AN OSTROWSKI TYPE TENSORIAL NORM INEQUALITY FOR CONTINUOUS FUNCTIONS OF SELFADJOINT OPERATORS IN HILBERT SPACES

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Abstract. Let H be a Hilbert space. Assume that f is continuously differentiable on I with $\|f'\|_{I,\infty} := \sup_{t\in I} |f'(t)| < \infty$ and A, B are selfadjoint operators with $\operatorname{Sp}(A), \operatorname{Sp}(B) \subset I$, then

$$\left\| f((1-\lambda) A \otimes 1 + \lambda 1 \otimes B) - \int_0^1 f((1-u) A \otimes 1 + u 1 \otimes B) du \right\|$$
$$\leq \|f'\|_{I,\infty} \left[\frac{1}{4} + \left(\lambda - \frac{1}{2} \right)^2 \right] \|1 \otimes B - A \otimes 1\|$$

for $\lambda \in [0,1]$. In particular, we have the midpoint inequality

$$\left\| f\left(\frac{A \otimes 1 + 1 \otimes B}{2} \right) - \int_0^1 f((1-u) A \otimes 1 + u 1 \otimes B) du \right\|$$
$$\leq \frac{1}{4} \|f'\|_{I,\infty} \|1 \otimes B - A \otimes 1\|.$$

1. Introduction

In 1938, A. Ostrowski [13], proved the following inequality concerning the distance between the integral mean $\frac{1}{b-a}\int_a^b f(t)\,dt$ and the value $f(x)$, $x \in [a,b]$.

Theorem 1. *Let $f : [a,b] \to \mathbb{R}$ be continuous on $[a,b]$ and differentiable on (a,b) such that $f' : (a,b) \to \mathbb{R}$ is bounded on (a,b), i.e., $\|f'\|_\infty := \sup_{t\in(a,b)} |f'(t)| < \infty$. Then*

$$(1.1)\qquad \left| f(x) - \frac{1}{b-a}\int_a^b f(t)\,dt \right| \leq \left[\frac{1}{4} + \left(\frac{x - \frac{a+b}{2}}{b-a} \right)^2 \right] \|f'\|_\infty (b-a),$$

for all $x \in [a,b]$ and the constant $\frac{1}{4}$ is the best possible.

If we take $x = \frac{a+b}{2}$, we get the *midpoint inequality*

$$\left| f\left(\frac{a+b}{2} \right) - \frac{1}{b-a}\int_a^b f(t)\,dt \right| \leq \frac{1}{4} \|f'\|_\infty (b-a),$$

with $\frac{1}{4}$ as best possible constant.

In order to extend this result for tensorial products of selfadjoint operators and norms, we need the following preparations.

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Let $I_1, ..., I_k$ be intervals from $\mathbb{R}$ and let $f : I_1 \times ... \times I_k \to \mathbb{R}$ be an essentially bounded real function defined on the product of the intervals. Let $A = (A_1, ..., A_n)$ be a k-tuple of bounded selfadjoint operators on Hilbert spaces $H_1, ..., H_k$ such that the spectrum of A_i is contained in I_i for $i = 1, ..., k$. We say that such a k-tuple is in the domain of f. If

$$A_i = \int_{I_i} \lambda_i dE_i (\lambda_i)$$

is the spectral resolution of A_i for $i = 1, ..., k$; by following [2], we define

$$f (A_1, ..., A_k) := \int_{I_1} ... \int_{I_k} f (\lambda_1, ..., \lambda_1) dE_1 (\lambda_1) \otimes ... \otimes dE_k (\lambda_k) \tag{1.2}$$

as a bounded selfadjoint operator on the tensorial product $H_1 \otimes ... \otimes H_k$.

If the Hilbert spaces are of finite dimension, then the above integrals become finite sums, and we may consider the functional calculus for arbitrary real functions. This construction [2] extends the definition of Korányi [7] for functions of two variables and have the property that

$$f (A_1, ..., A_k) = f_1(A_1) \otimes ... \otimes f_k(A_k),$$

whenever f can be separated as a product $f(t_1, ..., t_k) = f_1(t_1)...f_k(t_k)$ of k functions each depending on only one variable.

It is know that, *if f is super-multiplicative (sub-multiplicative)* on $[0, \infty)$, namely

$$f (st) \geq (\leq) f (s) f (t) \text{ for all } s, t \in [0, \infty)$$

and if f is continuous on $[0, \infty)$, then [10, p. 173]

$$f (A \otimes B) \geq (\leq) f (A) \otimes f (B) \text{ for all } A,\ B \geq 0. \tag{1.3}$$

This follows by observing that, if

$$A = \int_{[0,\infty)} tdE (t) \text{ and } B = \int_{[0,\infty)} sdF (s)$$

are the spectral resolutions of A and B, then

$$f (A \otimes B) = \int_{[0,\infty)} \int_{[0,\infty)} f (st) dE (t) \otimes dF (s) \tag{1.4}$$

for the continuous function f on $[0, \infty)$.

Recall the *geometric operator mean* for the positive operators $A,\ B > 0$

$$A\#_t B := A^{1/2}(A^{-1/2}BA^{-1/2})^t A^{1/2},$$

where $t \in [0, 1]$ and

$$A\#B := A^{1/2}(A^{-1/2}BA^{-1/2})^{1/2} A^{1/2}.$$

By the definitions of $\#$ and $\otimes$ we have

$$A\#B = B\#A \text{ and } (A\#B) \otimes (B\#A) = (A \otimes B) \# (B \otimes A).$$

In 2007, S. Wada [14] obtained the following *Callebaut type inequalities* for tensorial product

$$\begin{aligned} (A\#B) \otimes (A\#B) &\leq \frac{1}{2} [(A\#_\alpha B) \otimes (A\#_{1-\alpha} B) + (A\#_{1-\alpha} B) \otimes (A\#_\alpha B)] \\ &\leq \frac{1}{2} (A \otimes B + B \otimes A) \end{aligned} \tag{1.5}$$

for $A,\ B > 0$ and $\alpha \in [0, 1]$. For other similar results, see [1], [3] and [8]-[11].

Motivated by the above results, if f is continuously differentiable on I with $\|f'\|_{I,\infty} := \sup_{t\in I}|f'(t)| < \infty$ and A, B are selfadjoint operators with $\operatorname{Sp}(A)$, $\operatorname{Sp}(B) \subset I$, then

$$\begin{aligned}&\left\|f((1-\lambda)A\otimes 1+\lambda 1\otimes B)-\int_0^1 f((1-u)A\otimes 1+u1\otimes B)\,du\right\|\\&\le \|f'\|_{I,\infty}\left[\frac{1}{4}+\left(\lambda-\frac{1}{2}\right)^2\right]\|1\otimes B-A\otimes 1\|\end{aligned}$$

for $\lambda\in[0,1]$.

In particular, we have the midpoint inequality

$$\begin{aligned}&\left\|f\left(\frac{A\otimes 1+1\otimes B}{2}\right)-\int_0^1 f((1-u)A\otimes 1+u1\otimes B)\,du\right\|\\&\le \frac{1}{4}\|f'\|_{I,\infty}\|1\otimes B-A\otimes 1\|.\end{aligned}$$

2. Main Results

Recall the following property of the tensorial product

$$(AC)\otimes(BD)=(A\otimes B)(C\otimes D) \tag{2.1}$$

that holds for any $A,B,C,D\in B(H)$.

If we take $C=A$ and $D=B$, then we get

$$A^2\otimes B^2=(A\otimes B)^2.$$

By induction and using (2.1) we derive that

$$A^n\otimes B^n=(A\otimes B)^n \text{ for natural } n\ge 0. \tag{2.2}$$

In particular

$$A^n\otimes 1=(A\otimes 1)^n \text{ and } 1\otimes B^n=(1\otimes B)^n \tag{2.3}$$

for all $n\ge 0$.

We also observe that, by (2.1), the operators $A\otimes 1$ and $1\otimes B$ are commutative and

$$(A\otimes 1)(1\otimes B)=(1\otimes B)(A\otimes 1)=A\otimes B. \tag{2.4}$$

Moreover, for two natural numbers m, n we have

$$(A\otimes 1)^m(1\otimes B)^n=(1\otimes B)^n(A\otimes 1)^m=A^m\otimes B^n. \tag{2.5}$$

We have the following representation results for continuous functions:

Lemma 1. *Assume A and B are selfadjoint operators with $\operatorname{Sp}(A)\subset I$ and $\operatorname{Sp}(B)\subset J$. Let f,h be continuous on I, g,k continuous on J and φ continuous on an interval K that contains the sum of the intervals $h(I)+k(J)$, then*

$$\begin{aligned}&(f(A)\otimes 1+1\otimes g(B))\varphi(h(A)\otimes 1+1\otimes k(B))\\&=\int_I\int_J (f(t)+g(s))\varphi(h(t)+k(s))\,dE_t\otimes dF_s,\end{aligned} \tag{2.6}$$

where A and B have the spectral resolutions

$$A=\int_I t\,dE(t) \text{ and } B=\int_J s\,dF(s). \tag{2.7}$$

Proof. By Stone-Weierstrass, any continuous function can be approximated by a sequence of polynomials, therefore it suffices to prove the equality for the power function $\varphi(t)=t^{n}$ with n any natural number.

For natural number $n\geq 1$ we have

$$\begin{aligned}\mathcal{K}&:=\int_{I}\int_{J}\left(f(t)+g(s)\right)\left(h(t)+k(s)\right)^{n}dE_{t}\otimes dF_{s} \tag{2.8}\\&=\int_{I}\int_{J}\left(f(t)+g(s)\right)\sum_{m=0}^{n}C_{n}^{m}\left[h(t)\right]^{m}\left[k(s)\right]^{n-m}dE_{t}\otimes dF_{s}\\&=\sum_{m=0}^{n}C_{n}^{m}\int_{I}\int_{J}\left(f(t)+g(s)\right)\left[h(t)\right]^{m}\left[k(s)\right]^{n-m}dE_{t}\otimes dF_{s}\\&=\sum_{m=0}^{n}C_{n}^{m}\left[\int_{I}\int_{J}f(t)\left[h(t)\right]^{m}\left[k(s)\right]^{n-m}dE_{t}\otimes dF_{s}\right.\\&\left.+\int_{I}\int_{J}\left[h(t)\right]^{m}g(s)\left[k(s)\right]^{n-m}dE_{t}\otimes dF_{s}\right].\end{aligned}$$

Observe that

$$\begin{aligned}&\int_{I}\int_{J}f(t)\left[h(t)\right]^{m}\left[k(s)\right]^{n-m}dE_{t}\otimes dF_{s}\\&=f(A)\left[h(A)\right]^{m}\otimes\left[k(B)\right]^{n-m}=\left(f(A)\otimes 1\right)\left(\left[h(A)\right]^{m}\otimes\left[k(B)\right]^{n-m}\right)\\&=\left(f(A)\otimes 1\right)\left(\left[h(A)\right]^{m}\otimes 1\right)\left(1\otimes\left[k(B)\right]^{n-m}\right)\\&=\left(f(A)\otimes 1\right)\left(h(A)\otimes 1\right)^{m}\left(1\otimes k(B)\right)^{n-m}\end{aligned}$$

and

$$\begin{aligned}&\int_{I}\int_{J}\left[h(t)\right]^{m}g(s)\left[k(s)\right]^{n-m}dE_{t}\otimes dF_{s}\\&=\left[h(A)\right]^{m}\otimes\left(g(B)\left[k(B)\right]^{n-m}\right)=\left(1\otimes g(B)\right)\left(\left[h(A)\right]^{m}\otimes\left[k(B)\right]^{n-m}\right)\\&=\left(1\otimes g(B)\right)\left(\left[h(A)\right]^{m}\otimes 1\right)\left(1\otimes\left[k(B)\right]^{n-m}\right)\\&=\left(1\otimes g(B)\right)\left(h(A)\otimes 1\right)^{m}\left(1\otimes k(B)\right)^{n-m},\end{aligned}$$

with $h(A)\otimes 1$ and $1\otimes k(B)$ commutative.

Therefore

$$\begin{aligned}\mathcal{K}&=\left(f(A)\otimes 1+1\otimes g(B)\right)\sum_{m=0}^{n}C_{n}^{m}\left(h(A)\otimes 1\right)^{m}\left(1\otimes k(B)\right)^{n-m}\\&=\left(f(A)\otimes 1+1\otimes g(B)\right)\left(h(A)\otimes 1+1\otimes k(B)\right)^{n},\end{aligned}$$

for which the commutativity of $h(A)\otimes 1$ and $1\otimes k(B)$ has been employed. $\square$

Theorem 2. *Assume that f is continuously differentiable on I, A and B are selfadjoint operators with* $\operatorname{Sp}(A), \operatorname{Sp}(B) \subset I$, *then*

$$
\begin{aligned}
&f\left((1-\lambda) A \otimes 1+\lambda 1 \otimes B\right)-\int_0^1 f\left((1-u) A \otimes 1+u 1 \otimes B\right) du \\
&=\lambda^2(1 \otimes B-A \otimes 1) \\
&\times \int_0^1 u f'\left((1-u\lambda) A \otimes 1+u\lambda 1 \otimes B\right) du \\
&-(1-\lambda)^2(1 \otimes B-A \otimes 1) \\
&\times \int_0^1 u f'\left(u(1-\lambda) A \otimes 1+(1-(1-\lambda) u) 1 \otimes B\right) du,
\end{aligned} \tag{2.9}
$$

for all $\lambda \in [0,1]$. In particular, for $\lambda=\frac{1}{2}$, we have the midpoint identity

$$
\begin{aligned}
&f\left(\frac{A \otimes 1+1 \otimes B}{2}\right)-\int_0^1 f\left((1-u) A \otimes 1+u 1 \otimes B\right) du \\
&=\frac{1}{4}(1 \otimes B-A \otimes 1) \int_0^1 u\left[f'\left(\left(1-\frac{u}{2}\right) A \otimes 1+\frac{u}{2} 1 \otimes B\right)\right. \\
&\left.-f'\left(\frac{u}{2} A \otimes 1+\left(1-\frac{u}{2}\right) 1 \otimes B\right)\right] du.
\end{aligned} \tag{2.10}
$$

Proof. We start to the Montgomery identity for real valued absolutely continuous functions on $[a,b]$ that can be easily proved integrating by parts in the right side of the equality,

$$
(b-a) f(x)-\int_a^b f(t) dt=\int_a^x (t-a) f'(t) dt+\int_x^b (t-b) f'(t) dt \tag{2.11}
$$

for $a \leq x \leq b$.

If we use the change of variable $t=(1-u) a+u x$, then we have $dt=(x-a) du$ and

$$
\int_a^x (t-a) f'(t) dt=(x-a)^2 \int_0^1 u f'((1-u) a+u x) du.
$$

If we use the change of variable $t=(1-u) x+u b$, then we have $dt=(b-x) du$ and

$$
\int_x^b (t-b) f'(t) dt=-(b-x)^2 \int_0^1 (1-u) f'((1-u) x+u b) du.
$$

By (2.11) we get

$$
\begin{aligned}
&(b-a) f(x)-(b-a) \int_0^1 f((1-u) a+u b) du \\
&=(x-a)^2 \int_0^1 u f'((1-u) a+u x) du \\
&-(b-x)^2 \int_0^1 (1-u) f'((1-u) x+u b) du.
\end{aligned} \tag{2.12}
$$

If we take $x=(1-\lambda)a+\lambda b$, $\lambda\in[0,1]$ in (2.12), then we get

$$\begin{aligned}
&(b-a)f((1-\lambda)a+\lambda b)-(b-a)\int_0^1 f((1-u)a+ub)\,du \qquad (2.13)\\
&=(b-a)^2\lambda^2\int_0^1 uf'((1-u)a+u[(1-\lambda)a+\lambda b])\,du\\
&-(b-a)^2(1-\lambda)^2\int_0^1(1-u)f'((1-u)[(1-\lambda)a+\lambda b]+ub)\,du\\
&=(b-a)^2\lambda^2\int_0^1 uf'((1-u\lambda)a+u\lambda b)\,du\\
&-(b-a)^2(1-\lambda)^2\int_0^1(1-u)f'((1-u)(1-\lambda)a+(\lambda+(1-\lambda)u)b)\,du.
\end{aligned}$$

Therefore, for all $a,b\in I$ and $\lambda\in[0,1]$,

$$\begin{aligned}
&f((1-\lambda)a+\lambda b)-\int_0^1 f((1-u)a+ub)\,du \qquad (2.14)\\
&=(b-a)\lambda^2\int_0^1 uf'((1-u\lambda)a+u\lambda b)\,du\\
&-(b-a)(1-\lambda)^2\int_0^1(1-u)f'((1-u)(1-\lambda)a+(\lambda+(1-\lambda)u)b)\,du\\
&=\lambda^2(b-a)\int_0^1 uf'((1-u\lambda)a+u\lambda b)\,du\\
&-(1-\lambda)^2(b-a)\int_0^1 uf'(u(1-\lambda)a+(1-(1-\lambda)u)b)\,du,
\end{aligned}$$

where for the last equality we change the variable $1-u$ with u in the second previous integral.

Assume that A and B have the spectral resolutions

$$A=\int_I tdE(t)\ \text{ and } B=\int_I sdF(s).$$

If we take the integral $\int_I\int_I$ over $dE_t\otimes dF_s$ in (2.14), then we get

$$\begin{aligned}
&\int_I\int_I f((1-\lambda)t+\lambda s)\,dE_t\otimes dF_s \qquad (2.15)\\
&-\int_I\int_I\left(\int_0^1 f((1-u)t+us)\,du\right)dE_t\otimes dF_s\\
&=\lambda^2\int_I\int_I\left((s-t)\int_0^1 uf'((1-u\lambda)t+u\lambda s)\,du\right)dE_t\otimes dF_s\\
&-(1-\lambda)^2\\
&\times\int_I\int_I\left((s-t)\int_0^1 uf'(u(1-\lambda)t+(1-(1-\lambda)u)s)\,du\right)dE_t\otimes dF_s,
\end{aligned}$$

for all $\lambda\in[0,1]$.

By utilizing the Fubini's theorem and Lemma 1 for appropriate choices of the functions involved, we have successively

$$\int_I \int_I f((1-\lambda)t+\lambda s)\,dE_t \otimes dF_s = f((1-\lambda)A\otimes 1+\lambda 1\otimes B),$$

$$\begin{aligned}&\int_I \int_I \left(\int_0^1 f((1-u)t+us)\,du\right) dE_t \otimes dF_s \\ &= \int_0^1 \left(\int_I \int_I f((1-u)t+us)\,dE_t \otimes dF_s\right) du \\ &= \int_0^1 f((1-u)A\otimes 1+u1\otimes B)\,du,\end{aligned}$$

$$\begin{aligned}&\int_I \int_I \left((s-t)\int_0^1 uf'((1-u\lambda)t+u\lambda s)\,du\right) dE_t \otimes dF_s \\ &= \int_0^1 u\left(\int_I \int_I (s-t) f'((1-u\lambda)t+u\lambda s)\,dE_t \otimes dF_s\right) du \\ &= (1\otimes B-A\otimes 1)\int_0^1 uf'((1-u\lambda)A\otimes 1+u\lambda 1\otimes B)\,du\end{aligned}$$

and

$$\begin{aligned}&\int_I \int_I \left((s-t)\int_0^1 uf'(u(1-\lambda)t+(1-(1-\lambda)u)s)\,du\right) dE_t \otimes dF_s \\ &= \int_0^1 u\left(\int_I \int_I (s-t) f'((1-\lambda)t+(1-(1-\lambda)u)s)\,dE_t \otimes dF_s\right) du \\ &= (1\otimes B-A\otimes 1)\int_0^1 u(f'((1-\lambda)A\otimes 1+(1-(1-\lambda)u)1\otimes B))\,du.\end{aligned}$$

By employing (2.15), we then get the desired result (2.9). □

Theorem 3. *Assume that f is continuously differentiable on I with $\|f'\|_{I,\infty} := \sup_{t\in I}|f'(t)| < \infty$ and A, B are selfadjoint operators with $\operatorname{Sp}(A)$, $\operatorname{Sp}(B) \subset I$, then*

$$\begin{aligned}&\left\|f((1-\lambda)A\otimes 1+\lambda 1\otimes B)-\int_0^1 f((1-u)A\otimes 1+u1\otimes B)\,du\right\| \\ &\leq \|f'\|_{I,\infty}\left[\frac{1}{4}+\left(\lambda-\frac{1}{2}\right)^2\right]\|1\otimes B-A\otimes 1\|\end{aligned}\tag{2.16}$$

for $\lambda \in [0,1]$.

In particular, we have the midpoint inequality

$$\begin{aligned}&\left\|f\left(\frac{A\otimes 1+1\otimes B}{2}\right)-\int_0^1 f((1-u)A\otimes 1+u1\otimes B)\,du\right\| \\ &\leq \frac{1}{4}\|f'\|_{I,\infty}\|1\otimes B-A\otimes 1\|.\end{aligned}\tag{2.17}$$

Proof. If we take the operator norm and use the triangle inequality, we get

$$\begin{aligned}(2.18)\quad & \left\| f\left((1-\lambda)A\otimes 1+\lambda 1\otimes B\right)-\int_0^1 f\left((1-u)A\otimes 1+u1\otimes B\right)du\right\| \\ & \le \lambda^2\left\|1\otimes B-A\otimes 1\right\| \\ & \times\left\|\int_0^1 uf'\left((1-u\lambda)A\otimes 1+u\lambda 1\otimes B\right)du\right\| \\ & +(1-\lambda)^2\left\|1\otimes B-A\otimes 1\right\| \\ & \times\left\|\int_0^1 uf'\left(u(1-\lambda)A\otimes 1+(1-(1-\lambda)u)1\otimes B\right)du\right\|,\end{aligned}$$

for all $\lambda\in[0,1]$.

By the properties of the integral and norm, we have

$$\begin{aligned}(2.19)\quad & \left\|\int_0^1 uf'\left((1-u\lambda)A\otimes 1+u\lambda 1\otimes B\right)du\right\| \\ & \le\int_0^1 u\left\|f'\left((1-u\lambda)A\otimes 1+u\lambda 1\otimes B\right)\right\|du\end{aligned}$$

and

$$\begin{aligned}(2.20)\quad & \left\|\int_0^1 uf'\left(u(1-\lambda)A\otimes 1+(1-(1-\lambda)u)1\otimes B\right)du\right\| \\ & \le\int_0^1 u\left\|f'\left(u(1-\lambda)A\otimes 1+(1-(1-\lambda)u)1\otimes B\right)\right\|du.\end{aligned}$$

Observe that, by Lemma 1

$$\left|f'\left((1-u\lambda)A\otimes 1+u\lambda 1\otimes B\right)\right|=\int_I\int_I\left|f'\left((1-u\lambda)t+u\lambda s\right)\right|dE_t\otimes dF_s$$

for $u,\lambda\in[0,1]$.

Since

$$\left|f'\left((1-u\lambda)t+u\lambda s\right)\right|\le\left\|f'\right\|_{I,\infty}$$

for $u,\lambda\in[0,1]$ and $t,s\in I$.

If we take the integral $\int_I\int_I$ over $dE_t\otimes dF_s$, then we get

$$\begin{aligned}(2.21)\quad & \left|f'\left((1-u\lambda)A\otimes 1+u\lambda 1\otimes B\right)\right| \\ & =\int_I\int_I\left|f'\left((1-u\lambda)t+u\lambda s\right)\right|dE_t\otimes dF_s\le\left\|f'\right\|_{I,\infty}\int_I\int_I dE_t\otimes dF_s \\ & =\left\|f'\right\|_{I,\infty}\end{aligned}$$

for $u,\lambda\in[0,1]$. This implies that

$$\left\|f'\left((1-u\lambda)A\otimes 1+u\lambda 1\otimes B\right)\right\|\le\left\|f'\right\|_{I,\infty}$$

for $u,\lambda\in[0,1]$ which gives

$$\int_0^1 u\left\|f'\left((1-u\lambda)A\otimes 1+u\lambda 1\otimes B\right)\right\|du\le\left\|f'\right\|_{I,\infty}\int_0^1 udu=\frac{1}{2}\left\|f'\right\|_{I,\infty}$$

Similarly, we have

$$\int_0^1 u\left\|f'\left(u(1-\lambda)A\otimes 1+(1-(1-\lambda)u)1\otimes B\right)\right\|du\le\frac{1}{2}\left\|f'\right\|_{I,\infty}.$$

By (2.18)-(2.20) we derive

$$\begin{aligned}&\left\|f\left((1-\lambda)A\otimes 1+\lambda 1\otimes B\right)-\int_0^1 f\left((1-u)A\otimes 1+u1\otimes B\right)du\right\|\\&\leq\frac{1}{2}\left\|f'\right\|_{I,\infty}\left\|1\otimes B-A\otimes 1\right\|\left[\lambda^2+(1-\lambda)^2\right]\\&=\left\|1\otimes B-A\otimes 1\right\|\left[\frac{1}{4}+\left(\lambda-\frac{1}{2}\right)^2\right]\left\|f'\right\|_{I,\infty},\end{aligned}$$

which proves (2.16). □

3. Related Results

We start by the following result:

Theorem 4. *Assume that f is continuously differentiable on I with $|f'|$ is convex on I, A and B are selfadjoint operators with* $\operatorname{Sp}(A), \operatorname{Sp}(B)\subset I$*, then*

$$\begin{aligned}&\left\|f\left((1-\lambda)A\otimes 1+\lambda 1\otimes B\right)-\int_0^1 f\left((1-u)A\otimes 1+u1\otimes B\right)du\right\|\\&\leq\left\|1\otimes B-A\otimes 1\right\|\left[p(1-\lambda)\left\|f'(A)\right\|+p(\lambda)\left\|f'(B)\right\|\right],\end{aligned}\tag{3.1}$$

for $\lambda\in[0,1]$, where

$$p(\lambda)=\frac{1}{3}\left[\lambda^3-(1-\lambda)^3\right]+\frac{1}{2}(1-\lambda)^2,\ \lambda\in[0,1].$$

In particular, for $\lambda=\frac{1}{2}$, we get the midpoint inequality:

$$\begin{aligned}&\left\|f\left(\frac{A\otimes 1+1\otimes B}{2}\right)-\int_0^1 f\left((1-u)A\otimes 1+u1\otimes B\right)du\right\|\\&\leq\frac{1}{8}\left\|1\otimes B-A\otimes 1\right\|\left[\left\|f'(A)\right\|+\left\|f'(B)\right\|\right].\end{aligned}\tag{3.2}$$

Proof. Since $|f'|$ is convex on I, then we get

$$\left|f'\left((1-u\lambda)t+u\lambda s\right)\right|\leq(1-u\lambda)\left|f'(t)\right|+u\lambda\left|f'(s)\right|$$

for all for $u,\lambda\in[0,1]$ and $t,s\in I$.

If we take the integral $\int_I\int_I$ over $dE_t\otimes dF_s$, then we get

$$\begin{aligned}&\left|f'\left((1-u\lambda)A\otimes 1+u\lambda 1\otimes B\right)\right|\\&=\int_I\int_I\left|f'\left((1-u\lambda)t+u\lambda s\right)\right|dE_t\otimes dF_s\\&\leq\int_I\int_I\left[(1-u\lambda)\left|f'(t)\right|+u\lambda\left|f'(s)\right|\right]dE_t\otimes dF_s\\&=(1-u\lambda)\left|f'(A)\right|\otimes 1+u\lambda 1\otimes\left|f'(B)\right|\end{aligned}\tag{3.3}$$

for all for $u,\lambda\in[0,1]$.

If we take the norm in (3.3), then we get

$$\begin{aligned}&\left\|f'\left((1-u\lambda)A\otimes 1+u\lambda 1\otimes B\right)\right\|\\&\leq\left\|(1-u\lambda)\left|f'(A)\right|\otimes 1+u\lambda 1\otimes\left|f'(B)\right|\right\|\\&\leq(1-u\lambda)\left\|\left|f'(A)\right|\otimes 1\right\|+u\lambda\left\|1\otimes\left|f'(B)\right|\right\|\\&=(1-u\lambda)\left\|f'(A)\right\|+u\lambda\left\|f'(B)\right\|.\end{aligned}\tag{3.4}$$

Therefore,

$$\begin{aligned}&\int_0^1 u\left\|f'\left((1-u\lambda)A\otimes 1+u\lambda 1\otimes B\right)\right\|du\\&\leq\left\|f'(A)\right\|\int_0^1 u(1-u\lambda)\,du+\left\|f'(B)\right\|\lambda\int_0^1 u^2du\\&=\left(\frac{1}{2}-\frac{\lambda}{3}\right)\left\|f'(A)\right\|+\left\|f'(B)\right\|\frac{\lambda}{3}.\end{aligned}$$

Similarly,

$$\begin{aligned}&\int_0^1 u\left\|f'\left(u(1-\lambda)A\otimes 1+(1-(1-\lambda)u)1\otimes B\right)\right\|du\\&\leq\int_0^1 u\left[u(1-\lambda)\left\|f'(A)\right\|+(1-(1-\lambda)u)\left\|f'(B)\right\|\right]\\&=\frac{1}{3}(1-\lambda)\left\|f'(A)\right\|+\left(\frac{1}{2}-\frac{(1-\lambda)}{3}\right)\left\|f'(B)\right\|.\end{aligned}$$

From (2.180 we get

$$\begin{aligned}&\left\|f\left((1-\lambda)A\otimes 1+\lambda 1\otimes B\right)-\int_0^1 f\left((1-u)A\otimes 1+u1\otimes B\right)du\right\|\\&\leq\lambda^2\left\|1\otimes B-A\otimes 1\right\|\left[\left(\frac{1}{2}-\frac{\lambda}{3}\right)\left\|f'(A)\right\|+\left\|f'(B)\right\|\frac{\lambda}{3}\right]\\&+(1-\lambda)^2\left\|1\otimes B-A\otimes 1\right\|\\&\times\left[\frac{1}{3}(1-\lambda)\left\|f'(A)\right\|+\left(\frac{1}{2}-\frac{(1-\lambda)}{3}\right)\left\|f'(B)\right\|\right]\end{aligned}$$

$$\begin{aligned}&=\left\|1\otimes B-A\otimes 1\right\|\\&\times\left\{\lambda^2\left[\left(\frac{1}{2}-\frac{\lambda}{3}\right)\left\|f'(A)\right\|+\left\|f'(B)\right\|\frac{\lambda}{3}\right]\right.\\&\left.+(1-\lambda)^2\left[\frac{1}{3}(1-\lambda)\left\|f'(A)\right\|+\left(\frac{1}{2}-\frac{(1-\lambda)}{3}\right)\left\|f'(B)\right\|\right]\right\}\\&=\left\|1\otimes B-A\otimes 1\right\|\\&\times\left\{\left[\frac{1}{3}(1-\lambda)^3+\lambda^2\left(\frac{1}{2}-\frac{\lambda}{3}\right)\right]\left\|f'(A)\right\|\right.\\&\left.+\left[\frac{1}{3}\lambda^3+(1-\lambda)^2\left(\frac{1}{2}-\frac{1-\lambda}{3}\right)\right]\left\|f'(B)\right\|\right\},\end{aligned}$$

which gives the desired result (3.1). □

We recall that the function $g:I\to\mathbb{R}$ is *quasi-convex*, if

$$g\left((1-\lambda)t+\lambda s\right)\leq\max\left\{g(t),g(s)\right\}=\frac{1}{2}\left(g(t)+g(s)+\left|g(t)-g(s)\right|\right)$$

for all $t,s\in I$ and $\lambda\in[0,1]$.

Theorem 5. *Assume that f is continuously differentiable on I with $|f'|$ is quasi-convex on I, A and B are selfadjoint operators with $\operatorname{Sp}(A)$, $\operatorname{Sp}(B) \subset I$, then*

$$\begin{aligned}(3.5)\qquad & \left\| f\left((1-\lambda)A\otimes 1+\lambda 1\otimes B\right)-\int_0^1 f\left((1-u)A\otimes 1+u1\otimes B\right)du\right\| \\ & \le \frac{1}{2}\left[\frac{1}{4}+\left(\lambda-\frac{1}{2}\right)^2\right]\|1\otimes B-A\otimes 1\| \\ & \times\left(\||f'(A)|\otimes 1+1\otimes|f'(B)|\|+\||f'(A)|\otimes 1-1\otimes|f'(B)|\|\right)\end{aligned}$$

for $\lambda\in[0,1]$.

In particular, we have the midpoint inequality:

$$\begin{aligned}(3.6)\qquad & \left\| f\left(\frac{A\otimes 1+1\otimes B}{2}\right)-\int_0^1 f\left((1-u)A\otimes 1+u1\otimes B\right)du\right\| \\ & \le \frac{1}{8}\|1\otimes B-A\otimes 1\| \\ & \times\left(\||f'(A)|\otimes 1+1\otimes|f'(B)|\|+\||f'(A)|\otimes 1-1\otimes|f'(B)|\|\right).\end{aligned}$$

Proof. Since $|f'|$ is quasi-convex on I, then we get

$$|f'((1-u\lambda)t+u\lambda s)|\le\frac{1}{2}\left(|f'(t)|+|f'(s)|+||f'(t)|-|f'(s)||\right)$$

for all for $u,\lambda\in[0,1]$ and $t,s\in I$.

If we take the integral $\int_I\int_I$ over $dE_t\otimes dF_s$, then we get

$$\begin{aligned}(3.7)\qquad & |f'((1-u\lambda)A\otimes 1+u\lambda 1\otimes B)| \\ & =\int_I\int_I|f'((1-u\lambda)t+u\lambda s)|\,dE_t\otimes dF_s \\ & \le\frac{1}{2}\int_I\int_I\left(|f'(t)|+|f'(s)|+||f'(t)|-|f'(s)||\right)dE_t\otimes dF_s \\ & =\frac{1}{2}\left(|f'(A)|\otimes 1+1\otimes|f'(B)|+||f'(A)|\otimes 1-1\otimes|f'(B)||\right)\end{aligned}$$

for all for $u,\lambda\in[0,1]$.

If we take the norm, then we get

$$\begin{aligned} & \|f'((1-u\lambda)A\otimes 1+u\lambda 1\otimes B)\| \\ & \le\frac{1}{2}\left\|\left(|f'(A)|\otimes 1+1\otimes|f'(B)|+||f'(A)|\otimes 1-1\otimes|f'(B)||\right)\right\| \\ & \le\frac{1}{2}\left(\||f'(A)|\otimes 1+1\otimes|f'(B)|\|+\||f'(A)|\otimes 1-1\otimes|f'(B)|\|\right)\end{aligned}$$

for all for $u,\lambda\in[0,1]$.

Therefore

$$\begin{aligned} & \int_0^1 u\|f'((1-u\lambda)A\otimes 1+u\lambda 1\otimes B)\|\,du \\ & \le\frac{1}{2}\left(\||f'(A)|\otimes 1+1\otimes|f'(B)|\|+\||f'(A)|\otimes 1-1\otimes|f'(B)|\|\right)\int_0^1 udu \\ & =\frac{1}{4}\left(\||f'(A)|\otimes 1+1\otimes|f'(B)|\|+\||f'(A)|\otimes 1-1\otimes|f'(B)|\|\right)\end{aligned}$$

and, in a similar way

$$\int_0^1 u \left\| f'\left(u\left(1-\lambda\right)A\otimes 1+\left(1-\left(1-\lambda\right)u\right)1\otimes B\right)\right\| du$$
$$\leq \frac{1}{4}\left(\left\|\left|f'\left(A\right)\right|\otimes 1+1\otimes\left|f'\left(B\right)\right|\right\|+\left\|\left|f'\left(A\right)\right|\otimes 1-1\otimes\left|f'\left(B\right)\right|\right\|\right).$$

By utilizing (2.18) we then get

$$\left\| f\left(\left(1-\lambda\right)A\otimes 1+\lambda 1\otimes B\right)-\int_0^1 f\left(\left(1-u\right)A\otimes 1+u1\otimes B\right)du\right\|$$
$$\leq \lambda^2\left\|1\otimes B-A\otimes 1\right\|$$
$$\times\frac{1}{4}\left(\left\|\left|f'\left(A\right)\right|\otimes 1+1\otimes\left|f'\left(B\right)\right|\right\|+\left\|\left|f'\left(A\right)\right|\otimes 1-1\otimes\left|f'\left(B\right)\right|\right\|\right)$$
$$+\left(1-\lambda\right)^2\left\|1\otimes B-A\otimes 1\right\|$$
$$\times\frac{1}{4}\left(\left\|\left|f'\left(A\right)\right|\otimes 1+1\otimes\left|f'\left(B\right)\right|\right\|+\left\|\left|f'\left(A\right)\right|\otimes 1-1\otimes\left|f'\left(B\right)\right|\right\|\right)$$
$$=\frac{1}{4}\left(\lambda^2+\left(1-\lambda\right)^2\right)\left\|1\otimes B-A\otimes 1\right\|$$
$$\times\left(\left\|\left|f'\left(A\right)\right|\otimes 1+1\otimes\left|f'\left(B\right)\right|\right\|+\left\|\left|f'\left(A\right)\right|\otimes 1-1\otimes\left|f'\left(B\right)\right|\right\|\right)$$
$$=\frac{1}{2}\left[\frac{1}{4}+\left(\lambda-\frac{1}{2}\right)^2\right]\left\|1\otimes B-A\otimes 1\right\|$$
$$\times\left(\left\|\left|f'\left(A\right)\right|\otimes 1+1\otimes\left|f'\left(B\right)\right|\right\|+\left\|\left|f'\left(A\right)\right|\otimes 1-1\otimes\left|f'\left(B\right)\right|\right\|\right),$$

which proves the desired inequality (3.5). □

4. Examples

It is known that if U and V are commuting, i.e. $UV = VU$, then the exponential function satisfies the property

$$\exp\left(U\right)\exp\left(V\right)=\exp\left(V\right)\exp\left(U\right)=\exp\left(U+V\right).$$

Also, if U is invertible and $a, b \in \mathbb{R}$ with $a < b$ then

$$\int_a^b \exp\left(tU\right)dt=U^{-1}\left[\exp\left(bU\right)-\exp\left(aU\right)\right].$$

Moreover, if U and V are commuting and $V-U$ is invertible, then

$$\begin{aligned}\int_0^1 \exp\left(\left(1-s\right)U+sV\right)ds &= \int_0^1 \exp\left(s\left(V-U\right)\right)\exp\left(U\right)ds\\ &=\left(\int_0^1 \exp\left(s\left(V-U\right)\right)ds\right)\exp\left(U\right)\\ &=\left(V-U\right)^{-1}\left[\exp\left(V-U\right)-I\right]\exp\left(U\right)\\ &=\left(V-U\right)^{-1}\left[\exp\left(V\right)-\exp\left(U\right)\right].\end{aligned}$$

Since the operators $U = A\otimes 1$ and $V = 1\otimes B$ are commutative and if $1\otimes B - A\otimes 1$ is invertible, then

$$\begin{aligned}&\int_0^1 \exp\left((1-u)\,A\otimes 1 + u1\otimes B\right)du\\&= (1\otimes B - A\otimes 1)^{-1}\left[\exp(1\otimes B) - \exp(A\otimes 1)\right].\end{aligned}$$

If A, B are selfadjoint operators with $\operatorname{Sp}(A)$, $\operatorname{Sp}(B) \subset [m, M]$ and $1\otimes B - A\otimes 1$ is invertible, then by (2.16)

$$\begin{aligned}&\|\exp\left((1-\lambda)\,A\otimes 1 + \lambda 1\otimes B\right)\\&- (1\otimes B - A\otimes 1)^{-1}\left[\exp(1\otimes B) - \exp(A\otimes 1)\right]\Big\|\\&\leq \exp(M)\left[\frac{1}{4} + \left(\lambda - \frac{1}{2}\right)^2\right]\|1\otimes B - A\otimes 1\|,\end{aligned} \tag{4.1}$$

for $\lambda \in [0,1]$.

In particular,

$$\begin{aligned}&\left\|\exp\left(\frac{A\otimes 1 + 1\otimes B}{2}\right)\right.\\&- (1\otimes B - A\otimes 1)^{-1}\left[\exp(1\otimes B) - \exp(A\otimes 1)\right]\Big\|\\&\leq \frac{1}{4}\exp(M)\|1\otimes B - A\otimes 1\|.\end{aligned} \tag{4.2}$$

Since for $f(t) = \exp t$, $t \in \mathbb{R}$, $|f'|$ is convex, then by Theorem 4 we get

$$\begin{aligned}&\|\exp\left((1-\lambda)\,A\otimes 1 + \lambda 1\otimes B\right)\\&- (1\otimes B - A\otimes 1)^{-1}\left[\exp(1\otimes B) - \exp(A\otimes 1)\right]\Big\|\\&\leq \frac{1}{2}\left[\frac{1}{4} + \left(\lambda - \frac{1}{2}\right)^2\right]\|1\otimes B - A\otimes 1\|\\&\times\left(\|\exp(A)\otimes 1 + 1\otimes \exp(B)\| + \|\exp(A)\otimes 1 - 1\otimes \exp(B)\|\right)\end{aligned} \tag{4.3}$$

for $\lambda \in [0,1]$.

In particular,

$$\begin{aligned}&\left\|\exp\left(\frac{A\otimes 1 + 1\otimes B}{2}\right)\right.\\&- (1\otimes B - A\otimes 1)^{-1}\left[\exp(1\otimes B) - \exp(A\otimes 1)\right]\Big\|\\&\leq \frac{1}{8}\|1\otimes B - A\otimes 1\|\\&\times\left(\|\exp(A)\otimes 1 + 1\otimes \exp(B)\| + \|\exp(A)\otimes 1 - 1\otimes \exp(B)\|\right)\end{aligned} \tag{4.4}$$

provided that $1\otimes B - A\otimes 1$ is invertible.

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