle Ax,x\rangle}{\langle By,y\rangle}\right) \\ &\leq \ln\left(\frac{M_1M_2}{m_2m_1}\right) \\ &\quad \times \left\langle \frac{1}{\langle Ax,x\rangle}\left|A\otimes 1 - \frac{\langle Ax,x\rangle}{\langle By,y\rangle}\left(1\otimes B\right)\right|\left(x\otimes y\right),x\otimes y\right\rangle \end{aligned} \tag{4.9}$$

for $x,\, y \in H$ with $\|x\| = \|y\| = 1$.

By choosing other convex functions, one can derive several similar inequalities. The details are omitted.

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