INEQUALITIES FOR THE NORMALIZED DETERMINANTS OF TWO POSITIVE OPERATORS IN HILBERT SPACES

SILVESTRU SEVER DRAGOMIR[1,2]

Abstract. For positive invertible operators A on a Hilbert space H and a fixed unit vector $x \in H$, define the normalized determinant by $\Delta_x(A) := \exp \langle \ln Ax, x \rangle$. In this paper we prove among others that, if A, $B > 0$ then for x, $y \in H$ with $\|x\| = \|y\| = 1$, we have the inequalities

$$\exp\left(1 - \left\langle A^{-1}x, x\right\rangle \langle By, y\rangle\right) \leq \frac{\Delta_x(A)}{\Delta_y(B)} \leq \exp\left(\langle Ax, x\rangle \left\langle B^{-1}y, y\right\rangle - 1\right).$$

1. Introduction

Let $B(H)$ be the space of all bounded linear operators on a Hilbert space H, and I stands for the identity operator on H. An operator A in $B(H)$ is said to be positive (in symbol: $A \geq 0$) if $\langle Ax, x\rangle \geq 0$ for all $x \in H$. In particular, $A > 0$ means that A is positive and invertible. For a pair A, B of selfadjoint operators the order relation $A \geq B$ means as usual that $A - B$ is positive.

In 1998, Fujii et al. [3], [4], introduced the *normalized determinant* $\Delta_x(A)$ for positive invertible operators A on a Hilbert space H and a fixed unit vector $x \in H$, namely $\|x\| = 1$, defined by $\Delta_x(A) := \exp \langle \ln Ax, x\rangle$ and discussed it as a continuous geometric mean and observed some inequalities around the determinant from this point of view.

Some of the fundamental properties of normalized determinant are as follows, [3].

For each unit vector $x \in H$, see also [6], we have:

(i) *continuity*: the map $A \to \Delta_x(A)$ is norm continuous;
(ii) *bounds*: $\left\langle A^{-1}x, x\right\rangle^{-1} \leq \Delta_x(A) \leq \langle Ax, x\rangle$;
(iii) *continuous mean*: $\langle A^p x, x\rangle^{1/p} \downarrow \Delta_x(A)$ for $p \downarrow 0$ and $\langle A^p x, x\rangle^{1/p} \uparrow \Delta_x(A)$ for $p \uparrow 0$;
(iv) *power equality:* $\Delta_x(A^t) = \Delta_x(A)^t$ for all $t > 0$;
(v) *homogeneity*: $\Delta_x(tA) = t\Delta_x(A)$ and $\Delta_x(tI) = t$ for all $t > 0$;
(vi) *monotonicity*: $0 < A \leq B$ implies $\Delta_x(A) \leq \Delta_x(B)$;
(vii) *multiplicativity*: $\Delta_x(AB) = \Delta_x(A)\Delta_x(B)$ for commuting A and B;
(viii) *Ky Fan type inequality*: $\Delta_x((1-\alpha)A + \alpha B) \geq \Delta_x(A)^{1-\alpha}\Delta_x(B)^{\alpha}$ for $0 < \alpha < 1$.

1991 *Mathematics Subject Classification.* 47A63, 26D15, 46C05.
Key words and phrases. Positive operators, Normalized determinants, Inequalities.

RGMIA Res. Rep. Coll. 25 (2022), Art. 11, 10 pp, Received 10/02/22

We define the logarithmic mean of two positive numbers a, b by

$$L(a,b):=\begin{cases}\frac{b-a}{\ln b-\ln a} \text{ if } b\neq a,\\ \\ a \text{ if } b=a.\end{cases}$$

In [3] the authors obtained the following additive reverse inequality for the operator A which satisfy the condition $0<mI\leq A\leq MI$, where m,M are positive numbers,

$$0\leq\langle Ax,x\rangle-\Delta_x(A)\leq L(m,M)\left[\ln L(m,M)+\frac{M\ln m-m\ln M}{M-m}-1\right] \tag{1.1}$$

for all $x\in H$, $\|x\|=1$.

We recall that *Specht's ratio* is defined by [7]

$$S(h):=\begin{cases}\frac{h^{\frac{1}{h-1}}}{e\ln\left(h^{\frac{1}{h-1}}\right)} \text{ if } h\in(0,1)\cup(1,\infty),\\ \\ 1 \text{ if } h=1.\end{cases} \tag{1.2}$$

It is well known that $\lim_{h\to1}S(h)=1$, $S(h)=S\left(\frac{1}{h}\right)>1$ for $h>0$, $h\neq1$. The function is decreasing on $(0,1)$ and increasing on $(1,\infty)$.

In [4], the authors obtained the following multiplicative reverse inequality as well

$$1\leq\frac{\langle Ax,x\rangle}{\Delta_x(A)}\leq S\left(\frac{M}{m}\right) \tag{1.3}$$

for $0<mI\leq A\leq MI$ and $x\in H$, $\|x\|=1$.

2. Main Results

The first main result is as follows:

Theorem 1. *If A, $B>0$ then for x, $y\in H$ with $\|x\|=\|y\|=1$, we have the inequalities*

$$\exp\left(1-\left\langle A^{-1}x,x\right\rangle\langle By,y\rangle\right)\leq\frac{\Delta_x(A)}{\Delta_y(B)}\leq\exp\left(\langle Ax,x\rangle\left\langle B^{-1}y,y\right\rangle-1\right). \tag{2.1}$$

In particular,

$$\exp\left(1-\left\langle A^{-1}x,x\right\rangle\langle Ay,y\rangle\right)\leq\frac{\Delta_x(A)}{\Delta_y(A)}\leq\exp\left(\langle Ax,x\rangle\left\langle A^{-1}y,y\right\rangle-1\right) \tag{2.2}$$

and

$$\exp\left(1-\left\langle A^{-1}x,x\right\rangle\langle Bx,x\rangle\right)\leq\frac{\Delta_x(A)}{\Delta_x(B)}\leq\exp\left(\langle Ax,x\rangle\left\langle B^{-1}x,x\right\rangle-1\right). \tag{2.3}$$

Proof. In [1] we obtained the following result for two operators and a convex function:

Let I be an interval and $f:I\to\mathbb{R}$ be a convex and differentiable function on $\mathring{I}$ (the interior of I) whose derivative f' is continuous on $\mathring{I}$. If A and B are selfadjoint operators on the Hilbert space H with spectra $\operatorname{Sp}(A)$, $\operatorname{Sp}(B)\subset\mathring{I}$, then

$$\begin{aligned}\langle f'(A)x,x\rangle\langle By,y\rangle-\langle f'(A)Ax,x\rangle&\leq\langle f(B)y,y\rangle-\langle f(A)x,x\rangle\\&\leq\langle f'(B)By,y\rangle-\langle Ax,x\rangle\langle f'(B)y,y\rangle\end{aligned} \tag{2.4}$$

for any $x, y \in H$ with $\|x\| = \|y\| = 1$.

Let A, B two positive definite operators on H. By applying the above inequalities for the convex function $f(t) = -\ln t$, $t > 0$, then we have the inequalities

$$1 - \langle A^{-1}x, x\rangle \langle By, y\rangle \le \langle \ln Ax, x\rangle - \langle \ln By, y\rangle \le \langle Ax, x\rangle \langle B^{-1}y, y\rangle - 1 \tag{2.5}$$

for any $x, y \in H$ with $\|x\| = \|y\| = 1$.

By taking the exponential in (2.5) then we get

$$\exp\left(1 - \langle A^{-1}x, x\rangle \langle By, y\rangle\right) \le \frac{\exp\langle \ln Ax, x\rangle}{\exp\langle \ln By, y\rangle} \le \exp\left(\langle Ax, x\rangle \langle B^{-1}y, y\rangle - 1\right),$$

which gives (2.1). □

Corollary 1. *If A, $B > 0$ then for x, $y \in H$ with $\|x\| = \|y\| = 1$, we have the inequalities*

$$\exp\left(1 - \langle A^{-1}x, x\rangle \langle A^{-1}y, y\rangle\right) \le \Delta_x(A)\Delta_y(A) \le \exp\left(\langle Ax, x\rangle \langle Ay, y\rangle - 1\right) \tag{2.6}$$

and, in particular,

$$\exp\left[\frac{1}{2}\left(1 - \langle A^{-1}x, x\rangle^2\right)\right] \le \Delta_x(A) \le \exp\left[\frac{1}{2}\left(\langle Ax, x\rangle^2 - 1\right)\right]. \tag{2.7}$$

The proof follows by Theorem 1 by taking $B = A^{-1}$.

We also have:

Corollary 2. *If A, B, $C > 0$ then for x, $y \in H$ with $\|x\| = \|y\| = 1$, we have the inequalities*

$$\begin{aligned} &\exp\left(1 - \frac{1}{2}\langle By, y\rangle \left\langle \left(A^{-1} + C^{-1}\right)x, x\right\rangle\right) \\ &\le \frac{\int_0^1 \Delta_x((1-t)A + tC)dt}{\Delta_y(B)} \le \Phi(x, y, A, B, C) \end{aligned} \tag{2.8}$$

where

$$\Phi(x, y, A, B, C) := \begin{cases} \frac{\exp\left(\langle Cx, x\rangle \langle B^{-1}y, y\rangle - 1\right) - \exp\left(\langle Ax, x\rangle \langle B^{-1}y, y\rangle - 1\right)}{\left(\langle Cx, x\rangle - \langle Ax, x\rangle\right)\langle B^{-1}y, y\rangle} \\ \text{if } \langle Cx, x\rangle - \langle Ax, x\rangle \neq 0, \\ \\ \exp\left[\left(\langle Cx, x\rangle - \langle Ax, x\rangle\right)\langle B^{-1}y, y\rangle\right] \\ \text{if } \langle Cx, x\rangle - \langle Ax, x\rangle = 0 \end{cases}$$

and

$$\begin{aligned} &\exp\left(1 - \frac{1}{2}\langle Bx, x\rangle \left\langle \left(A^{-1} + C^{-1}\right)x, x\right\rangle\right) \\ &\le \frac{\int_0^1 \Delta_x((1-t)A + tC)dt}{\Delta_x(B)} \le \Phi(x, A, B, C), \end{aligned} \tag{2.9}$$

where

$$\Phi(x,A,B,C) := \begin{cases} \frac{\exp\left(\langle Cx,x\rangle\left\langle B^{-1}x,x\right\rangle-1\right)-\exp\left(\langle Ax,x\rangle\left\langle B^{-1}x,x\right\rangle-1\right)}{\left(\langle Cx,x\rangle-\langle Ax,x\rangle\right)\left\langle B^{-1}x,x\right\rangle} \\ \text{if } \langle Cx,x\rangle-\langle Ax,x\rangle\neq 0, \\ \\ \exp\left[\left(\langle Cx,x\rangle-\langle Ax,x\rangle\right)\left\langle B^{-1}x,x\right\rangle\right] \\ \text{if } \langle Cx,x\rangle-\langle Ax,x\rangle=0. \end{cases}$$

Proof. From (2.1) we get

$$\begin{aligned} &\exp\left(1-\left\langle\left((1-t)A+tC\right)^{-1}x,x\right\rangle\langle By,y\rangle\right) \\ &\leq\frac{\Delta_x((1-t)A+tC)}{\Delta_y(B)} \\ &\leq\exp\left(\langle\left((1-t)A+tC\right)x,x\rangle\left\langle B^{-1}y,y\right\rangle-1\right). \end{aligned}$$

for all $t\in[0,1]$ and $x,\,y\in H$ with $\|x\|=\|y\|=1$.

By taking the integral, we derive

$$\begin{aligned} &\int_0^1\exp\left(1-\left\langle\left((1-t)A+tC\right)^{-1}x,x\right\rangle\langle By,y\rangle\right)dt \\ &\leq\frac{\int_0^1\Delta_x((1-t)A+tC)dt}{\Delta_y(B)} \\ &\leq\int_0^1\exp\left(\langle\left((1-t)A+tC\right)x,x\rangle\left\langle B^{-1}y,y\right\rangle-1\right)dt \end{aligned}$$

for $x,\,y\in H$ with $\|x\|=\|y\|=1$.

We have

$$\begin{aligned} &\int_0^1\exp\left(\langle\left((1-t)A+tC\right)x,x\rangle\left\langle B^{-1}y,y\right\rangle-1\right)dt \\ &=\int_0^1\exp\left(\left(\left(\langle Cx,x\rangle-\langle Ax,x\rangle\right)t+\langle Ax,x\rangle\right)\left\langle B^{-1}y,y\right\rangle-1\right)dt \\ &=\int_0^1\exp\left(\left(\langle Cx,x\rangle-\langle Ax,x\rangle\right)\left\langle B^{-1}y,y\right\rangle t+\langle Ax,x\rangle\left\langle B^{-1}y,y\right\rangle-1\right)dt \\ &=\frac{\exp\left(\langle Cx,x\rangle\left\langle B^{-1}y,y\right\rangle-1\right)-\exp\left(\langle Ax,x\rangle\left\langle B^{-1}y,y\right\rangle-1\right)}{\left(\langle Cx,x\rangle-\langle Ax,x\rangle\right)\left\langle B^{-1}y,y\right\rangle} \end{aligned}$$

for $x,\,y\in H$ with $\|x\|=\|y\|=1$ and by Jensen's inequality for the exponential,

$$\begin{aligned} &\exp\left(1-\left\langle\left((1-t)A+tC\right)^{-1}x,x\right\rangle\langle By,y\rangle\right)dt \\ &\geq\exp\left(1-\langle By,y\rangle\left\langle\int_0^1\left((1-t)A+tC\right)^{-1}dtx,x\right\rangle\right) \end{aligned}$$

for $x,\,y\in H$ with $\|x\|=\|y\|=1$.

By Hermite-Hadamard inequality for the operator convex functions, we also have

$$\int_0^1\left((1-t)A+tC\right)^{-1}dt\leq\frac{1}{2}\left(A^{-1}+C^{-1}\right),$$

which gives

$$-\left\langle \int_0^1 ((1-t)A+tC)^{-1}\,dtx, x\right\rangle \geq -\left\langle \frac{1}{2}\left(A^{-1}+C^{-1}\right)x, x\right\rangle$$

for $x \in H$ with $\|x\| = 1$.

Therefore

$$\begin{aligned}&\exp\left(1-\langle By, y\rangle\left\langle \int_0^1 ((1-t)A+tC)^{-1}\,dtx, x\right\rangle\right)\\&\geq \exp\left(1-\frac{1}{2}\langle By, y\rangle\left\langle\left(A^{-1}+C^{-1}\right)x, x\right\rangle\right)\end{aligned}$$

for x, $y \in H$ with $\|x\| = \|y\| = 1$ and the inequality (2.8) is proved. □

Corollary 3. *If A, $C > 0$ then for $x \in H$ with $\|x\| = 1$, we have the inequalities*

$$\begin{aligned}&\exp\left(1-\frac{1}{4}\langle (A+C)x, x\rangle\left\langle\left(A^{-1}+C^{-1}\right)x, x\right\rangle\right)\\&\leq \frac{\int_0^1 \Delta_x((1-t)A+tC)dt}{\Delta_x(\frac{A+C}{2})} \leq \Psi(x, A, C),\end{aligned} \tag{2.10}$$

where

$$\Psi(x, A, BC) := \begin{cases}\dfrac{\exp\left(\langle Cx,x\rangle\left\langle\left(\frac{A+C}{2}\right)^{-1}x,x\right\rangle-1\right)-\exp\left(\langle Ax,x\rangle\left\langle\left(\frac{A+C}{2}\right)^{-1}x,x\right\rangle-1\right)}{(\langle Cx,x\rangle-\langle Ax,x\rangle)\left\langle\left(\frac{A+C}{2}\right)^{-1}x,x\right\rangle}\\ \text{if } \langle Cx, x\rangle - \langle Ax, x\rangle \neq 0,\\ \\ \exp\left[(\langle Cx, x\rangle - \langle Ax, x\rangle)\left\langle\left(\frac{A+C}{2}\right)^{-1}x, x\right\rangle\right]\\ \text{if } \langle Cx, x\rangle - \langle Ax, x\rangle = 0.\end{cases}$$

Further, we have:

Theorem 2. *If A, $B > 0$ then for x, $y \in H$ with $\|x\| = \|y\| = 1$, we have the inequalities*

$$\exp\left(1-\left\langle A^{-1}x, x\right\rangle\langle By, y\rangle\right) \leq \frac{\langle Ax, x\rangle}{\Delta_y(B)} \leq \exp\left(\langle Ax, x\rangle\left\langle B^{-1}y, y\right\rangle - 1\right). \tag{2.11}$$

In particular,

$$\exp\left(1-\left\langle A^{-1}x, x\right\rangle\langle Ay, y\rangle\right) \leq \frac{\langle Ax, x\rangle}{\Delta_y(A)} \leq \exp\left(\langle Ax, x\rangle\left\langle A^{-1}y, y\right\rangle - 1\right) \tag{2.12}$$

and

$$\exp\left(1-\left\langle A^{-1}x, x\right\rangle\langle Bx, x\rangle\right) \leq \frac{\langle Ax, x\rangle}{\Delta_x(B)} \leq \exp\left(\langle Ax, x\rangle\left\langle B^{-1}x, x\right\rangle - 1\right). \tag{2.13}$$

Proof. In [1] we obtained the following result for two operators and a convex function as well:

Let I be an interval and $f : I \to \mathbb{R}$ be a convex and differentiable function on $\mathring{I}$ (the interior of I) whose derivative f' is continuous on $\mathring{I}$. If A and B are selfadjoint operators on the Hilbert space H with $\operatorname{Sp}(A), \operatorname{Sp}(B) \subset \mathring{I}$, then

$$f'(\langle Ax,x\rangle)(\langle By,y\rangle - \langle Ax,x\rangle) \le \langle f(B)y,y\rangle - f(\langle Ax,x\rangle) \le \langle f'(B)By,y\rangle - \langle Ax,x\rangle \langle f'(B)y,y\rangle \tag{2.14}$$

for any $x, y \in H$ with $\|x\| = \|y\| = 1$.

If we write the inequality (2.14) for the convex function $f(t) = -\ln t$, $t > 0$, then we have the inequalities

$$1 - \langle Ax,x\rangle^{-1} \langle By,y\rangle \le \ln(\langle Ax,x\rangle) - \langle \ln By,y\rangle \le \langle Ax,x\rangle \left\langle B^{-1}y,y\right\rangle - 1 \tag{2.15}$$

for any $x, y \in H$ with $\|x\| = \|y\| = 1$.

If we take the exponential in (2.15), then we get

$$\exp\left(1 - \langle Ax,x\rangle^{-1}\langle By,y\rangle\right) \le \frac{\langle Ax,x\rangle}{\exp\langle \ln By,y\rangle} \le \exp\left[\langle Ax,x\rangle \left\langle B^{-1}y,y\right\rangle - 1\right]$$

for any $x, y \in H$ with $\|x\| = \|y\| = 1$. □

Remark 1. *If we take $B = A$ in (2.13) we get*

$$\exp\left(1 - \left\langle A^{-1}x,x\right\rangle \langle Ax,x\rangle\right) \le \frac{\langle Ax,x\rangle}{\Delta_x(A)} \le \exp\left(\langle Ax,x\rangle \left\langle A^{-1}x,x\right\rangle - 1\right). \tag{2.16}$$

for $x \in H$ with $\|x\| = 1$.

Since $\langle Ax,x\rangle \left\langle A^{-1}x,x\right\rangle - 1 > 0$ for $x \in H$ with $\|x\| = 1$, we observe that the first inequality in (2.16) is not as good as the second inequality in (ii) from the introduction.

By replacing A with A^{-1} and B with B^{-1} in Theorem 2 we can state:

Corollary 4. *If $A, B > 0$ then for $x, y \in H$ with $\|x\| = \|y\| = 1$, we have the inequalities*

$$\exp\left(1 - \langle Ax,x\rangle \left\langle B^{-1}y,y\right\rangle\right) \le \frac{\Delta_y(B)}{\langle A^{-1}x,x\rangle^{-1}} \le \exp\left(\left\langle A^{-1}x,x\right\rangle \langle By,y\rangle - 1\right). \tag{2.17}$$

In particular,

$$\exp\left(1 - \langle Ax,x\rangle \left\langle A^{-1}y,y\right\rangle\right) \le \frac{\Delta_y(A)}{\langle A^{-1}x,x\rangle^{-1}} \le \exp\left(\left\langle A^{-1}x,x\right\rangle \langle Ay,y\rangle - 1\right) \tag{2.18}$$

and

$$\exp\left(1 - \langle Ax,x\rangle \left\langle A^{-1}x,x\right\rangle\right) \le \frac{\Delta_x(A)}{\langle A^{-1}x,x\rangle^{-1}} \le \exp\left(\left\langle A^{-1}x,x\right\rangle \langle Ax,x\rangle - 1\right). \tag{2.19}$$

Since $\langle Ax,x\rangle \left\langle A^{-1}x,x\right\rangle - 1 > 0$ for $x \in H$ with $\|x\| = 1$, we observe that the first inequality in (2.16) is not as good as the first inequality in (ii) from the introduction.

3. Related Results

We also have:

Theorem 3. *Assume that the operators A, B satisfy the condition $0 < m \leq A$, $B \leq M$, then for $t \in [0,1]$*

$$\begin{aligned} &\left\langle \frac{A+B}{2}x, x\right\rangle \\ &\geq \sqrt{((1-t)\langle Ax,x\rangle + t\langle Bx,x\rangle)(t\langle Ax,x\rangle + (1-t)\langle Bx,x\rangle)} \\ &\geq \sqrt{\Delta_x((1-t)A+tB)\,\Delta_x((tA+(1-t)B))} \\ &\geq m^{\frac{M-\left\langle\frac{A+B}{2}x,x\right\rangle}{M-m}} M^{\frac{\left\langle\frac{A+B}{2}x,x\right\rangle - m}{M-m}}, \end{aligned} \tag{3.1}$$

for $x \in H$ with $\|x\| = 1$.

Proof. In [2] we obtained the following inequalities:

Let A and B selfadjoint operators on the Hilbert space H and assume that $\operatorname{Sp}(A), \operatorname{Sp}(B) \subseteq [m, M]$ for some scalars m, M with $m < M$. If f is a convex function on $[m, M]$, then

$$\begin{aligned} &f\left(\left\langle \frac{A+B}{2}x, x\right\rangle\right) \\ &\leq \frac{1}{2}\left[f((1-t)\langle Ax,x\rangle + t\langle Bx,x\rangle) + f(t\langle Ax,x\rangle + (1-t)\langle Bx,x\rangle)\right] \\ &\leq \left\langle \frac{1}{2}\left[f((1-t)A+tB) + f(tA+(1-t)B)\right]x, x\right\rangle \\ &\leq \frac{M-\left\langle\frac{A+B}{2}x,x\right\rangle}{M-m} f(m) + \frac{\left\langle\frac{A+B}{2}x,x\right\rangle - m}{M-m} f(M) \end{aligned} \tag{3.2}$$

for any $t \in [0,1]$ and each $x \in H$ with $\|x\| = 1$.

If we write the inequality (3.2) for the convex function $f(t) = -\ln t$, then we get for A, B with $0 < m \leq A$, $B \leq M$

$$\begin{aligned} &\ln\left(\left\langle \frac{A+B}{2}x, x\right\rangle\right) \\ &\geq \frac{1}{2}\left[\ln((1-t)\langle Ax,x\rangle + t\langle Bx,x\rangle) + \ln(t\langle Ax,x\rangle + (1-t)\langle Bx,x\rangle)\right] \\ &\geq \left\langle \frac{1}{2}\left[\ln((1-t)A+tB) + \ln(tA+(1-t)B)\right]x, x\right\rangle \\ &\geq \frac{M-\left\langle\frac{A+B}{2}x,x\right\rangle}{M-m}\ln(m) + \frac{\left\langle\frac{A+B}{2}x,x\right\rangle - m}{M-m}\ln(M), \end{aligned}$$

namely

$$\begin{aligned}
&\ln\left(\left\langle \frac{A+B}{2}x,x\right\rangle\right) \\
&\geq \ln\sqrt{\left(\left(1-t\right)\left\langle Ax,x\right\rangle + t\left\langle Bx,x\right\rangle\right)\left(t\left\langle Ax,x\right\rangle + \left(1-t\right)\left\langle Bx,x\right\rangle\right)} \\
&\geq \frac{1}{2}\left[\left\langle \ln\left(\left(1-t\right)A+tB\right)x,x\right\rangle + \left\langle \ln\left(tA+\left(1-t\right)B\right)x,x\right\rangle\right] \\
&\geq \ln\left(m^{\frac{M-\left\langle \frac{A+B}{2}x,x\right\rangle}{M-m}} M^{\frac{\left\langle \frac{A+B}{2}x,x\right\rangle - m}{M-m}}\right),
\end{aligned} \tag{3.3}$$

for any $t\in[0,1]$ and each $x\in H$ with $\|x\|=1$.

If we take the exponential in (3.3), we get

$$\begin{aligned}
&\left\langle \frac{A+B}{2}x,x\right\rangle \\
&\geq \sqrt{\left(\left(1-t\right)\left\langle Ax,x\right\rangle + t\left\langle Bx,x\right\rangle\right)\left(t\left\langle Ax,x\right\rangle + \left(1-t\right)\left\langle Bx,x\right\rangle\right)} \\
&\geq \sqrt{\exp\left\langle \ln\left(\left(1-t\right)A+tB\right)x,x\right\rangle \exp\left\langle \ln\left(tA+\left(1-t\right)B\right)x,x\right\rangle} \\
&\geq m^{\frac{M-\left\langle \frac{A+B}{2}x,x\right\rangle}{M-m}} M^{\frac{\left\langle \frac{A+B}{2}x,x\right\rangle - m}{M-m}},
\end{aligned}$$

which proves (3.1). □

Theorem 4. *Assume that A, $B>0$, then*

$$\begin{aligned}
\frac{\left\langle Ax,x\right\rangle + \left\langle By,y\right\rangle}{2} &\geq \exp\left(\int_0^1 \ln\left(\left(1-t\right)\left\langle Ax,x\right\rangle + t\left\langle By,y\right\rangle\right)dt\right) \\
&\geq \exp\left\langle \left[\int_0^1 \ln\left(\left(1-t\right)A + t\left\langle By,y\right\rangle I\right)dt\right]x,x\right\rangle \\
&\geq \sqrt{\Delta_x\left(A\right)\left\langle By,y\right\rangle} \geq \sqrt{\Delta_x\left(A\right)\Delta_y\left(B\right)}
\end{aligned} \tag{3.4}$$

and

$$\begin{aligned}
\frac{\left\langle Ax,x\right\rangle + \left\langle By,y\right\rangle}{2} &\geq \Delta_x\left(\frac{A+\left\langle By,y\right\rangle I}{2}\right) \\
&\geq \exp\left\langle \left[\int_0^1 \ln\left(\left(1-t\right)A + t\left\langle By,y\right\rangle I\right)dt\right]x,x\right\rangle
\end{aligned} \tag{3.5}$$

for each x, $y\in H$ with $\|x\|=\|y\|=1$.

Proof. Let A and B selfadjoint operators on the Hilbert space H and assume that $\operatorname{Sp}(A), \operatorname{Sp}(B)\subseteq \mathring{I}$. If f is a convex function on I, then

$$\begin{aligned}
f\left(\frac{\left\langle Ax,x\right\rangle + \left\langle By,y\right\rangle}{2}\right) &\leq \int_0^1 f\left(\left(1-t\right)\left\langle Ax,x\right\rangle + t\left\langle By,y\right\rangle\right)dt \\
&\leq \left\langle \left[\int_0^1 f\left(\left(1-t\right)A + t\left\langle By,y\right\rangle I\right)dt\right]x,x\right\rangle \\
&\leq \frac{1}{2}\left[\left\langle f\left(A\right)x,x\right\rangle + f\left(\left\langle By,y\right\rangle\right)\right] \\
&\leq \frac{1}{2}\left[\left\langle f\left(A\right)x,x\right\rangle + \left\langle f\left(B\right)y,y\right\rangle\right]
\end{aligned} \tag{3.6}$$

and

$$f\left(\frac{\langle Ax,x\rangle+\langle By,y\rangle}{2}\right)\leq\left\langle f\left(\frac{A+\langle By,y\rangle I}{2}\right)x,x\right\rangle \tag{3.7}$$
$$\leq\left\langle\left[\int_0^1 f\left((1-t)A+t\langle By,y\rangle I\right)dt\right]x,x\right\rangle$$

for each $x,y\in H$ with $\|x\|=\|y\|=1$.

If we write the inequality (3.6) for the convex function $f(t)=-\ln t$, we get

$$\ln\left(\frac{\langle Ax,x\rangle+\langle By,y\rangle}{2}\right)\geq\int_0^1\ln\left((1-t)\langle Ax,x\rangle+t\langle By,y\rangle\right)dt \tag{3.8}$$
$$\geq\left\langle\left[\int_0^1\ln\left((1-t)A+t\langle By,y\rangle I\right)dt\right]x,x\right\rangle$$
$$\geq\frac{1}{2}\left[\langle\ln Ax,x\rangle+\ln\langle By,y\rangle\right]$$
$$\geq\frac{1}{2}\left[\langle\ln Ax,x\rangle+\langle\ln By,y\rangle\right],$$

for each $x,y\in H$ with $\|x\|=\|y\|=1$.

If we take the exponential in (3.8), then we get

$$\frac{\langle Ax,x\rangle+\langle By,y\rangle}{2}\geq\exp\left(\int_0^1\ln\left((1-t)\langle Ax,x\rangle+t\langle By,y\rangle\right)dt\right) \tag{3.9}$$
$$\geq\exp\left\langle\left[\int_0^1\ln\left((1-t)A+t\langle By,y\rangle I\right)dt\right]x,x\right\rangle$$
$$\geq\exp\left(\frac{1}{2}\left[\langle\ln Ax,x\rangle+\ln\langle By,y\rangle\right]\right)$$
$$\geq\exp\left(\frac{1}{2}\left[\langle\ln Ax,x\rangle+\langle\ln By,y\rangle\right]\right),$$

and the inequality (3.4) is proved.

The inequality (3.5) follows by (3.7). □

Corollary 5. *Assume that* $A>0$, *then*

$$\langle Ax,x\rangle\geq\Delta_x\left(\frac{A+\langle Ax,x\rangle I}{2}\right) \tag{3.10}$$
$$\geq\exp\left\langle\left[\int_0^1\ln\left((1-t)A+t\langle Ax,x\rangle I\right)dt\right]x,x\right\rangle$$
$$\geq\sqrt{\langle Ax,x\rangle\Delta_x(A)}\geq\Delta_x(A),$$

for all $x\in H$ *with* $\|x\|=1$.

References

[1] S. S. Dragomir, Some inequalities for convex functions of selfadjoint operators in Hilbert spaces, *Filomat* **23**:3 (2009), 81–92.

[2] S. S. Dragomir, Hermite-Hadamard's type inequalities for convex functions of selfadjoint operators in Hilbert spaces, *Linear Algebra Appl.*, **436** (2012), no. 5, 1503–1515; Preprint *RGMIA Res. Rep. Coll.*, **13** (2010). Issue 2, Art. 1, 2010.

[3] J. I. Fujii and Y. Seo, Determinant for positive operators, *Sci. Math.*, **1** (1998), 153–156.

[4] J. I. Fujii, S. Izumino and Y. Seo, Determinant for positive operators and Specht's Theorem, *Sci. Math.*, **1** (1998), 307–310.
[5] T. Furuta, J. Mičić-Hot, J. Pečarić and Y. Seo, *Mond-Pečarić Method in Operator Inequalities*, Element, Croatia.
[6] S. Hiramatsu and Y. Seo, Determinant for positive operators and Oppenheim's inequality, *J. Math. Inequal.*, Volume 15 (2021), Number 4, 1637–1645.
[7] W. Specht, Zer Theorie der elementaren Mittel, *Math. Z.* **74** (1960), pp. 91-98.

[1]Mathematics, College of Engineering & Science, Victoria University, PO Box 14428, Melbourne City, MC 8001, Australia.
E-mail address: sever.dragomir@vu.edu.au
URL: http://rgmia.org/dragomir

[2]DST-NRF Centre of Excellence in the Mathematical, and Statistical Sciences, School of Computer Science, & Applied Mathematics, University of the Witwatersrand,, Private Bag 3, Johannesburg 2050, South Africa