# LOWER AND UPPER BOUNDS FOR TENSORIAL PERSPECTIVE FOR CONVEX FUNCTIONS OF SELFADJOINT OPERATORS IN HILBERT SPACES

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Abstract. Let $H$ be a Hilbert space. Assume that $f:[0,\infty)\to\mathbb{R}$ is continuous and $A,\ B>0$. We define the *tensorial perspective* for the function $f$ and the pair of operators $(A,B)$ by

$$\mathcal{P}_{f,\otimes}(A,B):=(1\otimes B)\,f\left(A\otimes B^{-1}\right).$$

In this paper we show among others that, if $f$ is convex on $(0,\infty)$, $A,\ B>0$ with spectra $\operatorname{Sp}(A)\subset I$, $\operatorname{Sp}(B)\subset J$ and such that $0<\gamma\leq\frac{t}{s}\leq\Gamma$ for $t\in I$ and $s\in J$, then

$$\begin{aligned}
0&\leq\left[\frac{f(\gamma)+f(\Gamma)}{2}-f\left(\frac{\gamma+\Gamma}{2}\right)\right]\\
&\times\left[1\otimes B-\frac{2}{\Gamma-\gamma}\left|A\otimes 1-\frac{\gamma+\Gamma}{2}1\otimes B\right|\right]\\
&\leq\frac{f(\gamma)+f(\Gamma)}{2}1\otimes B+\frac{f(\Gamma)-f(\gamma)}{\Gamma-\gamma}\left(A\otimes 1-\frac{\gamma+\Gamma}{2}1\otimes B\right)\\
&-\mathcal{P}_{f,\otimes}(A,B)\\
&\leq\left[\frac{f(\gamma)+f(\Gamma)}{2}-f\left(\frac{\gamma+\Gamma}{2}\right)\right]\\
&\times\left[1\otimes B-\frac{2}{\Gamma-\gamma}\left|A\otimes 1-\frac{\gamma+\Gamma}{2}1\otimes B\right|\right].
\end{aligned}$$

## 1. Introduction

Let $I_1,...,I_k$ be intervals from $\mathbb{R}$ and let $f:I_1\times...\times I_k\to\mathbb{R}$ be an essentially bounded real function defined on the product of the intervals. Let $A=(A_1,...,A_n)$ be a $k$-tuple of bounded selfadjoint operators on Hilbert spaces $H_1,...,H_k$ such that the spectrum of $A_i$ is contained in $I_i$ for $i=1,...,k$. We say that such a $k$-tuple is in the domain of $f$. If

$$A_i=\int_{I_i}\lambda_i dE_i(\lambda_i)$$

is the spectral resolution of $A_i$ for $i=1,...,k$; by following [2], we define

$$f(A_1,...,A_k):=\int_{I_1}...\int_{I_k}f(\lambda_1,...,\lambda_1)\,dE_1(\lambda_1)\otimes...\otimes dE_k(\lambda_k)\tag{1.1}$$

as a bounded selfadjoint operator on the tensorial product $H_1\otimes...\otimes H_k$.

If the Hilbert spaces are of finite dimension, then the above integrals become finite sums, and we may consider the functional calculus for arbitrary real functions.

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This construction [2] extends the definition of Korányi [7] for functions of two variables and have the property that

$$f(A_1,...,A_k) = f_1(A_1) \otimes ... \otimes f_k(A_k),$$

whenever $f$ can be separated as a product $f(t_1,...,t_k) = f_1(t_1)...f_k(t_k)$ of $k$ functions each depending on only one variable.

It is know that, *if $f$ is super-multiplicative (sub-multiplicative)* on $[0,\infty)$, namely

$$f(st) \geq (\leq) f(s) f(t) \text{ for all } s,t \in [0,\infty)$$

and if $f$ is continuous on $[0,\infty)$, then [10, p. 173]

$$f(A \otimes B) \geq (\leq) f(A) \otimes f(B) \text{ for all } A,\ B \geq 0. \tag{1.2}$$

This follows by observing that, if

$$A = \int_{[0,\infty)} t dE(t) \text{ and } B = \int_{[0,\infty)} s dF(s)$$

are the spectral resolutions of $A$ and $B$, then

$$f(A \otimes B) = \int_{[0,\infty)} \int_{[0,\infty)} f(st)\, dE(t) \otimes dF(s) \tag{1.3}$$

for the continuous function $f$ on $[0,\infty)$.

Recall the *geometric operator mean* for the positive operators $A,\ B > 0$

$$A\#_t B := A^{1/2}(A^{-1/2}BA^{-1/2})^t A^{1/2},$$

where $t \in [0,1]$ and

$$A\#B := A^{1/2}(A^{-1/2}BA^{-1/2})^{1/2} A^{1/2}.$$

By the definitions of $\#$ and $\otimes$ we have

$$A\#B = B\#A \text{ and } (A\#B) \otimes (B\#A) = (A \otimes B) \# (B \otimes A).$$

In 2007, S. Wada [13] obtained the following *Callebaut type inequalities* for tensorial product

$$\begin{aligned}(A\#B) \otimes (A\#B) &\leq \frac{1}{2}\left[(A\#_\alpha B) \otimes (A\#_{1-\alpha} B) + (A\#_{1-\alpha} B) \otimes (A\#_\alpha B)\right] \\ &\leq \frac{1}{2}(A \otimes B + B \otimes A)\end{aligned} \tag{1.4}$$

for $A,\ B > 0$ and $\alpha \in [0,1]$.

Assume that $f : [0,\infty) \to \mathbb{R}$ is continuous and $A,\ B > 0$. We define the *tensorial perspective* for the function $f$ and the pair of operators $(A,B)$

$$\mathcal{P}_{f,\otimes}(A,B) := (1 \otimes B) f\left(A \otimes B^{-1}\right).$$

Motivated by the above results, in this paper we show among others that, if $f$ is convex on $(0,\infty)$, $A,\ B > 0$ with spectra $\operatorname{Sp}(A) \subset I$, $\operatorname{Sp}(B) \subset J$ and such that

$0 < \gamma \leq \frac{t}{s} \leq \Gamma$ for $t \in I$ and $s \in J$, then

$$\begin{aligned}
0 &\leq \left[\frac{f(\gamma)+f(\Gamma)}{2} - f\left(\frac{\gamma+\Gamma}{2}\right)\right] \\
&\times \left[1 \otimes B - \frac{2}{\Gamma-\gamma}\left|A \otimes 1 - \frac{\gamma+\Gamma}{2} 1 \otimes B\right|\right] \\
&\leq \frac{f(\gamma)+f(\Gamma)}{2} 1 \otimes B + \frac{f(\Gamma)-f(\gamma)}{\Gamma-\gamma}\left(A \otimes 1 - \frac{\gamma+\Gamma}{2} 1 \otimes B\right) \\
&- \mathcal{P}_{f,\otimes}(A,B) \\
&\leq \left[\frac{f(\gamma)+f(\Gamma)}{2} - f\left(\frac{\gamma+\Gamma}{2}\right)\right] \\
&\times \left[1 \otimes B - \frac{2}{\Gamma-\gamma}\left|A \otimes 1 - \frac{\gamma+\Gamma}{2} 1 \otimes B\right|\right].
\end{aligned}$$

## 2. Some Preliminary Facts

Recall the following property of the tensorial product

$$(AC) \otimes (BD) = (A \otimes B)(C \otimes D) \tag{2.1}$$

that holds for any $A, B, C, D \in B(H)$.

If we take $C = A$ and $D = B$, then we get

$$A^2 \otimes B^2 = (A \otimes B)^2.$$

By induction and using (2.1) we derive that

$$A^n \otimes B^n = (A \otimes B)^n \text{ for natural } n \geq 0. \tag{2.2}$$

In particular

$$A^n \otimes 1 = (A \otimes 1)^n \text{ and } 1 \otimes B^n = (1 \otimes B)^n \tag{2.3}$$

for all $n \geq 0$.

We also observe that, by (2.1), the operators $A \otimes 1$ and $1 \otimes B$ are commutative and

$$(A \otimes 1)(1 \otimes B) = (1 \otimes B)(A \otimes 1) = A \otimes B. \tag{2.4}$$

Moreover, for two natural numbers $m$, $n$ we have

$$(A \otimes 1)^m (1 \otimes B)^n = (1 \otimes B)^n (A \otimes 1)^m = A^m \otimes B^n. \tag{2.5}$$

According with the properties of tensorial products and functional calculus for continuous functions of selfadjoint operators, we have

$$\begin{aligned}
\mathcal{P}_{f,\otimes}(A,B) &= (1 \otimes B) f\left((A \otimes 1)(1 \otimes B)^{-1}\right) \\
&= f\left((A \otimes 1)(1 \otimes B)^{-1}\right)(1 \otimes B) \\
&= f\left((1 \otimes B)^{-1}(A \otimes 1)\right)(1 \otimes B),
\end{aligned}$$

due to the commutativity of $A \otimes 1$ and $1 \otimes B$.

In the following, we consider the spectral resolutions of $A$ and $B$ given by

$$A = \int_{[0,\infty)} t dE(t) \text{ and } B = \int_{[0,\infty)} s dF(s). \tag{2.6}$$

We have the following representation result for continuous functions:

**Lemma 1.** *Assume that* $f:[0,\infty)\to\mathbb{R}$ *is continuous and* $A,\ B>0$, *then*

$$\mathcal{P}_{f,\otimes}(A,B)=\int_{[0,\infty)}\int_{[0,\infty)}sf\left(\frac{t}{s}\right)dE(t)\otimes dF(s). \tag{2.7}$$

*Proof.* By Stone-Weierstrass theorem, any continuous function can be approximated by a sequence of polynomials, therefore it suffices to prove the equality for the power function $\varphi(t)=t^n$ with $n$ any natural number.

We have that

$$\begin{aligned}&\int_{[0,\infty)}\int_{[0,\infty)}s\varphi\left(\frac{t}{s}\right)dE(t)\otimes dF(s)\\&=\int_{[0,\infty)}\int_{[0,\infty)}s\left(\frac{t}{s}\right)^n dE(t)\otimes dF(s)\\&=\int_{[0,\infty)}\int_{[0,\infty)}t^n s^{1-n}dE(t)\otimes dF(s)\\&=A^n\otimes B^{1-n}=A^n\otimes BB^{-n}=(1\otimes B)\left(A^n\otimes B^{-n}\right)\\&=(1\otimes B)\left(A\otimes B^{-1}\right)^n=\mathcal{P}_{\varphi,\otimes}(A,B),\end{aligned}$$

which shows that (2.7) holds for the power function.

This proves the lemma. $\square$

We assume in the following that $A,\ B>0$.

If we consider the function $\Pi_r(u)=u^r-1,\ u\geq 0,\ r>0$, then we have

$$\begin{aligned}\mathcal{P}_{\Pi_r,\otimes}(A,B)&:=(1\otimes B)\,\Pi_r\left(A\otimes B^{-1}\right)\\&=(1\otimes B)\left[\left(A\otimes B^{-1}\right)^r-1\right]\\&=(A\otimes 1)^r(1\otimes B)^{1-r}-1\otimes B.\end{aligned}$$

If we take $f=-\ln(\cdot)$, then we get

$$\begin{aligned}\mathcal{P}_{-\ln(\cdot),\otimes}(A,B)&:=-(1\otimes B)\ln\left(A\otimes B^{-1}\right)\\&=-\ln\left((1\otimes B)^{-1}(A\otimes 1)\right)(1\otimes B)\\&=(1\otimes B)\left[\ln(1\otimes B)-\ln(A\otimes 1)\right].\end{aligned}$$

If we take $f=(\cdot)\ln(\cdot)$, then we get

$$\begin{aligned}\mathcal{P}_{(\cdot)\ln(\cdot),\otimes}(A,B)&:=(1\otimes B)\left(A\otimes B^{-1}\right)\ln\left(A\otimes B^{-1}\right)\\&=(A\otimes 1)\left[\ln(A\otimes 1)-\ln(1\otimes B)\right].\end{aligned}$$

If we take $f=|\cdot-\alpha|,\ \alpha\in\mathbb{R}$, then

$$\begin{aligned}\mathcal{P}_{|\cdot-\alpha|,\otimes}(A,B)&=\int_{[0,\infty)}\int_{[0,\infty)}s\left|\frac{t}{s}-\alpha\right|dE(t)\otimes dF(s)\\&=\int_{[0,\infty)}\int_{[0,\infty)}|t-\alpha s|\,dE(t)\otimes dF(s)\\&=|A\otimes 1-\alpha 1\otimes B|,\end{aligned}$$

where for the last equality we used the result obtained in [5],

$$\psi\left(h\left(A\right)\otimes 1+1\otimes k\left(B\right)\right)=\int_{I}\int_{J}\psi\left(h\left(t\right)+k\left(s\right)\right)dE\left(t\right)\otimes dF\left(s\right), \tag{2.8}$$

here $A$ and $B$ are selfadjoint operators with $\operatorname{Sp}\left(A\right)\subset I$ and $\operatorname{Sp}\left(B\right)\subset J$, $h$ is continuous on $I$, $k$ is continuous on $J$ and $\psi$ is continuous on an interval $U$ that contains the sum of the intervals $h\left(I\right)+k\left(J\right)$, while $A$ and $B$ have the spectral resolutions

$$A=\int_{I}tdE\left(t\right)\text{ and }B=\int_{J}sdF\left(s\right).$$

For $f=\left|\cdot-1\right|$ we get

$$\mathcal{P}_{\left|\cdot-1\right|,\otimes}\left(A,B\right)=\left|A\otimes 1-1\otimes B\right|.$$

Consider the $q$-logarithm defined by

$$\ln_{q}u=\begin{cases}\frac{u^{1-q}-1}{1-q}\text{ if }q\neq 1,\\ \\ \ln u\text{ if }q=1.\end{cases}$$

For $q\neq 1$ we define

$$\begin{aligned}\mathcal{P}_{\ln_{q},\otimes}\left(A,B\right)&:=\left(1\otimes B\right)\ln_{q}\left(\left(A\otimes 1\right)\left(1\otimes B\right)^{-1}\right)\\&=\frac{\left(A\otimes 1\right)^{1-q}\left(1\otimes B\right)^{q}-1\otimes B}{1-q}.\end{aligned}\tag{2.9}$$

Let $f$ be a continuous function defined on the interval $I$ of real numbers, $B$ a selfadjoint operator on the Hilbert space $H$ and $A$ a positive invertible operator on $H$. Assume that the spectrum $\operatorname{Sp}\left(A^{-1/2}BA^{-1/2}\right)\subset\mathring{I}$. Then by using the continuous functional calculus, we can define the *perspective* $\mathcal{P}_{f}\left(B,A\right)$ by setting

$$\mathcal{P}_{f}\left(B,A\right):=A^{1/2}f\left(A^{-1/2}BA^{-1/2}\right)A^{1/2}.$$

If $A$ and $B$ are commutative, then

$$\mathcal{P}_{f}\left(B,A\right)=Af\left(BA^{-1}\right)$$

provided $\operatorname{Sp}\left(BA^{-1}\right)\subset\mathring{I}$.

It is well known that (see for instance [8]), if $f$ is an *operator convex function* defined in the positive half-line, then the mapping

$$\left(B,A\right)\mapsto\mathcal{P}_{f}\left(B,A\right)$$

*defined in pairs of positive definite operators, is operator convex.*

The following inequality is also of interest, see [12]:

**Theorem 1.** *Assume that $f$ is nonnegative and operator monotone on $[0,\infty)$. If $A\geq C>0$ and $B\geq D>0$, then*

$$\mathcal{P}_{f}\left(A,B\right)\geq\mathcal{P}_{f}\left(C,D\right). \tag{2.10}$$

In the recent paper [6] we obtained the following result for the tensorial perspective:

**Theorem 2.** *If $f$ is an operator convex function defined in the positive half-line, then $\mathcal{P}_{f,\otimes}$ is operator convex in pairs of positive definite operators as well. If $A \geq C > 0$ and $B \geq D > 0$, then also*

$$\mathcal{P}_{f,\otimes}(A,B) \geq \mathcal{P}_{f,\otimes}(C,D). \tag{2.11}$$

## 3. Main Results

Our first main result is as follows:

**Theorem 3.** *Assume that $f$ is convex on $(0,\infty)$, $A$, $B>0$ with spectra $\operatorname{Sp}(A) \subset I$, $\operatorname{Sp}(B) \subset J$ and such that $0 < \gamma \leq \frac{t}{s} \leq \Gamma$ for $t \in I$ and $s \in J$, then*

$$\begin{aligned} 0 &\leq \left[\frac{f(\gamma)+f(\Gamma)}{2} - f\left(\frac{\gamma+\Gamma}{2}\right)\right] \\ &\times \left[1 \otimes B - \frac{2}{\Gamma-\gamma}\left|A \otimes 1 - \frac{\gamma+\Gamma}{2} 1 \otimes B\right|\right] \\ &\leq \frac{f(\gamma)+f(\Gamma)}{2} 1 \otimes B + \frac{f(\Gamma)-f(\gamma)}{\Gamma-\gamma}\left(A \otimes 1 - \frac{\gamma+\Gamma}{2} 1 \otimes B\right) \\ &- \mathcal{P}_{f,\otimes}(A,B) \\ &\leq \left[\frac{f(\gamma)+f(\Gamma)}{2} - f\left(\frac{\gamma+\Gamma}{2}\right)\right] \\ &\times \left[1 \otimes B - \frac{2}{\Gamma-\gamma}\left|A \otimes 1 - \frac{\gamma+\Gamma}{2} 1 \otimes B\right|\right]. \end{aligned} \tag{3.1}$$

*Proof.* Recall the following result obtained by Dragomir in 2006 [4] that provides a refinement and a reverse for the weighted Jensen's discrete inequality:

$$\begin{aligned} 0 &\leq n \min_{j\in\{1,2,\ldots,n\}} \{p_j\} \left[\frac{1}{n}\sum_{j=1}^n f(x_j) - f\left(\frac{1}{n}\sum_{j=1}^n x_j\right)\right] \\ &\leq \frac{1}{P_n}\sum_{j=1}^n p_j f(x_j) - f\left(\frac{1}{P_n}\sum_{j=1}^n p_j x_j\right) \\ &\leq n \max_{j\in\{1,2,\ldots,n\}} \{p_j\} \left[\frac{1}{n}\sum_{j=1}^n f(x_j) - f\left(\frac{1}{n}\sum_{j=1}^n x_j\right)\right], \end{aligned} \tag{3.2}$$

where $f : C \to \mathbb{R}$ is a convex function defined on convex subset $C$ of the linear space $X$, $\{x_j\}_{j\in\{1,2,\ldots,n\}}$ are vectors in $C$ and $\{p_j\}_{j\in\{1,2,\ldots,n\}}$ are nonnegative numbers with $P_n = \sum_{j=1}^n p_j > 0$.

For $n = 2$, we deduce from (3.2) that

$$\begin{aligned} 0 &\leq 2\min\{\nu, 1-\nu\}\left[\frac{f(x)+f(y)}{2} - f\left(\frac{x+y}{2}\right)\right] \\ &\leq \nu f(x) + (1-\nu) f(y) - f[\nu x + (1-\nu) y] \\ &\leq 2\max\{\nu, 1-\nu\}\left[\frac{f(x)+f(y)}{2} - f\left(\frac{x+y}{2}\right)\right] \end{aligned} \tag{3.3}$$

for any $x$, $y \in \mathbb{R}$ and $\nu \in [0,1]$.

Let $u \in [\gamma, \Gamma]$ and take $x = \Gamma$, $y = \gamma$ and $\nu = \frac{u-\gamma}{\Gamma-\gamma}$. Then

$$\nu x + (1-\nu)\, y = \frac{u-\gamma}{\Gamma-\gamma}\Gamma + \frac{\Gamma-u}{\Gamma-\gamma}\gamma = u$$

and by (3.3) we get

$$\begin{aligned} (3.4) \qquad 0 &\le 2\min\left\{\frac{u-\gamma}{\Gamma-\gamma}, \frac{\Gamma-u}{\Gamma-\gamma}\right\}\left[\frac{f(\gamma)+f(\Gamma)}{2} - f\left(\frac{\gamma+\Gamma}{2}\right)\right] \\ &\le \frac{u-\gamma}{\Gamma-\gamma} f(\Gamma) + \frac{\Gamma-u}{\Gamma-\gamma} f(\gamma) - f(u) \\ &\le 2\max\left\{\frac{u-\gamma}{\Gamma-\gamma}, \frac{\Gamma-u}{\Gamma-\gamma}\right\}\left[\frac{f(\gamma)+f(\Gamma)}{2} - f\left(\frac{\gamma+\Gamma}{2}\right)\right] \end{aligned}$$

for any $u \in [\gamma, \Gamma]$.

Since

$$\min\left\{\frac{u-\gamma}{\Gamma-\gamma}, \frac{\Gamma-u}{\Gamma-\gamma}\right\} = \frac{1}{2} - \frac{1}{\Gamma-\gamma}\left|u - \frac{\gamma+\Gamma}{2}\right|$$

and

$$\max\left\{\frac{u-\gamma}{\Gamma-\gamma}, \frac{\Gamma-u}{\Gamma-\gamma}\right\} = \frac{1}{2} + \frac{1}{\Gamma-\gamma}\left|u - \frac{\gamma+\Gamma}{2}\right|$$

then by (3.4) we get

$$\begin{aligned} (3.5) \qquad 0 &\le \left(1 - \frac{2}{\Gamma-\gamma}\left|u - \frac{\gamma+\Gamma}{2}\right|\right)\left[\frac{f(\gamma)+f(\Gamma)}{2} - f\left(\frac{\gamma+\Gamma}{2}\right)\right] \\ &\le \frac{u-\gamma}{\Gamma-\gamma} f(\Gamma) + \frac{\Gamma-u}{\Gamma-\gamma} f(\gamma) - f(u) \\ &\le \left(1 + \frac{2}{\Gamma-\gamma}\left|u - \frac{\gamma+\Gamma}{2}\right|\right)\left[\frac{f(\gamma)+f(\Gamma)}{2} - f\left(\frac{\gamma+\Gamma}{2}\right)\right] \end{aligned}$$

for any $u \in [\gamma, \Gamma]$.

Let $u = \frac{t}{s} \in [\gamma, \Gamma] \subset (0, \infty)$. Then by (3.5) we get

$$\begin{aligned} (3.6) \qquad 0 &\le \left(1 - \frac{2}{\Gamma-\gamma}\left|\frac{t}{s} - \frac{\gamma+\Gamma}{2}\right|\right)\left[\frac{f(\gamma)+f(\Gamma)}{2} - f\left(\frac{\gamma+\Gamma}{2}\right)\right] \\ &\le \frac{\frac{t}{s}-\gamma}{\Gamma-\gamma} f(\Gamma) + \frac{\Gamma-\frac{t}{s}}{\Gamma-\gamma} f(\gamma) - f\left(\frac{t}{s}\right) \\ &\le \left(1 + \frac{2}{\Gamma-\gamma}\left|\frac{t}{s} - \frac{\gamma+\Gamma}{2}\right|\right)\left[\frac{f(\gamma)+f(\Gamma)}{2} - f\left(\frac{\gamma+\Gamma}{2}\right)\right] \end{aligned}$$

for $t \in I$ and $s \in J$.

If we multiply (3.6) by $s > 0$, then we get

$$\begin{aligned} (3.7) \qquad 0 &\le \left(s - \frac{2}{\Gamma-\gamma}\left|t - \frac{\gamma+\Gamma}{2}s\right|\right)\left[\frac{f(\gamma)+f(\Gamma)}{2} - f\left(\frac{\gamma+\Gamma}{2}\right)\right] \\ &\le \frac{t-\gamma s}{\Gamma-\gamma} f(\Gamma) + \frac{\Gamma s - t}{\Gamma-\gamma} f(\gamma) - sf\left(\frac{t}{s}\right) \\ &\le \left(s + \frac{2}{\Gamma-\gamma}\left|t - \frac{\gamma+\Gamma}{2}s\right|\right)\left[\frac{f(\gamma)+f(\Gamma)}{2} - f\left(\frac{\gamma+\Gamma}{2}\right)\right] \end{aligned}$$

for $t \in I$ and $s \in J$.

We consider the spectral resolutions of $A$ and $B$ given by

$$A=\int_I tdE(t) \text{ and } B=\int_J sdF(s).$$

If we take the double integral $\int_I \int_J$ over $dE(t) \otimes dF(s)$ in (3.7), then we get

$$\begin{aligned}
0 &\leq \left[\frac{f(\gamma)+f(\Gamma)}{2}-f\left(\frac{\gamma+\Gamma}{2}\right)\right] \\
&\times \int_I \int_J \left(s-\frac{2}{\Gamma-\gamma}\left|t-\frac{\gamma+\Gamma}{2}s\right|\right) dE(t) \otimes dF(s) \\
&\leq \int_I \int_J \left[\frac{t-\gamma s}{\Gamma-\gamma} f(\Gamma)+\frac{\Gamma s-t}{\Gamma-\gamma} f(\gamma)-s f\left(\frac{t}{s}\right)\right] dE(t) \otimes dF(s) \\
&\leq \left[\frac{f(\gamma)+f(\Gamma)}{2}-f\left(\frac{\gamma+\Gamma}{2}\right)\right] \\
&\times \int_I \int_J \left(s+\frac{2}{\Gamma-\gamma}\left|t-\frac{\gamma+\Gamma}{2}s\right|\right) dE(t) \otimes dF(s).
\end{aligned} \tag{3.8}$$

Observe that

$$\begin{aligned}
&\int_I \int_J \left(s-\frac{2}{\Gamma-\gamma}\left|t-\frac{\gamma+\Gamma}{2}s\right|\right) dE(t) \otimes dF(s) \\
&=\int_I \int_J s dE(t) \otimes dF(s)-\frac{2}{\Gamma-\gamma} \int_I \int_J\left|t-\frac{\gamma+\Gamma}{2}s\right| dE(t) \otimes dF(s) \\
&=1 \otimes B-\frac{2}{\Gamma-\gamma}\left|A \otimes 1-\frac{\gamma+\Gamma}{2} 1 \otimes B\right|,
\end{aligned}$$

$$\begin{aligned}
&\int_I \int_J \left[\frac{t-\gamma s}{\Gamma-\gamma} f(\Gamma)+\frac{\Gamma s-t}{\Gamma-\gamma} f(\gamma)-s f\left(\frac{t}{s}\right)\right] dE(t) \otimes dF(s) \\
&=\frac{f(\Gamma)}{\Gamma-\gamma} \int_I \int_J (t-\gamma s) dE(t) \otimes dF(s) \\
&+\frac{f(\gamma)}{\Gamma-\gamma} \int_I \int_J (\Gamma s-t) dE(t) \otimes dF(s) \\
&-\int_I \int_J s f\left(\frac{t}{s}\right) dE(t) \otimes dF(s) \\
&=\frac{f(\Gamma)}{\Gamma-\gamma}(A \otimes 1-\gamma 1 \otimes B)+\frac{f(\gamma)}{\Gamma-\gamma}(\Gamma 1 \otimes B-A \otimes 1) \\
&-\mathcal{P}_{f, \otimes}(A, B)
\end{aligned}$$

and

$$\begin{aligned}
&\int_I \int_J \left(s+\frac{2}{\Gamma-\gamma}\left|t-\frac{\gamma+\Gamma}{2}s\right|\right) dE(t) \otimes dF(s) \\
&=1 \otimes B+\frac{2}{\Gamma-\gamma}\left|A \otimes 1-\frac{\gamma+\Gamma}{2} 1 \otimes B\right|.
\end{aligned}$$

Also, observe that

$$
\begin{aligned}
& f(\Gamma)\frac{A\otimes 1-\gamma 1\otimes B}{\Gamma-\gamma}+f(\gamma)\frac{\Gamma 1\otimes B-A\otimes 1}{\Gamma-\gamma} \\
&=\frac{f(\gamma)+f(\Gamma)}{2}1\otimes B \\
&+f(\gamma)\left(\frac{\Gamma 1\otimes B-A\otimes 1}{\Gamma-\gamma}-\frac{1}{2}1\otimes B\right) \\
&+f(\Gamma)\left(\frac{A\otimes 1-\gamma 1\otimes B}{\Gamma-\gamma}-\frac{1}{2}1\otimes B\right) \\
&=\frac{f(\gamma)+f(\Gamma)}{2}1\otimes B \\
&-f(\gamma)\left(\frac{A\otimes 1-\frac{\gamma+\Gamma}{2}1\otimes B}{\Gamma-\gamma}\right)+f(\Gamma)\left(\frac{A\otimes 1-\frac{\gamma+\Gamma}{2}1\otimes B}{\Gamma-\gamma}\right) \\
&=\frac{f(\gamma)+f(\Gamma)}{2}1\otimes B+\frac{f(\Gamma)-f(\gamma)}{\Gamma-\gamma}\left(A\otimes 1-\frac{\gamma+\Gamma}{2}1\otimes B\right).
\end{aligned}
$$

By making use of (3.8) we derive the desired result (3.1). □

**Theorem 4.** *Assume that $f$ is convex on $(0,\infty)$, $A$, $B>0$ with spectra $\operatorname{Sp}(A)\subset I$, $\operatorname{Sp}(B)\subset J$ and such that $0<\gamma\le\frac{t}{s}\le\Gamma$ for $t\in I$ and $s\in J$, then*

$$
\begin{aligned}
(3.9)\qquad 0 &\le \frac{f(\gamma)+f(\Gamma)}{2}1\otimes B+\frac{f(\Gamma)-f(\gamma)}{\Gamma-\gamma}\left(A\otimes 1-\frac{\gamma+\Gamma}{2}1\otimes B\right) \\
&\quad -\mathcal{P}_{f,\otimes}(A,B) \\
&\le \begin{cases} \frac{\sup_{u\in(\gamma,\Gamma)}\Psi_f(u;\gamma,\Gamma)}{\Gamma-\gamma} \\ \times\left[\frac{1}{4}(\Gamma-\gamma)^2 1\otimes B-\left(A\otimes 1-\frac{\gamma+\Gamma}{2}1\otimes B\right)^2\left(1\otimes B^{-1}\right)\right] \\ \\ \frac{1}{4}(\Gamma-\gamma)\mathcal{P}_{\Psi_f(\cdot;\gamma,\Gamma),\otimes}(A,B) \end{cases} \\
&\le \begin{cases} \frac{f'_-(\Gamma)-f'_+(\gamma)}{\Gamma-\gamma} \\ \times\left[\frac{1}{4}(\Gamma-\gamma)^2 1\otimes B-\left(A\otimes 1-\frac{\gamma+\Gamma}{2}1\otimes B\right)^2\left(1\otimes B^{-1}\right)\right] \\ \\ \frac{1}{4}(\Gamma-\gamma)\sup_{u\in(\gamma,\Gamma)}\Psi_f(u;\gamma,\Gamma)(1\otimes B) \end{cases} \\
&\le \frac{1}{4}(\Gamma-\gamma)\left[f'_-(\Gamma)-f'_+(\gamma)\right](1\otimes B),
\end{aligned}
$$

*where*

$$\Psi_f(u;\gamma,\Gamma)=\frac{f(\Gamma)-f(u)}{\Gamma-u}-\frac{f(u)-f(\gamma)}{u-\gamma}\ge 0,\ \ u\in(\gamma,\Gamma).$$

*Proof.* By denoting

$$\Delta_f(u;\gamma,\Gamma):=\frac{(u-\gamma)f(\Gamma)+(\Gamma-u)f(\gamma)}{\Gamma-\gamma}-f(u)\ge 0,\quad u\in[\gamma,\Gamma]$$

we have

$$\begin{aligned}
\Delta_f(u;\gamma,\Gamma) &= \frac{(u-\gamma)f(\Gamma)+(\Gamma-u)f(\gamma)-(\Gamma-\gamma)f(u)}{\Gamma-\gamma} \\
&= \frac{(u-\gamma)f(\Gamma)+(\Gamma-u)f(\gamma)-(\Gamma-u+u-\gamma)f(u)}{\Gamma-\gamma} \\
&= \frac{(u-\gamma)[f(\Gamma)-f(u)]-(\Gamma-u)[f(u)-f(\gamma)]}{\Gamma-\gamma} \\
&= \frac{(\Gamma-u)(u-\gamma)}{\Gamma-\gamma}\Psi_f(u;\gamma,\Gamma),
\end{aligned} \tag{3.10}$$

for $u\in(\gamma,\Gamma)$.

Since $f$ is a convex function, then we have

$$\begin{aligned}
\sup_{u\in(\gamma,\Gamma)}\Psi_f(u;\gamma,\Gamma) &= \sup_{u\in(\gamma,\Gamma)}\left[\frac{f(\Gamma)-f(u)}{\Gamma-u}-\frac{f(u)-f(\gamma)}{u-\gamma}\right] \\
&\le \sup_{u\in(\gamma,\Gamma)}\left[\frac{f(\Gamma)-f(u)}{\Gamma-u}\right]+\sup_{u\in(\gamma,\Gamma)}\left[-\frac{f(u)-f(\gamma)}{u-\gamma}\right] \\
&= \sup_{u\in(\gamma,\Gamma)}\left[\frac{f(\Gamma)-f(u)}{\Gamma-u}\right]-\inf_{u\in(\gamma,\Gamma)}\left[\frac{f(u)-f(\gamma)}{u-\gamma}\right] \\
&= f'_-(\Gamma)-f'_+(\gamma).
\end{aligned}$$

Therefore

$$\begin{aligned}
&\frac{(\Gamma-u)(u-\gamma)}{\Gamma-\gamma}\Psi_f(u;\gamma,\Gamma) \\
&\le \begin{cases} \frac{(\Gamma-u)(u-\gamma)}{\Gamma-\gamma}\sup_{u\in(\gamma,\Gamma)}\Psi_f(u;\gamma,\Gamma) \\ \\ \Psi_f(u;\gamma,\Gamma)\sup_{u\in(\gamma,\Gamma)}\left[\frac{(\Gamma-u)(u-\gamma)}{\Gamma-\gamma}\right] \end{cases} = \begin{cases} \frac{(\Gamma-u)(u-\gamma)}{\Gamma-\gamma}\left[f'_-(\Gamma)-f'_+(\gamma)\right] \\ \\ \frac{1}{4}(\Gamma-\gamma)\Psi_f(u;\gamma,\Gamma) \end{cases} \\
&\le \frac{1}{4}(\Gamma-\gamma)\left[f'_-(\Gamma)-f'_+(\gamma)\right]
\end{aligned}$$

for any $u\in(\gamma,\Gamma)$.

By making use of (3.10) we get

$$\begin{aligned}
0 &\le \frac{(u-\gamma)f(\Gamma)+(\Gamma-u)f(\gamma)}{\Gamma-\gamma}-f(u) \\
&\le \begin{cases} \frac{(\Gamma-u)(u-\gamma)}{\Gamma-\gamma}\sup_{u\in(\gamma,\Gamma)}\Psi_f(u;\gamma,\Gamma) \\ \\ \frac{1}{4}(\Gamma-\gamma)\Psi_f(u;\gamma,\Gamma) \end{cases} \le \begin{cases} \frac{(\Gamma-u)(u-\gamma)}{\Gamma-\gamma}\left[f'_-(\Gamma)-f'_+(\gamma)\right] \\ \\ \frac{1}{4}(\Gamma-\gamma)\sup_{u\in(\gamma,\Gamma)}\Psi_f(u;\gamma,\Gamma) \end{cases} \\
&\le \frac{1}{4}(\Gamma-\gamma)\left[f'_-(\Gamma)-f'_+(\gamma)\right]
\end{aligned}$$

and since, as in the proof of Theorem 2,

$$\begin{aligned}
&\frac{(u-\gamma)f(\Gamma)+(\Gamma-u)f(\gamma)}{\Gamma-\gamma}-f(u) \\
&= \frac{f(\gamma)+f(\Gamma)}{2}+\frac{f(\Gamma)-f(\gamma)}{\Gamma-\gamma}\left(u-\frac{\gamma+\Gamma}{2}\right)-f(u),
\end{aligned}$$

and

$$(\Gamma - u)(u-\gamma) = \frac{1}{4}(\Gamma-\gamma)^2 - \left(u - \frac{\gamma+\Gamma}{2}\right)^2$$

then we have the following inequality of interest in itself

$$\begin{aligned}
(3.11) \qquad 0 &\le \frac{f(\gamma)+f(\Gamma)}{2} + \frac{f(\Gamma)-f(\gamma)}{\Gamma-\gamma}\left(u-\frac{\gamma+\Gamma}{2}\right) - f(u) \\
&\le \begin{cases} \frac{\sup_{u\in(\gamma,\Gamma)}\Psi_f(u;\gamma,\Gamma)}{\Gamma-\gamma}\left[\frac{1}{4}(\Gamma-\gamma)^2 - \left(u-\frac{\gamma+\Gamma}{2}\right)^2\right] \\ \\ \frac{1}{4}(\Gamma-\gamma)\Psi_f(u;\gamma,\Gamma) \end{cases} \\
&\le \begin{cases} \frac{f'_-(\Gamma)-f'_+(\gamma)}{\Gamma-\gamma}\left[\frac{1}{4}(\Gamma-\gamma)^2 - \left(u-\frac{\gamma+\Gamma}{2}\right)^2\right] \\ \\ \frac{1}{4}(\Gamma-\gamma)\sup_{u\in(\gamma,\Gamma)}\Psi_f(u;\gamma,\Gamma) \end{cases} \\
&\le \frac{1}{4}(\Gamma-\gamma)\left[f'_-(\Gamma)-f'_+(\gamma)\right]
\end{aligned}$$

for any $u \in (\gamma,\Gamma)$.

Let $t \in I$ and $s \in J$. Take $u = \frac{t}{s} \in (\gamma,\Gamma)$ in (3.11) and multiply by $s>0$ to get

$$\begin{aligned}
(3.12) \qquad 0 &\le \frac{f(\gamma)+f(\Gamma)}{2}s + \frac{f(\Gamma)-f(\gamma)}{\Gamma-\gamma}\left(t-\frac{\gamma+\Gamma}{2}s\right) - sf\left(\frac{t}{s}\right) \\
&\le \begin{cases} \frac{\sup_{u\in(\gamma,\Gamma)}\Psi_f(u;\gamma,\Gamma)}{\Gamma-\gamma}\left[\frac{1}{4}(\Gamma-\gamma)^2 s - \left(t-\frac{\gamma+\Gamma}{2}s\right)^2 s^{-1}\right] \\ \\ \frac{1}{4}(\Gamma-\gamma)s\Psi_f\left(\frac{t}{s};\gamma,\Gamma\right) \end{cases} \\
&\le \begin{cases} \frac{f'_-(\Gamma)-f'_+(\gamma)}{\Gamma-\gamma}\left[\frac{1}{4}(\Gamma-\gamma)^2 s - \left(t-\frac{\gamma+\Gamma}{2}s\right)^2 s^{-1}\right] \\ \\ \frac{1}{4}(\Gamma-\gamma)\sup_{u\in(\gamma,\Gamma)}\Psi_f(u;\gamma,\Gamma)s \end{cases} \\
&\le \frac{1}{4}(\Gamma-\gamma)\left[f'_-(\Gamma)-f'_+(\gamma)\right]s.
\end{aligned}$$

We consider the spectral resolutions of $A$ and $B$ given by

$$A = \int_I t dE(t) \text{ and } B = \int_J s dF(s).$$

If we take the double integral $\int_I \int_J$ over $dE(t) \otimes dF(s)$ in (3.12), then we get

$$\begin{aligned}
(3.13) \qquad 0 &\le \frac{f(\gamma)+f(\Gamma)}{2}\int_I\int_J s dE(t)\otimes dF(s) \\
&+ \frac{f(\Gamma)-f(\gamma)}{\Gamma-\gamma}\int_I\int_J\left(t-\frac{\gamma+\Gamma}{2}s\right)E(t)\otimes dF(s) \\
&- \int_I\int_J sf\left(\frac{t}{s}\right)dE(t)\otimes dF(s)
\end{aligned}$$

$$\leq \begin{cases} \frac{\sup_{u\in(\gamma,\Gamma)}\Psi_f(u;\gamma,\Gamma)}{\Gamma-\gamma} \\ \times\left[\frac{1}{4}(\Gamma-\gamma)^2\int_I\int_J sE(t)\otimes dF(s) - \int_I\int_J\left(t-\frac{\gamma+\Gamma}{2}s\right)^2 s^{-1}E(t)\otimes dF(s)\right] \\ \\ \frac{1}{4}(\Gamma-\gamma)\int_I\int_J s\Psi_f\left(\frac{t}{s};\gamma,\Gamma\right)E(t)\otimes dF(s) \end{cases}$$

$$\leq \begin{cases} \frac{f'_-(\Gamma)-f'_+(\gamma)}{\Gamma-\gamma} \\ \times\left[\frac{1}{4}(\Gamma-\gamma)^2\int_I\int_J sE(t)\otimes dF(s) - \int_I\int_J\left(t-\frac{\gamma+\Gamma}{2}s\right)^2 s^{-1}E(t)\otimes dF(s)\right] \\ \\ \frac{1}{4}(\Gamma-\gamma)\sup_{u\in(\gamma,\Gamma)}\int_I\int_J sE(t)\otimes dF(s) \end{cases}$$

$$\leq \frac{1}{4}(\Gamma-\gamma)\left[f'_-(\Gamma)-f'_+(\gamma)\right]\int_I\int_J sE(t)\otimes dF(s).$$

Since, by the commutativity of $A\otimes 1$ and $1\otimes B$,

$$\begin{aligned} &\int_I\int_J\left(t-\frac{\gamma+\Gamma}{2}s\right)^2 s^{-1}E(t)\otimes dF(s) \\ &= \int_I\int_J\left[t^2-2ts\frac{\gamma+\Gamma}{2}+\left(\frac{\gamma+\Gamma}{2}\right)^2 s^2\right]s^{-1}E(t)\otimes dF(s) \\ &= \int_I\int_J\left[t^2s^{-1}-2t\frac{\gamma+\Gamma}{2}+\left(\frac{\gamma+\Gamma}{2}\right)^2 s\right]E(t)\otimes dF(s) \\ &= A^2\otimes B^{-1}-2\left(\frac{\gamma+\Gamma}{2}\right)(A\otimes 1)+\left(\frac{\gamma+\Gamma}{2}\right)^2(1\otimes B) \\ &= (A\otimes 1)^2(1\otimes B)^{-1}-2\left(\frac{\gamma+\Gamma}{2}\right)(A\otimes 1)(1\otimes B)(1\otimes B)^{-1} \\ &+\left(\frac{\gamma+\Gamma}{2}\right)^2(1\otimes B)^2(1\otimes B)^{-1} \\ &= \left(A\otimes 1-\frac{\gamma+\Gamma}{2}1\otimes B\right)^2(1\otimes B)^{-1}, \end{aligned}$$

hence by (3.13) we get (3.9). $\square$

## 4. Some Examples

Let $A,\ B>0$ with spectra $\operatorname{Sp}(A)\subset I$, $\operatorname{Sp}(B)\subset J$ and such that $0<\gamma\leq\frac{t}{s}\leq\Gamma$ for $t\in I$ and $s\in J$. Consider the convex function $f_r(u)=u^r$, $u>0$, $r\in(-\infty,0)\cup[1,\infty)$, then by (3.1) we get

$$\begin{aligned} 0 &\leq \left[\frac{\gamma^r+\Gamma^r}{2}-\left(\frac{\gamma+\Gamma}{2}\right)^r\right] \\ &\times\left[1\otimes B-\frac{2}{\Gamma-\gamma}\left|A\otimes 1-\frac{\gamma+\Gamma}{2}1\otimes B\right|\right] \end{aligned} \tag{4.1}$$

$$
\begin{aligned}
&\leq \frac{\gamma^r+\Gamma^r}{2} 1 \otimes B+\frac{\Gamma^r-\gamma^r}{\Gamma-\gamma}\left(A \otimes 1-\frac{\gamma+\Gamma}{2} 1 \otimes B\right) \\
&-\mathcal{P}_{f_r, \otimes}(A, B) \\
&\leq\left[\frac{\gamma^r+\Gamma^r}{2}-\left(\frac{\gamma+\Gamma}{2}\right)^r\right] \\
&\times\left[1 \otimes B+\frac{2}{\Gamma-\gamma}\left|A \otimes 1-\frac{\gamma+\Gamma}{2} 1 \otimes B\right|\right],
\end{aligned}
$$

where

$$
\begin{aligned}
\mathcal{P}_{f_{r, \otimes}}(A, B) &:=(1 \otimes B) f_r\left(A \otimes B^{-1}\right)=(1 \otimes B)\left(A \otimes B^{-1}\right)^r \\
&=(A \otimes 1)^r(1 \otimes B)^{1-r} .
\end{aligned}
$$

For $r=2$ we get

$$
\begin{aligned}
0 &\leq \frac{1}{4}(\Gamma-\gamma)^2\left[1 \otimes B-\frac{2}{\Gamma-\gamma}\left|A \otimes 1-\frac{\gamma+\Gamma}{2} 1 \otimes B\right|\right] \\
&\leq \frac{\gamma^2+\Gamma^2}{2} 1 \otimes B+(\Gamma+\gamma)\left(A \otimes 1-\frac{\gamma+\Gamma}{2} 1 \otimes B\right) \\
&-\mathcal{P}_{f_2, \otimes}(A, B) \\
&\leq \frac{1}{4}(\Gamma-\gamma)^2\left[1 \otimes B+\frac{2}{\Gamma-\gamma}\left|A \otimes 1-\frac{\gamma+\Gamma}{2} 1 \otimes B\right|\right],
\end{aligned} \tag{4.2}
$$

where

$$
\mathcal{P}_{f_{2, \otimes}}(A, B):=(A \otimes 1)^2(1 \otimes B)^{-1}=\left(A^2 \otimes 1\right)\left(1 \otimes B^{-1}\right) .
$$

For $r=-1$, we derive

$$
\begin{aligned}
0 &\leq \frac{(\Gamma-\gamma)^2}{2 \Gamma \gamma(\Gamma+\gamma)}\left[1 \otimes B-\frac{2}{\Gamma-\gamma}\left|A \otimes 1-\frac{\gamma+\Gamma}{2} 1 \otimes B\right|\right] \\
&\leq \frac{\gamma+\Gamma}{2 \Gamma \gamma} 1 \otimes B+\frac{1}{\Gamma \gamma}\left(A \otimes 1-\frac{\gamma+\Gamma}{2} 1 \otimes B\right) \\
&-\mathcal{P}_{f_{-1}, \otimes}(A, B) \\
&\leq \frac{(\Gamma-\gamma)^2}{2 \Gamma \gamma(\Gamma+\gamma)}\left[1 \otimes B+\frac{2}{\Gamma-\gamma}\left|A \otimes 1-\frac{\gamma+\Gamma}{2} 1 \otimes B\right|\right],
\end{aligned} \tag{4.3}
$$

where

$$
\mathcal{P}_{f_{-1, \otimes}}(A, B):=(A \otimes 1)^{-1}(1 \otimes B)^2=\left(A^{-1} \otimes 1\right)\left(1 \otimes B^2\right) .
$$

From (3.9) we also get

$$
\begin{aligned}
0 &\leq \frac{\gamma^r+\Gamma^r}{2} 1 \otimes B+\frac{\Gamma^r-\gamma^r}{\Gamma-\gamma}\left(A \otimes 1-\frac{\gamma+\Gamma}{2} 1 \otimes B\right) \\
&-\mathcal{P}_{f_r, \otimes}(A, B) \\
&\leq r \frac{\Gamma^{r-1}-\gamma^{r-1}}{\Gamma-\gamma} \\
&\times\left[\frac{1}{4}(\Gamma-\gamma)^2 1 \otimes B-\left(A \otimes 1-\frac{\gamma+\Gamma}{2} 1 \otimes B\right)^2\left(1 \otimes B^{-1}\right)\right] \\
&\leq \frac{1}{4} r(\Gamma-\gamma)\left(\Gamma^{r-1}-\gamma^{r-1}\right)(1 \otimes B),
\end{aligned} \tag{4.4}
$$

for $r \in (-\infty, 0) \cup [1, \infty)$.

For $r = 2$, we derive

$$
\begin{aligned}
(4.5) \qquad 0 &\leq \frac{\gamma^2 + \Gamma^2}{2} 1 \otimes B + (\Gamma + \gamma) \left( A \otimes 1 - \frac{\gamma + \Gamma}{2} 1 \otimes B \right) \\
&- \mathcal{P}_{f_2, \otimes} (A, B) \\
&\leq 2 \left[ \frac{1}{4} (\Gamma - \gamma)^2 1 \otimes B - \left( A \otimes 1 - \frac{\gamma + \Gamma}{2} 1 \otimes B \right)^2 \left( 1 \otimes B^{-1} \right) \right] \\
&\leq \frac{1}{2} (\Gamma - \gamma)^2 (1 \otimes B),
\end{aligned}
$$

while for $r = -1$,

$$
\begin{aligned}
(4.6) \qquad 0 &\leq \frac{\gamma + \Gamma}{2\Gamma\gamma} 1 \otimes B + \frac{1}{\Gamma\gamma} \left( A \otimes 1 - \frac{\gamma + \Gamma}{2} 1 \otimes B \right) - \mathcal{P}_{f_{-1}, \otimes} (A, B) \\
&\leq \frac{\gamma + \Gamma}{\gamma^2 \Gamma^2} \\
&\times \left[ \frac{1}{4} (\Gamma - \gamma)^2 1 \otimes B - \left( A \otimes 1 - \frac{\gamma + \Gamma}{2} 1 \otimes B \right)^2 \left( 1 \otimes B^{-1} \right) \right] \\
&\leq \frac{1}{4} \left( \frac{\Gamma + \gamma}{\Gamma^2 \gamma^2} \right) (\Gamma - \gamma)^2 (1 \otimes B).
\end{aligned}
$$

Consider the convex function $f = -\ln(\cdot)$, then we get from (3.1) that

$$
\begin{aligned}
(4.7) \qquad 0 &\leq \ln \left( \frac{\gamma + \Gamma}{2\sqrt{\gamma\Gamma}} \right) \\
&\times \left[ 1 \otimes B - \frac{2}{\Gamma - \gamma} \left| A \otimes 1 - \frac{\gamma + \Gamma}{2} 1 \otimes B \right| \right] \\
&\leq (1 \otimes B) \ln \left( A \otimes B^{-1} \right) \\
&- \ln \sqrt{\gamma\Gamma} (1 \otimes B) - \ln \left( \frac{\Gamma}{\gamma} \right)^{\frac{1}{\Gamma - \gamma}} \left( A \otimes 1 - \frac{\gamma + \Gamma}{2} 1 \otimes B \right) \\
&\leq \ln \left( \frac{\gamma + \Gamma}{2\sqrt{\gamma\Gamma}} \right) \\
&\times \left[ 1 \otimes B - \frac{2}{\Gamma - \gamma} \left| A \otimes 1 - \frac{\gamma + \Gamma}{2} 1 \otimes B \right| \right],
\end{aligned}
$$

while from (3.9)

$$
\begin{aligned}
(4.8) \qquad 0 &\leq (1 \otimes B) \ln \left( A \otimes B^{-1} \right) \\
&- \ln \sqrt{\gamma\Gamma} (1 \otimes B) - \ln \left( \frac{\Gamma}{\gamma} \right)^{\frac{1}{\Gamma - \gamma}} \left( A \otimes 1 - \frac{\gamma + \Gamma}{2} 1 \otimes B \right) \\
&\leq \frac{1}{\Gamma\gamma} \left[ \frac{1}{4} (\Gamma - \gamma)^2 1 \otimes B - \left( A \otimes 1 - \frac{\gamma + \Gamma}{2} 1 \otimes B \right)^2 \left( 1 \otimes B^{-1} \right) \right] \\
&\leq \frac{1}{4\Gamma\gamma} (\Gamma - \gamma)^2 (1 \otimes B).
\end{aligned}
$$

If $0 < m_1 \leq A \leq M_1$ and $0 < m_2 \leq B \leq M_2$ then we can take $\gamma = \frac{m_1}{M_2} < \frac{M_1}{m_2} = \Gamma$ in the above inequalities and obtain the corresponding bounds. We omit the details.

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