A TRAPEZOID TYPE TENSORIAL NORM INEQUALITY FOR CONTINUOUS FUNCTIONS OF SELFADJOINT OPERATORS IN HILBERT SPACES

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Abstract. Let H be a Hilbert space. Assume that f is continuously differentiable on I with $\|f'\|_{I,\infty} := \sup_{t\in I}|f'(t)| < \infty$ and A, B are selfadjoint operators with $\operatorname{Sp}(A), \operatorname{Sp}(B) \subset I$, then

$$\left\|(1-\lambda) f(A) \otimes 1 + \lambda 1 \otimes f(B) - \int_0^1 f((1-u) A \otimes 1 + u 1 \otimes B) \, du\right\|$$
$$\leq \|f'\|_{I,\infty} \left[\frac{1}{4} + \left(\lambda - \frac{1}{2}\right)^2\right] \|1 \otimes B - A \otimes 1\|$$

for $\lambda \in [0,1]$. In particular, we have the midpoint inequality

$$\left\|\frac{f(A) \otimes 1 + 1 \otimes f(B)}{2} - \int_0^1 f((1-u) A \otimes 1 + u 1 \otimes B) \, du\right\|$$
$$\leq \frac{1}{4} \|f'\|_{I,\infty} \|1 \otimes B - A \otimes 1\|.$$

1. Introduction

Assume that the function $f : [a,b] \to \mathbb{R}$ is absolutely continuous on $[a,b]$, then we have the *generalized trapezoid inequality*, see for instance [4, p. 90]

(1.1)
$$\left|\frac{(b-x) f(b) + (x-a) f(a)}{b-a} - \frac{1}{b-a} \int_a^b f(t) \, dt\right|$$
$$\leq \left[\frac{1}{4} + \left.\frac{x - \frac{a+b}{2}}{b-a}\right)^2\right] \|f'\|_\infty (b-a),$$

for all $x \in [a,b]$ and the constant $\frac{1}{4}$ is the best possible.

For $x = \frac{a+b}{2}$ we get the *trapezoid inequality*

$$\left|\frac{f(b) + f(a)}{2} - \frac{1}{b-a} \int_a^b f(t) \, dt\right| \leq \frac{1}{4} \|f'\|_\infty (b-a),$$

with $\frac{1}{4}$ as best possible contstant.

In order to extend this result for tensorial products of selfadjoint operators and norms, we need the following preparations.

Let $I_1, ..., I_k$ be intervals from $\mathbb{R}$ and let $f : I_1 \times ... \times I_k \to \mathbb{R}$ be an essentially bounded real function defined on the product of the intervals. Let $A = (A_1, ..., A_n)$

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be a k-tuple of bounded selfadjoint operators on Hilbert spaces $H_1, ..., H_k$ such that the spectrum of A_i is contained in I_i for $i = 1, ..., k$. We say that such a k-tuple is in the domain of f. If

$$A_i = \int_{I_i} \lambda_i dE_i (\lambda_i)$$

is the spectral resolution of A_i for $i = 1, ..., k$; by following [2], we define

$$f (A_1, ..., A_k) := \int_{I_1} ... \int_{I_k} f (\lambda_1, ..., \lambda_1) dE_1 (\lambda_1) \otimes ... \otimes dE_k (\lambda_k) \tag{1.2}$$

as a bounded selfadjoint operator on the tensorial product $H_1 \otimes ... \otimes H_k$.

If the Hilbert spaces are of finite dimension, then the above integrals become finite sums, and we may consider the functional calculus for arbitrary real functions. This construction [2] extends the definition of Korányi [7] for functions of two variables and have the property that

$$f (A_1, ..., A_k) = f_1(A_1) \otimes ... \otimes f_k(A_k),$$

whenever f can be separated as a product $f(t_1, ..., t_k) = f_1(t_1)...f_k(t_k)$ of k functions each depending on only one variable.

It is know that, *if f is super-multiplicative (sub-multiplicative)* on $[0, \infty)$, namely

$$f (st) \geq (\leq) f (s) f (t) \text{ for all } s, t \in [0, \infty)$$

and if f is continuous on $[0, \infty)$, then [10, p. 173]

$$f (A \otimes B) \geq (\leq) f (A) \otimes f (B) \text{ for all } A,\ B \geq 0. \tag{1.3}$$

This follows by observing that, if

$$A = \int_{[0,\infty)} tdE (t) \text{ and } B = \int_{[0,\infty)} sdF (s)$$

are the spectral resolutions of A and B, then

$$f (A \otimes B) = \int_{[0,\infty)} \int_{[0,\infty)} f (st) dE (t) \otimes dF (s) \tag{1.4}$$

for the continuous function f on $[0, \infty)$.

Recall the *geometric operator mean* for the positive operators $A,\ B > 0$

$$A\#_t B := A^{1/2}(A^{-1/2}BA^{-1/2})^t A^{1/2},$$

where $t \in [0, 1]$ and

$$A\#B := A^{1/2}(A^{-1/2}BA^{-1/2})^{1/2}A^{1/2}.$$

By the definitions of $\#$ and $\otimes$ we have

$$A\#B = B\#A \text{ and } (A\#B) \otimes (B\#A) = (A \otimes B) \# (B \otimes A).$$

In 2007, S. Wada [14] obtained the following *Callebaut type inequalities* for tensorial product

$$\begin{aligned} (A\#B) \otimes (A\#B) &\leq \frac{1}{2} [(A\#_\alpha B) \otimes (A\#_{1-\alpha} B) + (A\#_{1-\alpha} B) \otimes (A\#_\alpha B)] \\ &\leq \frac{1}{2} (A \otimes B + B \otimes A) \end{aligned} \tag{1.5}$$

for $A,\ B > 0$ and $\alpha \in [0, 1]$. For other similar results, see [1], [3] and [8]-[11].

Motivated by the above results, if f is continuously differentiable on I with $\|f'\|_{I,\infty} := \sup_{t\in I} |f'(t)| < \infty$ and A, B are selfadjoint operators with $\operatorname{Sp}(A)$, $\operatorname{Sp}(B) \subset I$, then

$$\left\|(1-\lambda) f(A)\otimes 1 + \lambda 1\otimes f(B) - \int_0^1 f((1-u)A\otimes 1 + u1\otimes B)\,du\right\|$$
$$\le \|f'\|_{I,\infty}\left[\frac{1}{4}+\left(\lambda-\frac{1}{2}\right)^2\right]\|1\otimes B - A\otimes 1\|$$

for $\lambda \in [0,1]$. In particular, we have the midpoint inequality

$$\left\|\frac{f(A)\otimes 1 + 1\otimes f(B)}{2} - \int_0^1 f((1-u)A\otimes 1 + u1\otimes B)\,du\right\|$$
$$\le \frac{1}{4}\|f'\|_{I,\infty}\|1\otimes B - A\otimes 1\|.$$

2. Main Results

Recall the following property of the tensorial product

$$(AC)\otimes(BD) = (A\otimes B)(C\otimes D) \tag{2.1}$$

that holds for any $A, B, C, D \in B(H)$.

If we take $C = A$ and $D = B$, then we get

$$A^2\otimes B^2 = (A\otimes B)^2.$$

By induction and using (2.1) we derive that

$$A^n\otimes B^n = (A\otimes B)^n \text{ for natural } n\ge 0. \tag{2.2}$$

In particular

$$A^n\otimes 1 = (A\otimes 1)^n \text{ and } 1\otimes B^n = (1\otimes B)^n \tag{2.3}$$

for all $n \ge 0$.

We also observe that, by (2.1), the operators $A\otimes 1$ and $1\otimes B$ are commutative and

$$(A\otimes 1)(1\otimes B) = (1\otimes B)(A\otimes 1) = A\otimes B. \tag{2.4}$$

Moreover, for two natural numbers m, n we have

$$(A\otimes 1)^m (1\otimes B)^n = (1\otimes B)^n (A\otimes 1)^m = A^m\otimes B^n. \tag{2.5}$$

We have the following representation results for continuous functions:

Lemma 1. *Assume A and B are selfadjoint operators with $\operatorname{Sp}(A)\subset I$ and $\operatorname{Sp}(B)\subset J$. Let f, h be continuous on I, g, k continuous on J and φ continuous on an interval K that contains the sum of the intervals $h(I)+k(J)$, then*

$$(f(A)\otimes 1 + 1\otimes g(B))\,\varphi(h(A)\otimes 1 + 1\otimes k(B))$$
$$= \int_I\int_J (f(t)+g(s))\,\varphi(h(t)+k(s))\,dE_t\otimes dF_s, \tag{2.6}$$

where A and B have the spectral resolutions

$$A = \int_I t\,dE(t) \text{ and } B = \int_J s\,dF(s). \tag{2.7}$$

Proof. By Stone-Weierstrass, any continuous function can be approximated by a sequence of polynomials, therefore it suffices to prove the equality for the power function $\varphi(t) = t^n$ with n any natural number.

For natural number $n \geq 1$ we have

$$\begin{aligned}
\mathcal{K} &:= \int_I \int_J (f(t) + g(s))(h(t) + k(s))^n \, dE_t \otimes dF_s \\
&= \int_I \int_J (f(t) + g(s)) \sum_{m=0}^{n} C_n^m [h(t)]^m [k(s)]^{n-m} \, dE_t \otimes dF_s \\
&= \sum_{m=0}^{n} C_n^m \int_I \int_J (f(t) + g(s)) [h(t)]^m [k(s)]^{n-m} \, dE_t \otimes dF_s \\
&= \sum_{m=0}^{n} C_n^m \left[\int_I \int_J f(t) [h(t)]^m [k(s)]^{n-m} \, dE_t \otimes dF_s \right. \\
&\quad \left. + \int_I \int_J [h(t)]^m g(s) [k(s)]^{n-m} \, dE_t \otimes dF_s \right].
\end{aligned} \tag{2.8}$$

Observe that

$$\begin{aligned}
&\int_I \int_J f(t) [h(t)]^m [k(s)]^{n-m} \, dE_t \otimes dF_s \\
&= f(A) [h(A)]^m \otimes [k(B)]^{n-m} = (f(A) \otimes 1) \left([h(A)]^m \otimes [k(B)]^{n-m} \right) \\
&= (f(A) \otimes 1) ([h(A)]^m \otimes 1) \left(1 \otimes [k(B)]^{n-m} \right) \\
&= (f(A) \otimes 1) (h(A) \otimes 1)^m (1 \otimes k(B))^{n-m}
\end{aligned}$$

and

$$\begin{aligned}
&\int_I \int_J [h(t)]^m g(s) [k(s)]^{n-m} \, dE_t \otimes dF_s \\
&= [h(A)]^m \otimes \left(g(B) [k(B)]^{n-m} \right) = (1 \otimes g(B)) \left([h(A)]^m \otimes [k(B)]^{n-m} \right) \\
&= (1 \otimes g(B)) ([h(A)]^m \otimes 1) \left(1 \otimes [k(B)]^{n-m} \right) \\
&= (1 \otimes g(B)) (h(A) \otimes 1)^m (1 \otimes k(B))^{n-m},
\end{aligned}$$

with $h(A) \otimes 1$ and $1 \otimes k(B)$ commutative.

Therefore

$$\begin{aligned}
\mathcal{K} &= (f(A) \otimes 1 + 1 \otimes g(B)) \sum_{m=0}^{n} C_n^m (h(A) \otimes 1)^m (1 \otimes k(B))^{n-m} \\
&= (f(A) \otimes 1 + 1 \otimes g(B)) (h(A) \otimes 1 + 1 \otimes k(B))^n,
\end{aligned}$$

for which the commutativity of $h(A) \otimes 1$ and $1 \otimes k(B)$ has been employed. □

We have the following representation result:

Theorem 1. *Assume that f is continuously differentiable on I, A and B are selfadjoint operators with* $\operatorname{Sp}(A), \operatorname{Sp}(B) \subset I$, *then*

$$(2.9) \qquad (1-\lambda) 1 \otimes f(B) + \lambda f(A) \otimes 1 - \int_0^1 f((1-u) A \otimes 1 + u 1 \otimes B) du$$
$$= (1 \otimes B - A \otimes 1) \int_0^1 (u-\lambda) f'((1-u) A \otimes 1 + u 1 \otimes B) du$$

for all $\lambda \in [0,1]$.

In particular, we have the trapezoid identity

$$(2.10) \qquad \frac{1 \otimes f(B) + f(A) \otimes 1}{2} - \int_0^1 f((1-u) A \otimes 1 + u 1 \otimes B) du$$
$$= (1 \otimes B - A \otimes 1) \int_0^1 \left(u - \frac{1}{2}\right) f'((1-u) A \otimes 1 + u 1 \otimes B) du$$

Proof. Integrating by parts in the Lebesgue integral, we have

$$(2.11) \qquad \int_a^b (t-x) f'(t) dt = (t-x) f(t)|_a^b - \int_a^b f(t) dt$$
$$= (b-x) f(b) + (x-a) f(a) - \int_a^b f(t) dt$$

for $a \leq x \leq b$ and f absolutely continuous on $[a,b]$.

If we take $x = (1-\lambda) a + \lambda b$, $\lambda \in [0,1]$ and change the variable $t = (1-u) a + ub$, then $dt = (b-a) du$ and by (2.11) we derive

$$(1-\lambda)(b-a) f(b) + \lambda (b-a) f(a) - (b-a) \int_0^1 f((1-u) a + ub) du$$
$$= (b-a)^2 \int_a^b (u-\lambda) f'((1-u) a + ub) du,$$

namely

$$(2.12) \qquad (1-\lambda) f(b) + \lambda f(a) - \int_0^1 f((1-u) a + ub) du$$
$$= (b-a) \int_0^1 (u-\lambda) f'((1-u) a + ub) du,$$

for all a, $b \in I$ and $\lambda \in [0,1]$.

Assume that A and B have the spectral resolutions

$$A = \int_I t dE(t) \text{ and } B = \int_I s dF(s).$$

If we take the integral $\int_I \int_I$ over $dE_t \otimes dF_s$ in (2.12), then we get

$$(2.13) \qquad \int_I \int_I \left[(1-\lambda) f(s) + \lambda f(t) - \int_0^1 f((1-u) t + us) du\right] dE_t \otimes dF_s$$
$$= \int_I \int_I \left[(s-t) \int_0^1 (u-\lambda) f'((1-u) t + us)\right] dE_t \otimes dF_s.$$

By utilizing Fubini's theorem and Lemma 1 we derive

$$\begin{aligned}(2.14)\quad & \int_I \int_I \left[(1-\lambda) f(s) + \lambda f(t) - \int_0^1 f((1-u)t + us)\, du \right] dE_t \otimes dF_s \\ &= (1-\lambda) \int_I \int_I f(s)\, dE_t \otimes dF_s + \lambda \int_I \int_I f(t)\, dE_t \otimes dF_s \\ &- \int_0^1 \left(\int_I \int_I (f((1-u)t + us))\, dE_t \otimes dF_s \right) du \\ &= (1-\lambda) 1 \otimes f(B) + \lambda f(A) \otimes 1 - \int_0^1 f((1-u) A \otimes 1 + u 1 \otimes B)\, du\end{aligned}$$

and

$$\begin{aligned}(2.15)\quad & \int_I \int_I \left[(s-t) \int_0^1 (u-\lambda) f'((1-u)t + us)\, du \right] dE_t \otimes dF_s \\ &= \int_0^1 (u-\lambda) \left[\int_I \int_I (s-t) f'((1-u)t + us)\, dE_t \otimes dF_s \right] du \\ &= \int_0^1 (u-\lambda)(1 \otimes B - A \otimes 1) f'((1-u) A \otimes 1 + u 1 \otimes B)\, du \\ &= (1 \otimes B - A \otimes 1) \int_0^1 (u-\lambda) f'((1-u) A \otimes 1 + u 1 \otimes B)\, du.\end{aligned}$$

Therefore, by (2.13)-(2.15) we get the desired identity (2.9). □

We have the following generalized trapezoid inequality:

Theorem 2. *Assume that f is continuously differentiable on I with $\|f'\|_{I,\infty} := \sup_{t \in I} |f'(t)| < \infty$ and A, B are selfadjoint operators with $\operatorname{Sp}(A)$, $\operatorname{Sp}(B) \subset I$, then*

$$\begin{aligned}(2.16)\quad & \left\| (1-\lambda) 1 \otimes f(B) + \lambda f(A) \otimes 1 - \int_0^1 f((1-u) A \otimes 1 + u 1 \otimes B)\, du \right\| \\ &\leq \|1 \otimes B - A \otimes 1\| \left[\frac{1}{4} + \left(\lambda - \frac{1}{2} \right)^2 \right] \|f'\|_{I,\infty}\end{aligned}$$

for all $\lambda \in [0,1]$.

In particular, we have the trapezoid inequality

$$\begin{aligned}(2.17)\quad & \left\| \frac{1 \otimes f(B) + f(A) \otimes 1}{2} - \int_0^1 f((1-u) A \otimes 1 + u 1 \otimes B)\, du \right\| \\ &\leq \frac{1}{4} \|f'\|_{I,\infty} \|1 \otimes B - A \otimes 1\|.\end{aligned}$$

Proof. If we take the norm in the identity (2.9) and use the properties of the integral, then we get

$$\begin{aligned}(2.18)\quad &\left\|(1-\lambda)\,1\otimes f(B)+\lambda f(A)\otimes 1-\int_0^1 f((1-u)\,A\otimes 1+u1\otimes B)\,du\right\| \\ &=\left\|(1\otimes B-A\otimes 1)\int_0^1 (u-\lambda)\,f'((1-u)\,A\otimes 1+u1\otimes B)\,du\right\| \\ &\le \|1\otimes B-A\otimes 1\|\left\|\int_0^1 (u-\lambda)\,f'((1-u)\,A\otimes 1+u1\otimes B)\,du\right\| \\ &\le \|1\otimes B-A\otimes 1\|\int_0^1 |u-\lambda|\,\|f'((1-u)\,A\otimes 1+u1\otimes B)\|\,du\end{aligned}$$

for all $\lambda\in[0,1]$.

Observe that, by Lemma 1

$$|f'((1-u)\,A\otimes 1+u1\otimes B)|=\int_I\int_I |f'((1-u)\,A\otimes 1+u1\otimes B)|\,dE_t\otimes dF_s$$

for $u,\lambda\in[0,1]$.

Since

$$|f'((1-u)\,A\otimes 1+u1\otimes B)|\le \|f'\|_{I,\infty}$$

for $u,\lambda\in[0,1]$ and $t,s\in I$.

If we take the integral $\int_I\int_I$ over $dE_t\otimes dF_s$, then we get

$$\begin{aligned}(2.19)\quad &|f'((1-u)\,A\otimes 1+u1\otimes B)| \\ &=\int_I\int_I |f'((1-u)\,A\otimes 1+u1\otimes B)|\,dE_t\otimes dF_s\le \|f'\|_{I,\infty}\int_I\int_I dE_t\otimes dF_s \\ &=\|f'\|_{I,\infty}\end{aligned}$$

for $u,\lambda\in[0,1]$. This implies that

$$\|f'((1-u)\,A\otimes 1+u1\otimes B)\|\le \|f'\|_{I,\infty}$$

for $u,\lambda\in[0,1]$ which gives

$$\begin{aligned}&\int_0^1 |u-\lambda|\,\|f'((1-u)\,A\otimes 1+u1\otimes B)\|\,du \\ &\le \|f'\|_{I,\infty}\int_0^1 |u-\lambda|\,du=\|f'\|_{I,\infty}\frac{(1-\lambda)^2+\lambda^2}{2} \\ &=\|1\otimes B-A\otimes 1\|\left[\frac{1}{4}+\left(\lambda-\frac{1}{2}\right)^2\right]\|f'\|_{I,\infty},\end{aligned}$$

which proves (2.16). □

3. Related Results

We start by the following result:

Theorem 3. *Assume that f is continuously differentiable on I with $|f'|$ is convex on I, A and B are selfadjoint operators with* $\operatorname{Sp}(A), \operatorname{Sp}(B) \subset I$, *then*

$$\left\| (1-\lambda) 1 \otimes f(B) + \lambda f(A) \otimes 1 - \int_0^1 f((1-u) A \otimes 1 + u 1 \otimes B) du \right\| \tag{3.1}$$
$$\leq \|1 \otimes B - A \otimes 1\| [p(1-\lambda) \|f'(A)\| + p(\lambda) \|f'(B)\|],$$

for $\lambda \in [0,1]$, *where*

$$q(\lambda) := \frac{1}{6}\left(2\lambda^3 - 3\lambda + 2\right).$$

In particular, we have the trapezoid inequality

$$\left\| \frac{1 \otimes f(B) + f(A) \otimes 1}{2} - \int_0^1 f((1-u) A \otimes 1 + u 1 \otimes B) du \right\| \tag{3.2}$$
$$\leq \frac{1}{8} \|1 \otimes B - A \otimes 1\| (\|f'(A)\| + \|f'(B)\|).$$

Proof. Since $|f'|$ is convex on I, then

$$|f'((1-u)t + us)| \leq (1-u)|f'(t)| + u|f'(s)|$$

for all $t, s \in I$ and $u \in [0,1]$.

If we take the integral $\int_I \int_I$ over $dE_t \otimes dF_s$, then we get

$$\begin{aligned}
&\int_I \int_I |f'((1-u)t + us)| dE_t \otimes dF_s \\
&\leq \int_I \int_I [(1-u)|f'(t)| + u|f'(s)|] dE_t \otimes dF_s \\
&= (1-u) \int_I \int_I |f'(t)| dE_t \otimes dF_s + u \int_I \int_I |f'(s)| dE_t \otimes dF_s,
\end{aligned}$$

namely

$$|f'((1-u) A \otimes 1 + u 1 \otimes B)| \leq (1-u)|f'(A)| \otimes 1 + u|f'(B)| \otimes 1 \tag{3.3}$$

for all $u \in [0,1]$.

If we take the norm in (3.3), then we get

$$\begin{aligned}
\|f'((1-u) A \otimes 1 + u 1 \otimes B)\| &\leq \|(1-u)|f'(A)| \otimes 1 + u|f'(B)| \otimes 1\| \\
&\leq (1-u)\||f'(A)| \otimes 1\| + u\||f'(B)| \otimes 1\| \\
&= (1-u)\|f'(A)\| + u\|f'(B)\|
\end{aligned} \tag{3.4}$$

for all $u \in [0,1]$.

By (2.18) and (3.4) we derive

$$\begin{aligned}
&\left\| (1-\lambda) 1 \otimes f(B) + \lambda f(A) \otimes 1 - \int_0^1 f((1-u) A \otimes 1 + u 1 \otimes B) du \right\| \\
&\leq \|1 \otimes B - A \otimes 1\| \int_0^1 |u-\lambda| \|f'((1-u) A \otimes 1 + u 1 \otimes B)\| du \\
&\leq \|1 \otimes B - A \otimes 1\| \int_0^1 |u-\lambda| [(1-u)\|f'(A)\| + u\|f'(B)\|] du \\
&= \|1 \otimes B - A \otimes 1\| \\
&\times \left[\|f'(A)\| \int_0^1 |u-\lambda|(1-u) du + \|f'(B)\| \int_0^1 u|u-\lambda| du\right],
\end{aligned} \tag{3.5}$$

for $\lambda \in [0,1]$.

Observe that

$$\int_0^1 u\left|u-\lambda\right| du = \frac{1}{6}\left(2\lambda^3 - 3\lambda + 2\right) = q\left(\lambda\right)$$

and

$$\int_0^1 \left(1-u\right)\left|u-\lambda\right| du = p\left(1-\lambda\right)$$

for $\lambda \in [0,1]$.

By utilising (3.5) we derive (3.1). □

We recall that the function $g : I \to \mathbb{R}$ is *quasi-convex*, if

$$g\left(\left(1-\lambda\right)t + \lambda s\right) \le \max\left\{g\left(t\right), g\left(s\right)\right\} = \frac{1}{2}\left(g\left(t\right) + g\left(s\right) + \left|g\left(t\right) - g\left(s\right)\right|\right)$$

for all $t, s \in I$ and $\lambda \in [0,1]$.

Theorem 4. *Assume that f is continuously differentiable on I with $|f'|$ is quasi-convex on I, A and B are selfadjoint operators with* $\operatorname{Sp}(A), \operatorname{Sp}(B) \subset I$, *then*

$$\begin{aligned}(3.6)\quad & \left\|\left(1-\lambda\right)1\otimes f\left(B\right) + \lambda f\left(A\right)\otimes 1 - \int_0^1 f\left(\left(1-u\right)A\otimes 1 + u1\otimes B\right)du\right\| \\ & \le \frac{1}{2}\left\|1\otimes B - A\otimes 1\right\|\left[\frac{1}{4} + \left(\lambda - \frac{1}{2}\right)^2\right] \\ & \times \left(\left\|\left|f'\left(A\right)\right|\otimes 1 + 1\otimes\left|f'\left(B\right)\right|\right\| + \left\|\left|f'\left(A\right)\right|\otimes 1 - 1\otimes\left|f'\left(B\right)\right|\right\|\right)\end{aligned}$$

In particular,

$$\begin{aligned}(3.7)\quad & \left\|\frac{1\otimes f\left(B\right) + f\left(A\right)\otimes 1}{2} - \int_0^1 f\left(\left(1-u\right)A\otimes 1 + u1\otimes B\right)du\right\| \\ & \le \frac{1}{8}\left\|1\otimes B - A\otimes 1\right\| \\ & \times \left(\left\|\left|f'\left(A\right)\right|\otimes 1 + 1\otimes\left|f'\left(B\right)\right|\right\| + \left\|\left|f'\left(A\right)\right|\otimes 1 - 1\otimes\left|f'\left(B\right)\right|\right\|\right).\end{aligned}$$

Proof. Since $|f'|$ is quasi-convex on I, then we get

$$\left|f'\left(\left(1-u\right)t + us\right)\right| \le \frac{1}{2}\left(\left|f'\left(t\right)\right| + \left|f'\left(s\right)\right| + \left|\left|f'\left(t\right)\right| - \left|f'\left(s\right)\right|\right|\right)$$

for all for $u \in [0,1]$ and $t, s \in I$.

If we take the integral $\int_I \int_I$ over $dE_t \otimes dF_s$, then we get

$$\begin{aligned}& \int_I \int_I \left|f'\left(\left(1-u\right)t + us\right)\right| dE_t \otimes dF_s \\ & \le \frac{1}{2}\int_I \int_I \left(\left|f'\left(t\right)\right| + \left|f'\left(s\right)\right| + \left|\left|f'\left(t\right)\right| - \left|f'\left(s\right)\right|\right|\right) dE_t \otimes dF_s\end{aligned}$$

namely

$$\begin{aligned}& \left|f'\left(\left(1-u\right)A\otimes 1 + u1\otimes B\right)\right| \\ & \le \frac{1}{2}\left(\left|f'\left(A\right)\right|\otimes 1 + 1\otimes\left|f'\left(B\right)\right| + \left|\left|f'\left(A\right)\right|\otimes 1 - 1\otimes\left|f'\left(B\right)\right|\right|\right)\end{aligned}$$

for all for $u \in [0,1]$.

If we take the norm, then we get

$$\begin{aligned}(3.8)\qquad & \|f'((1-u)A\otimes 1+u1\otimes B)\| \\ & \leq \frac{1}{2}\|(|f'(A)|\otimes 1+1\otimes |f'(B)|+||f'(A)|\otimes 1-1\otimes |f'(B)||)\| \\ & \leq \frac{1}{2}(\||f'(A)|\otimes 1+1\otimes |f'(B)|\|+\||f'(A)|\otimes 1-1\otimes |f'(B)|\|)\end{aligned}$$

for all for $u\in[0,1]$.

By (2.18) and (3.8)

$$\begin{aligned}& \left\|(1-\lambda)1\otimes f(B)+\lambda f(A)\otimes 1-\int_0^1 f((1-u)A\otimes 1+u1\otimes B)\,du\right\| \\ & \leq \|1\otimes B-A\otimes 1\|\int_0^1 |u-\lambda|\,\|f'((1-u)A\otimes 1+u1\otimes B)\|\,du \\ & \leq \frac{1}{2}\|1\otimes B-A\otimes 1\| \\ & \times\int_0^1 |u-\lambda|(\||f'(A)|\otimes 1+1\otimes |f'(B)|\|+\||f'(A)|\otimes 1-1\otimes |f'(B)|\|) \\ & =\frac{1}{2}\|1\otimes B-A\otimes 1\|\left[\frac{1}{4}+\left(\lambda-\frac{1}{2}\right)^2\right] \\ & \times(\||f'(A)|\otimes 1+1\otimes |f'(B)|\|+\||f'(A)|\otimes 1-1\otimes |f'(B)|\|)\end{aligned}$$

for all $\lambda\in[0,1]$ and the inequality (3.6) is proved. □

4. Examples

It is known that if U and V are commuting, i.e. $UV=VU$, then the exponential function satisfies the property

$$\exp(U)\exp(V)=\exp(V)\exp(U)=\exp(U+V).$$

Also, if U is invertible and $a,b\in\mathbb{R}$ with $a<b$ then

$$\int_a^b \exp(tU)\,dt=U^{-1}[\exp(bU)-\exp(aU)].$$

Moreover, if U and V are commuting and $V-U$ is invertible, then

$$\begin{aligned}\int_0^1 \exp((1-s)U+sV)\,ds &= \int_0^1 \exp(s(V-U))\exp(U)\,ds \\ &= \left(\int_0^1 \exp(s(V-U))\,ds\right)\exp(U) \\ &= (V-U)^{-1}[\exp(V-U)-I]\exp(U) \\ &= (V-U)^{-1}[\exp(V)-\exp(U)].\end{aligned}$$

Since the operators $U=A\otimes 1$ and $V=1\otimes B$ are commutative and if $1\otimes B-A\otimes 1$ is invertible, then

$$\begin{aligned}&\int_0^1 \exp((1-u)A\otimes 1+u1\otimes B)\,du \\ &=(1\otimes B-A\otimes 1)^{-1}[\exp(1\otimes B)-\exp(A\otimes 1)].\end{aligned}$$

If A, B are selfadjoint operators with $\operatorname{Sp}(A), \operatorname{Sp}(B) \subset [m, M]$ and $1 \otimes B - A \otimes 1$ is invertible, then by (2.16)

$$\begin{aligned}(4.1)\quad & \|(1-\lambda)\exp(A)\otimes 1 + \lambda 1\otimes \exp(B) \\ & - (1\otimes B - A\otimes 1)^{-1}[\exp(1\otimes B) - \exp(A\otimes 1)]\| \\ & \leq \exp(M)\left[\frac{1}{4} + \left(\lambda - \frac{1}{2}\right)^2\right]\|1\otimes B - A\otimes 1\|,\end{aligned}$$

for $\lambda \in [0,1]$.

In particular,

$$\begin{aligned}(4.2)\quad & \left\|\frac{\exp(A)\otimes 1 + 1\otimes \exp B}{2}\right. \\ & \left. - (1\otimes B - A\otimes 1)^{-1}[\exp(1\otimes B) - \exp(A\otimes 1)]\right\| \\ & \leq \frac{1}{4}\exp(M)\|1\otimes B - A\otimes 1\|.\end{aligned}$$

Since for $f(t) = \exp t$, $t \in \mathbb{R}$, $|f'|$ is convex, then by Theorem 3 we get

$$\begin{aligned}(4.3)\quad & \|(1-\lambda)\exp(A)\otimes 1 + \lambda 1\otimes \exp(B) \\ & - (1\otimes B - A\otimes 1)^{-1}[\exp(1\otimes B) - \exp(A\otimes 1)]\| \\ & \leq \frac{1}{2}\left[\frac{1}{4} + \left(\lambda - \frac{1}{2}\right)^2\right]\|1\otimes B - A\otimes 1\| \\ & \times (\|\exp(A)\otimes 1 + 1\otimes \exp(B)\| + \|\exp(A)\otimes 1 - 1\otimes \exp(B)\|)\end{aligned}$$

for $\lambda \in [0,1]$.

In particular,

$$\begin{aligned}(4.4)\quad & \left\|\frac{\exp(A)\otimes 1 + 1\otimes \exp B}{2}\right. \\ & \left. - (1\otimes B - A\otimes 1)^{-1}[\exp(1\otimes B) - \exp(A\otimes 1)]\right\| \\ & \leq \frac{1}{8}\|1\otimes B - A\otimes 1\| \\ & \times (\|\exp(A)\otimes 1 + 1\otimes \exp(B)\| + \|\exp(A)\otimes 1 - 1\otimes \exp(B)\|)\end{aligned}$$

provided that $1 \otimes B - A \otimes 1$ is invertible.

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