TENSORIAL UPPER AND LOWER BOUNDS FOR TAYLOR'S EXPANSION OF FUNCTIONS OF SELFADJOINT OPERATORS IN HILBERT SPACES

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Abstract. Let H be a Hilbert space. In this paper we show among others that, if f is of class C^{2m} on the open interval I and such that $\gamma_{2m} \leq f^{(2m)} \leq \Gamma_{2m}$ for some constants γ_{2m}, Γ_{2m}, A and B are selfadjoint operators with $\operatorname{Sp}(A), \operatorname{Sp}(B) \subset I$, then we have the inequalities

$$\begin{aligned}
&\frac{1}{(2m)!}(A \otimes 1 - 1 \otimes B)^{2m} \gamma_{2m} \\
&\leq f(A) \otimes 1 - \sum_{k=0}^{2m-1} \frac{1}{k!}(A \otimes 1 - 1 \otimes B)^{k} \left(1 \otimes f^{(k)}(B)\right) \\
&\leq \frac{1}{(2m)!}(A \otimes 1 - 1 \otimes B)^{2m} \Gamma_{2m}.
\end{aligned}$$

Some examples for logarithm and exponential functions are also provided.

1. Introduction

The following theorem is well known in the literature as Taylor's formula or Taylor's theorem with the integral remainder.

Theorem 1. *Let $I \subset \mathbb{R}$ be a closed interval, $c \in I$ and let n be a positive integer. If $f : I \longrightarrow \mathbb{C}$ is such that the n-derivative $f^{(n)}$ is absolutely continuous on I, then for each $y \in I$*

$$f(y) = T_n(f; c, y) + R_n(f; c, y), \tag{1.1}$$

where $T_n(f; c, y)$ is Taylor's polynomial, i.e.,

$$T_n(f; c, y) := \sum_{k=0}^{n} \frac{(y-c)^k}{k!} f^{(k)}(c). \tag{1.2}$$

Note that $f^{(0)} := f$ and $0! := 1$ and the remainder is given by

$$R_n(f; c, y) := \frac{1}{n!} \int_c^y (y-t)^n f^{(n+1)}(t)\, dt. \tag{1.3}$$

A simple proof of this theorem can be achieved by mathematical induction using the integration by parts formula in the Lebesgue integral.

In order to extend this result for tensorial products of selfadjoint operators and norms, we need the following preparations.

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Let $I_1, ..., I_k$ be intervals from $\mathbb{R}$ and let $f : I_1 \times ... \times I_k \to \mathbb{R}$ be an essentially bounded real function defined on the product of the intervals. Let $A = (A_1, ..., A_n)$ be a k-tuple of bounded selfadjoint operators on Hilbert spaces $H_1, ..., H_k$ such that the spectrum of A_i is contained in I_i for $i = 1, ..., k$. We say that such a k-tuple is in the domain of f. If

$$A_i = \int_{I_i} \lambda_i dE_i (\lambda_i)$$

is the spectral resolution of A_i for $i = 1, ..., k$; by following [2], we define

$$f (A_1, ..., A_k) := \int_{I_1} ... \int_{I_k} f (\lambda_1, ..., \lambda_1) dE_1 (\lambda_1) \otimes ... \otimes dE_k (\lambda_k) \tag{1.4}$$

as a bounded selfadjoint operator on the tensorial product $H_1 \otimes ... \otimes H_k$.

If the Hilbert spaces are of finite dimension, then the above integrals become finite sums, and we may consider the functional calculus for arbitrary real functions. This construction [2] extends the definition of Korányi [5] for functions of two variables and have the property that

$$f (A_1, ..., A_k) = f_1(A_1) \otimes ... \otimes f_k(A_k),$$

whenever f can be separated as a product $f(t_1, ..., t_k) = f_1(t_1)...f_k(t_k)$ of k functions each depending on only one variable.

It is know that, *if* f *is super-multiplicative (sub-multiplicative)* on $[0, \infty)$, namely

$$f (st) \geq (\leq) f (s) f (t) \text{ for all } s, t \in [0, \infty)$$

and if f is continuous on $[0, \infty)$, then [8, p. 173]

$$f (A \otimes B) \geq (\leq) f (A) \otimes f (B) \text{ for all } A,\ B \geq 0. \tag{1.5}$$

This follows by observing that, if

$$A = \int_{[0,\infty)} tdE (t) \text{ and } B = \int_{[0,\infty)} sdF (s)$$

are the spectral resolutions of A and B, then

$$f (A \otimes B) = \int_{[0,\infty)} \int_{[0,\infty)} f (st) dE (t) \otimes dF (s) \tag{1.6}$$

for the continuous function f on $[0, \infty)$.

Recall the *geometric operator mean* for the positive operators $A,\ B > 0$

$$A\#_t B := A^{1/2}(A^{-1/2}BA^{-1/2})^t A^{1/2},$$

where $t \in [0, 1]$ and

$$A\#B := A^{1/2}(A^{-1/2}BA^{-1/2})^{1/2}A^{1/2}.$$

By the definitions of $\#$ and $\otimes$ we have

$$A\#B = B\#A \text{ and } (A\#B) \otimes (B\#A) = (A \otimes B) \# (B \otimes A).$$

In 2007, S. Wada [11] obtained the following *Callebaut type inequalities* for tensorial product

$$\begin{aligned} (A\#B) \otimes (A\#B) &\leq \frac{1}{2} [(A\#_\alpha B) \otimes (A\#_{1-\alpha} B) + (A\#_{1-\alpha} B) \otimes (A\#_\alpha B)] \\ &\leq \frac{1}{2} (A \otimes B + B \otimes A) \end{aligned} \tag{1.7}$$

for $A,\ B > 0$ and $\alpha \in [0, 1]$. For other similar results, see [1], [3] and [6]-[9].

Motivated by the above results, in this paper we show among others that, if f is of class C^{2m} on the open interval I and such that $\gamma_{2m} \leq f^{(2m)} \leq \Gamma_{2m}$ for some constants γ_{2m}, Γ_{2m}, A and B are selfadjoint operators with $\operatorname{Sp}(A)$, $\operatorname{Sp}(B) \subset I$, then we have the inequalities

$$\begin{aligned}
&\frac{1}{(2m)!}\left(A\otimes 1-1\otimes B\right)^{2m}\gamma_{2m} \\
&\leq f(A)\otimes 1-\sum_{k=0}^{2m-1}\frac{1}{k!}\left(A\otimes 1-1\otimes B\right)^{k}\left(1\otimes f^{(k)}(B)\right) \\
&\leq \frac{1}{(2m)!}\left(A\otimes 1-1\otimes B\right)^{2m}\Gamma_{2m}.
\end{aligned}$$

Some examples for logarithm and exponential functions are also provided.

2. Main Results

Recall the following property of the tensorial product

$$(AC)\otimes(BD)=(A\otimes B)(C\otimes D) \tag{2.1}$$

that holds for any $A, B, C, D \in B(H)$.

If we take $C = A$ and $D = B$, then we get

$$A^2\otimes B^2=(A\otimes B)^2.$$

By induction and using (2.1) we derive that

$$A^n\otimes B^n=(A\otimes B)^n \text{ for natural } n\geq 0. \tag{2.2}$$

In particular

$$A^n\otimes 1=(A\otimes 1)^n \text{ and } 1\otimes B^n=(1\otimes B)^n \tag{2.3}$$

for all $n \geq 0$.

We also observe that, by (2.1), the operators $A \otimes 1$ and $1 \otimes B$ are commutative and

$$(A\otimes 1)(1\otimes B)=(1\otimes B)(A\otimes 1)=A\otimes B. \tag{2.4}$$

Moreover, for two natural numbers m, n we have

$$(A\otimes 1)^m(1\otimes B)^n=(1\otimes B)^n(A\otimes 1)^m=A^m\otimes B^n. \tag{2.5}$$

We have the following representation results for continuous functions:

Lemma 1. *Assume A and B are selfadjoint operators with $\operatorname{Sp}(A) \subset I$ and $\operatorname{Sp}(B) \subset J$. Let f, h be continuous on I, g, k continuous on J and φ and ψ continuous on an interval K that contains the product of the intervals $f(I)\,g(J)$, $k(I)\,k(J)$, then*

$$\varphi(f(A)\otimes g(B))\,\psi(h(A)\otimes k(B))=\int_I\int_J\varphi(f(t)\,g(s))\,\psi(h(t)\,k(s))\,dE_t\otimes dF_s \tag{2.6}$$

where A and B have the spectral resolutions

$$A=\int_I t\,dE(t) \text{ and } B=\int_J s\,dF(s). \tag{2.7}$$

Proof. By Stone-Weierstrass, any continuous function can be approximated by a sequence of polynomials, therefore it suffices to prove the equality for the power function $\varphi(t) = t^m$ and $\psi(t) = t^n$ with n and m any natural numbers.

We have

$$\begin{aligned}
&\int_I \int_J (f(t) g(s))^m (h(t) k(s))^n \, dE_t \otimes dF_s \\
&= \int_I \int_J [f(t)]^m [g(s)]^m [h(t)]^n [k(s)]^n \, dE_t \otimes dF_s \\
&= \int_I \int_J [f(t)]^m [h(t)]^n [g(s)]^m [k(s)]^n \, dE_t \otimes dF_s \\
&= ([f(A)]^m [h(A)]^n) \otimes ([g(B)]^m [k(B)]^n) \\
&= ([f(A)]^m \otimes [g(B)]^m) ([h(A)]^n \otimes [k(B)]^n) \\
&= (f(A) \otimes g(B))^m (h(A) \otimes k(B))^n
\end{aligned}$$

and the equality (2.6) is that proved. $\square$

The additive version is as follows:

Lemma 2. *Assume A and B are selfadjoint operators with* $\mathrm{Sp}(A) \subset I$, $\mathrm{Sp}(B) \subset J$ *and having the spectral resolutions (2.7). Let f, h be continuous on I, g, k continuous on J and φ and ψ continuous on an interval K that contains the sum of the intervals $f(I) + g(J)$, $k(I) + k(J)$, then*

$$\begin{aligned}
(2.8) \qquad &\varphi(f(A) \otimes 1 + 1 \otimes g(B)) \psi(h(A) \otimes 1 + 1 \otimes k(B)) \\
&= \int_I \int_J \varphi(f(t) + g(s)) \psi(h(t) + k(s)) \, dE_t \otimes dF_s.
\end{aligned}$$

Proof. Let a, b, c and d positive continuous functions such that $f(t) = \ln a(t)$, $h(t) = \ln c(t)$ for $t \in I$ and $g(s) = \ln b(s)$, $k(s) = \ln d(s)$ for $s \in J$. Then

$$\begin{aligned}
(2.9) \qquad &\int_I \int_J \varphi(f(t) + g(s)) \psi(h(t) + k(s)) \, dE_t \otimes dF_s \\
&= \int_I \int_J \varphi(\ln a(t) + \ln b(s)) \psi(\ln c(t) + \ln d(s)) \, dE_t \otimes dF_s \\
&= \int_I \int_J (\varphi \circ \ln)(a(t) b(s)) (\psi \circ \ln)(c(t) d(s)) \, dE_t \otimes dF_s.
\end{aligned}$$

If we use Lemma 1 for the functions $\varphi \circ \ln$ and $(\psi \circ \ln)$, we get

$$\begin{aligned}
(2.10) \qquad &\int_I \int_J (\varphi \circ 2.8 \ln)(a(t) b(s)) (\psi \circ \ln)(c(t) d(s)) \, dE_t \otimes dF_s \\
&= (\varphi \circ \ln)(a(A) \otimes b(B)) (\psi \circ \ln)(c(A) \otimes d(B)) \\
&= \varphi[\ln(a(A) \otimes b(B))] \psi[\ln(c(A) \otimes d(B))].
\end{aligned}$$

Now, observe that, by the commutativity of the operators $a(A) \otimes 1$ and $1 \otimes b(B)$,

$$\begin{aligned}
\ln(a(A) \otimes b(B)) &= \ln[(a(A) \otimes 1)(1 \otimes b(B))] \\
&= \ln(a(A) \otimes 1) + \ln(1 \otimes b(B)) \\
&= [\ln a(A)] \otimes 1 + 1 \otimes \ln b(B) \quad \text{(by (2.6))} \\
&= f(A) \otimes 1 + 1 \otimes g(B)
\end{aligned}$$

and, similarly

$$\ln\left(c\left(A\right)\otimes d\left(B\right)\right)=h\left(A\right)\otimes 1+1\otimes k\left(B\right).$$

By utilising (2.9) and (2.10) we then get the desired representation (2.8) $\square$

We have the following Taylor representation:

Lemma 3. *Assume that f is of class C^{n+1} on the open interval I, A and B are selfadjoint operators with* $\operatorname{Sp}(A)$, $\operatorname{Sp}(B)\subset I$, *then we have the representation*

$$\begin{aligned} f\left(A\right)\otimes 1 &=\sum_{k=0}^{n}\frac{1}{k!}\left(A\otimes 1-1\otimes B\right)^{k}\left(1\otimes f^{(k)}\left(B\right)\right) \\ &+\frac{1}{n!}\left(A\otimes 1-1\otimes B\right)^{n+1} \\ &\times\int_{0}^{1}\left(1-u\right)^{n}f^{(n+1)}\left(\left(1-u\right)1\otimes B+uA\otimes 1\right)du. \end{aligned}\tag{2.11}$$

Proof. Using Taylor's representation with the integral remainder (1.1) we can write the following two identities

$$f\left(x\right)=\sum_{k=0}^{n}\frac{1}{k!}f^{(k)}\left(a\right)\left(x-a\right)^{k}+\frac{1}{n!}\int_{a}^{x}f^{(n+1)}\left(y\right)\left(x-y\right)^{n}dy\tag{2.12}$$

for $x,a\in I$.

For any integrable function h on an interval and any distinct numbers c, d in that interval, we have, by the change of variable $y=\left(1-u\right)c+ud$, $u\in[0,1]$ that

$$\int_{c}^{d}h\left(y\right)dy=\left(d-c\right)\int_{0}^{1}h\left(\left(1-u\right)c+ud\right)du.$$

Therefore,

$$\begin{aligned} &\int_{a}^{x}f^{(n+1)}\left(y\right)\left(x-y\right)^{n}dy \\ &=\left(x-a\right)\int_{0}^{1}f^{(n+1)}\left(\left(1-u\right)a+ux\right)\left(x-\left(1-u\right)a-ux\right)^{n}du \\ &=\left(x-a\right)^{n+1}\int_{0}^{1}f^{(n+1)}\left(\left(1-u\right)a+ux\right)\left(1-u\right)^{n}du \end{aligned}$$

and the identity (2.12) becomes

$$\begin{aligned} f\left(t\right)&=\sum_{k=0}^{n}\frac{1}{k!}f^{(k)}\left(s\right)\left(t-s\right)^{k} \\ &+\frac{1}{n!}\left(t-s\right)^{n+1}\int_{0}^{1}f^{(n+1)}\left(\left(1-u\right)s+ut\right)\left(1-u\right)^{n}du, \end{aligned}\tag{2.13}$$

for all t, $s\in I$.

If A and B have the spectral resolutions

$$A=\int_{I}tdE\left(t\right)\text{ and }B=\int_{I}sdF\left(s\right),$$

then by taking the integral $\int_I \int_I$ over $dE_t \otimes dF_s$, we get

$$\begin{aligned}
&\int_I \int_I f(t)\, dE_t \otimes dF_s \\
&= \int_I \int_I \left(\sum_{k=0}^{n} \frac{1}{k!} f^{(k)}(s)(t-s)^k \right) dE_t \otimes dF_s \\
&+ \frac{1}{n!} \int_I \int_I (t-s)^{n+1} \left(\int_0^1 f^{(n+1)}((1-u)s + ut)(1-u)^n du \right) dE_t \otimes dF_s.
\end{aligned} \tag{2.14}$$

We have

$$\int_I \int_I f(t)\, dE_t \otimes dF_s = f(A) \otimes 1$$

and, by (2.8)

$$\begin{aligned}
&\int_I \int_I \left(\sum_{k=0}^{n} \frac{1}{k!} f^{(k)}(s)(t-s)^k \right) dE_t \otimes dF_s \\
&= \sum_{k=0}^{n} \frac{1}{k!} \int_I \int_I \left((t-s)^k f^{(k)}(s) \right) dE_t \otimes dF_s \\
&= \sum_{k=0}^{n} \frac{1}{k!} (A \otimes 1 - 1 \otimes B)^k \left(1 \otimes f^{(k)}(B) \right).
\end{aligned}$$

By Fubini's theorem and (2.8) we also get

$$\begin{aligned}
&\int_I \int_I (t-s)^{n+1} \left(\int_0^1 f^{(n+1)}((1-u)s + ut)(1-u)^n du \right) dE_t \otimes dF_s \\
&= \int_0^1 (1-u)^n \left(\int_I \int_I (t-s)^{n+1} \left(f^{(n+1)}((1-u)s + ut) \right) dE_t \otimes dF_s \right) du \\
&= (A \otimes 1 - 1 \otimes B)^{n+1} \int_0^1 (1-u)^n \left(f^{(n+1)}((1-u)1 \otimes B + uA \otimes 1) \right) du
\end{aligned}$$

and by (2.14) we obtain the desired result (2.11). $\square$

Our first main result is as follows:

Theorem 2. *Assume that f is of class C^{2m} on the open interval I and such that $\gamma_{2m} \le f^{(2m)} \le \Gamma_{2m}$ for some constants γ_{2m}, Γ_{2m}, A and B are selfadjoint operators with $\operatorname{Sp}(A), \operatorname{Sp}(B) \subset I$ then we have the inequalities*

$$\begin{aligned}
&\frac{1}{(2m)!} (A \otimes 1 - 1 \otimes B)^{2m} \gamma_{2m} \\
&\le f(A) \otimes 1 - \sum_{k=0}^{2m-1} \frac{1}{k!} (A \otimes 1 - 1 \otimes B)^k \left(1 \otimes f^{(k)}(B) \right) \\
&\le \frac{1}{(2m)!} (A \otimes 1 - 1 \otimes B)^{2m} \Gamma_{2m}
\end{aligned} \tag{2.15}$$

Proof. For $n = 2m - 1$ with $m \geq 1$ in (2.11), we have

$$f(A) \otimes 1 = \sum_{k=0}^{2m-1} \frac{1}{k!} (A \otimes 1 - 1 \otimes B)^k \left(1 \otimes f^{(k)}(B)\right) + \frac{1}{(2m-1)!} (A \otimes 1 - 1 \otimes B)^{2m} \times \int_0^1 (1-u)^{2m-1} f^{(2m)}((1-u) 1 \otimes B + uA \otimes 1)\, du. \tag{2.16}$$

From the assumption of boundedness, we have that

$$\gamma_{2m} \leq f^{(2m)}((1-u) s + ut) \leq \Gamma_{2m} \tag{2.17}$$

for all $t,\ s \in I$ and $u \in [0,1]$.

Now, if we multiply the inequality (2.17) by $(t-s)^{2m} \geq 0$ with $t,\ s \in I$, then we get

$$\gamma_{2m} (t-s)^{2m} \leq (t-s)^{2m} f^{(2m)}((1-u) s + ut) \leq \Gamma_{2m} (t-s)^{2m}. \tag{2.18}$$

By taking the integral $\int_I \int_I$ over $dE_t \otimes dF_s$ in (2.18) we obtain

$$\gamma_{2m} \int_I \int_I (t-s)^{2m} dE_t \otimes dF_s \leq \int_I \int_I (t-s)^{2m} f^{(2m)}((1-u) s + ut)\, dE_t \otimes dF_s \leq \Gamma_{2m} \int_I \int_I (t-s)^{2m} dE_t \otimes dF_s. \tag{2.19}$$

By using Lemma 2 we derive

$$\gamma_{2m} (A \otimes 1 - 1 \otimes B)^{2m} \leq (A \otimes 1 - 1 \otimes B)^{2m} f^{(2m)}((1-u) 1 \otimes B + uA \otimes 1) \leq \Gamma_{2m} (A \otimes 1 - 1 \otimes B)^{2m}.$$

Further, if we multiply this inequality by $(1-u)^{2m-1} \geq 0$, $u \in [0,1]$ and integrate, then we get

$$\gamma_{2m} \int_0^1 (1-u)^{2m-1} du\, (A \otimes 1 - 1 \otimes B)^{2m} \leq \int_0^1 (1-u)^{2m-1} (A \otimes 1 - 1 \otimes B)^{2m} f^{(2m)}((1-u) 1 \otimes B + uA \otimes 1)\, du \leq \Gamma_{2m} \int_0^1 (1-u)^{2m-1} du\, (A \otimes 1 - 1 \otimes B)^{2m},$$

which gives that

$$\frac{\gamma_{2m}}{2m} (A \otimes 1 - 1 \otimes B)^{2m} \leq (A \otimes 1 - 1 \otimes B)^{2m} \int_0^1 (1-u)^{2m-1} f^{(2m)}((1-u) 1 \otimes B + uA \otimes 1)\, du \leq \frac{\Gamma_{2m}}{2m} (A \otimes 1 - 1 \otimes B)^{2m},$$

namely

$$\begin{aligned}&\frac{\gamma_{2m}}{(2m)!}(A\otimes 1-1\otimes B)^{2m}\\&\leq\frac{1}{(2m-1)!}(A\otimes 1-1\otimes B)^{2m}\\&\times\int_0^1(1-u)^{2m-1}f^{(2m)}((1-u)\,1\otimes B+uA\otimes 1)\,du\\&\leq\frac{\Gamma_{2m}}{(2m)!}(A\otimes 1-1\otimes B)^{2m}.\end{aligned}$$

By utilising the identity (2.16) we derive the desired double inequality (2.15). □

As a particular case, we have:

Corollary 1. *Assume that f is of class C^2 on the open interval I and such that $\gamma_2\leq f''\leq\Gamma_{2m}$ for some constants γ_2,Γ_2, A and B are selfadjoint operators with $\operatorname{Sp}(A)$, $\operatorname{Sp}(B)\subset I$, then we have the inequalities*

$$\begin{aligned}&\frac{1}{2}(A\otimes 1-1\otimes B)^2\gamma_2\\&\leq f(A)\otimes 1-1\otimes f(B)-(A\otimes 1-1\otimes B)(1\otimes f'(B))\\&\leq\frac{1}{2}(A\otimes 1-1\otimes B)^2\Gamma_2.\end{aligned}\tag{2.20}$$

We also have:

Theorem 3. *Assume that f is of class C^{2m} on the open interval I and such that $f^{(2m)}$ is convex (concave) on I, A and B are selfadjoint operators with $\operatorname{Sp}(A)$, $\operatorname{Sp}(B)\subset I$ then we have the inequality*

$$\begin{aligned}&f(A)\otimes 1-\sum_{k=0}^{2m-1}\frac{1}{k!}(A\otimes 1-1\otimes B)^k\left(1\otimes f^{(k)}(B)\right)\\&\leq(\geq)\frac{1}{(2m)!}(A\otimes 1-1\otimes B)^{2m}\left[\frac{2m\left(1\otimes f^{(2m)}(B)\right)+f^{(2m)}(A)\otimes 1}{2m+1}\right],\end{aligned}\tag{2.21}$$

Proof. Since $f^{(2m)}$ is convex on I, then

$$f^{(2m)}((1-u)s+ut)\leq(1-u)f^{(2m)}(s)+uf^{(2m)}(t)\tag{2.22}$$

for all t, $s\in I$ and $u\in[0,1]$.

Now, if we multiply the inequality (2.22) by $(t-s)^{2m}\geq 0$ with t, $s\in I$, then we get

$$\begin{aligned}&(t-s)^{2m}f^{(2m)}((1-u)s+ut)\\&\leq(1-u)(t-s)^{2m}f^{(2m)}(s)+u(t-s)^{2m}f^{(2m)}(t)\end{aligned}\tag{2.23}$$

for all t, $s\in I$ and $u\in[0,1]$.

By taking the integral $\int_I\int_I$ over $dE_t\otimes dF_s$ in (2.23) we obtain

$$\begin{aligned}&\int_I\int_I(t-s)^{2m}f^{(2m)}((1-u)s+ut)\,dE_t\otimes dF_s\\&\leq\int_I\int_I\left[(1-u)(t-s)^{2m}f^{(2m)}(s)+u(t-s)^{2m}f^{(2m)}(t)\right]dE_t\otimes dF_s\end{aligned}\tag{2.24}$$

for all $u \in [0,1]$.

By using Lemma 2 we derive that

$$\begin{aligned}
&\int_I \int_I (t-s)^{2m} f^{(2m)}((1-u)s+ut)\, dE_t \otimes dF_s \\
&= (A \otimes 1 - 1 \otimes B)^{2m} f^{(2m)}((1-u)\, 1 \otimes B + uA \otimes 1)
\end{aligned}$$

and

$$\begin{aligned}
&\int_I \int_I \left[(1-u)(t-s)^{2m} f^{(2m)}(s) + u(t-s)^{2m} f^{(2m)}(t)\right] dE_t \otimes dF_s \\
&= (1-u) \int_I \int_I (t-s)^{2m} f^{(2m)}(s)\, dE_t \otimes dF_s \\
&+ u \int_I \int_I (t-s)^{2m} f^{(2m)}(t)\, dE_t \otimes dF_s \\
&= (1-u)(A \otimes 1 - 1 \otimes B)^{2m} \left(1 \otimes f^{(2m)}(B)\right) \\
&+ u (A \otimes 1 - 1 \otimes B)^{2m} \left(f^{(2m)}(A) \otimes 1\right)
\end{aligned}$$

and by (2.24) we obtain

$$\begin{aligned}
&(A \otimes 1 - 1 \otimes B)^{2m} f^{(2m)}((1-u)\, 1 \otimes B + uA \otimes 1) \\
&\leq (1-u)(A \otimes 1 - 1 \otimes B)^{2m} \left(1 \otimes f^{(2m)}(B)\right) \\
&+ u (A \otimes 1 - 1 \otimes B)^{2m} \left(f^{(2m)}(A) \otimes 1\right)
\end{aligned}$$

for all $u \in [0,1]$.

If we multiply by $(1-u)^{2m-1}$ and integrate, then we get

$$\begin{aligned}
&(A \otimes 1 - 1 \otimes B)^{2m} \int_0^1 (1-u)^{2m-1} f^{(2m)}((1-u)\, 1 \otimes B + uA \otimes 1)\, du \\
&\leq \int_0^1 (1-u)^{2m} du\, (A \otimes 1 - 1 \otimes B)^{2m} \left(1 \otimes f^{(2m)}(B)\right) \\
&+ \int_0^1 u (1-u)^{2m-1} du\, (A \otimes 1 - 1 \otimes B)^{2m} \left(f^{(2m)}(A) \otimes 1\right) \\
&= \frac{1}{2m+1} (A \otimes 1 - 1 \otimes B)^{2m} \left(1 \otimes f^{(2m)}(B)\right) \\
&+ \frac{1}{2m(2m+1)} (A \otimes 1 - 1 \otimes B)^{2m} \left(f^{(2m)}(A) \otimes 1\right) \\
&= \frac{1}{2m+1} (A \otimes 1 - 1 \otimes B)^{2m} \left[1 \otimes f^{(2m)}(B) + \frac{1}{2m} f^{(2m)}(A) \otimes 1\right] \\
&= \frac{1}{2m} (A \otimes 1 - 1 \otimes B)^{2m} \left[\frac{2m 1 \otimes f^{(2m)}(B) + f^{(2m)}(A) \otimes 1}{2m+1}\right].
\end{aligned}$$

If we multiply by $\frac{1}{(2m-1)!}$ we derive

$$\begin{aligned}&\frac{1}{(2m-1)!}\left(A\otimes 1-1\otimes B\right)^{2m}\\&\times\int_0^1 (1-u)^{2m-1} f^{(2m)}\left((1-u)\,1\otimes B+uA\otimes 1\right)du\\&\leq\frac{1}{(2m)!}\left(A\otimes 1-1\otimes B\right)^{2m}\left[\frac{2m1\otimes f^{(2m)}(B)+f^{(2m)}(A)\otimes 1}{2m+1}\right]\end{aligned}$$

and by (2.16) we derive (2.21). □

Corollary 2. *Assume that f is of class C^2 on the open interval I and such that f'' is convex (concave) on I, A and B are selfadjoint operators with $\operatorname{Sp}(A), \operatorname{Sp}(B)\subset I$, then we have the inequality*

$$\begin{aligned}&f(A)\otimes 1-1\otimes f(B)-(A\otimes 1-1\otimes B)(1\otimes f'(B))\\&\leq(\geq)\,\frac{1}{2}(A\otimes 1-1\otimes B)^2\left[\frac{2(1\otimes f''(B))+f''(A)\otimes 1}{3}\right].\end{aligned}\tag{2.25}$$

We recall that the function $g: I\to\mathbb{R}$ is *quasi-convex*, if

$$g((1-\lambda)t+\lambda s)\leq\max\{g(t), g(s)\}=\frac{1}{2}(g(t)+g(s)+|g(t)-g(s)|)$$

for all t, $s\in I$ and $\lambda\in[0,1]$.

Theorem 4. *Assume that f is of class C^{2m} on the open interval I and such that $f^{(2m)}$ is quasi-convex on I, A and B are selfadjoint operators with $\operatorname{Sp}(A)$, $\operatorname{Sp}(B)\subset I$ then we have the inequality*

$$\begin{aligned}&f(A)\otimes 1-\sum_{k=0}^{2m-1}\frac{1}{k!}(A\otimes 1-1\otimes B)^k\left(1\otimes f^{(k)}(B)\right)\\&\leq\frac{1}{(2m)!}(A\otimes 1-1\otimes B)^{2m}\\&\times\left[f^{(2m)}(A)\otimes 1+1\otimes f^{(2m)}(B)+\left|f^{(2m)}(A)\otimes 1-1\otimes f^{(2m)}(B)\right|\right].\end{aligned}\tag{2.26}$$

Proof. Since $f^{(2m)}$ is quasi-convex on I, then

$$f^{(2m)}((1-u)s+ut)\leq\frac{1}{2}\left(f^{(2m)}(t)+f^{(2m)}(s)+\left|f^{(2m)}(t)-f^{(2m)}(s)\right|\right)$$

for all t, $s\in I$ and $u\in[0,1]$.

If we multiply by $(t-s)^{2m}$, we get

$$\begin{aligned}&(t-s)^{2m} f^{(2m)}((1-u)s+ut)\\&\leq\frac{1}{2}(t-s)^{2m}\left(f^{(2m)}(t)+f^{(2m)}(s)+\left|f^{(2m)}(t)-f^{(2m)}(s)\right|\right)\end{aligned}\tag{2.27}$$

for all t, $s\in I$ and $u\in[0,1]$.

By taking the integral $\int_I \int_I$ over $dE_t \otimes dF_s$ in (2.27) we obtain

$$\begin{aligned}
&\int_I \int_I (t-s)^{2m} f^{(2m)}((1-u)s+ut)\, dE_t \otimes dF_s \\
&\leq \frac{1}{2} \int_I \int_I (t-s)^{2m} \left(f^{(2m)}(t) + f^{(2m)}(s) + \left| f^{(2m)}(t) - f^{(2m)}(s) \right| \right) \\
&\times dE_t \otimes dF_s.
\end{aligned} \tag{2.28}$$

Since, by Lemma 2,

$$\begin{aligned}
&\int_I \int_I (t-s)^{2m} f^{(2m)}((1-u)s+ut)\, dE_t \otimes dF_s \\
&= (A\otimes 1 - 1\otimes B)^{2m} f^{(2m)}((1-u)1\otimes B + uA\otimes 1)
\end{aligned}$$

and

$$\begin{aligned}
&\int_I \int_I (t-s)^{2m} \left(f^{(2m)}(t) + f^{(2m)}(s) + \left| f^{(2m)}(t) - f^{(2m)}(s) \right| \right) dE_t \otimes dF_s \\
&= \int_I \int_I (t-s)^{2m} f^{(2m)}(t)\, dE_t \otimes dF_s + \int_I \int_I (t-s)^{2m} f^{(2m)}(s)\, dE_t \otimes dF_s \\
&+ \int_I \int_I (t-s)^{2m} \left| f^{(2m)}(t) - f^{(2m)}(s) \right| dE_t \otimes dF_s \\
&= (A\otimes 1 - 1\otimes B)^{2m} f^{(2m)}(A)\otimes 1 + (A\otimes 1 - 1\otimes B)^{2m} \left(1\otimes f^{(2m)}(B) \right) \\
&+ (A\otimes 1 - 1\otimes B)^{2m} \left| f^{(2m)}(A)\otimes 1 - 1\otimes f^{(2m)}(B) \right| \\
&= (A\otimes 1 - 1\otimes B)^{2m} \\
&\times \left[f^{(2m)}(A)\otimes 1 + 1\otimes f^{(2m)}(B) + \left| f^{(2m)}(A)\otimes 1 - 1\otimes f^{(2m)}(B) \right| \right],
\end{aligned}$$

then by (2.28) we get

$$\begin{aligned}
&(A\otimes 1 - 1\otimes B)^{2m} f^{(2m)}((1-u)1\otimes B + uA\otimes 1) \\
&\leq (A\otimes 1 - 1\otimes B)^{2m} \\
&\times \left[f^{(2m)}(A)\otimes 1 + 1\otimes f^{(2m)}(B) + \left| f^{(2m)}(A)\otimes 1 - 1\otimes f^{(2m)}(B) \right| \right]
\end{aligned}$$

for $u \in [0,1]$.

If we multiply by $(1-u)^{2m-1}$ and integrate, then we get

$$\begin{aligned}
&(A\otimes 1 - 1\otimes B)^{2m} \int_0^1 (1-u)^{2m-1} f^{(2m)}((1-u)1\otimes B + uA\otimes 1)\, du \\
&\leq \frac{1}{2m} (A\otimes 1 - 1\otimes B)^{2m} \\
&\times \left[f^{(2m)}(A)\otimes 1 + 1\otimes f^{(2m)}(B) + \left| f^{(2m)}(A)\otimes 1 - 1\otimes f^{(2m)}(B) \right| \right].
\end{aligned}$$

Finally, if we use (2.16) we derive the desired result (2.26). □

Corollary 3. *Assume that f is of class C^2 on the open interval I and such that f'' is quasi-convex on I, A and B are selfadjoint operators with* $\mathrm{Sp}(A)$, $\mathrm{Sp}(B) \subset I$, *then we have the inequality*

$$
\begin{aligned}
& f(A)\otimes 1-1\otimes f(B)-(A\otimes 1-1\otimes B)(1\otimes f'(B)) \\
& \leq \frac{1}{2}(A\otimes 1-1\otimes B)^2 \\
& \times\left[f''(A)\otimes 1+1\otimes f''(B)+\left|f''(A)\otimes 1-1\otimes f''(B)\right|\right]
\end{aligned} \tag{2.29}
$$

3. Some Examples

We consider the function $f(t)=\ln t$, $t>0$. Then

$$
f^{(k)}(t)=\frac{(-1)^{k-1}(k-1)!}{t^k},\ k\geq 1,\ t>0.
$$

From (2.16) we then get for $m\geq 1$ that

$$
\begin{aligned}
\ln(A)\otimes 1 &= 1\otimes\ln B+\sum_{k=1}^{2m-1}\frac{(-1)^{k-1}}{k}(A\otimes 1-1\otimes B)^k\left(1\otimes B^{-k}\right) \\
& -(A\otimes 1-1\otimes B)^{2m} \\
& \times\int_0^1(1-s)^{2m-1}\left[(1-u)1\otimes B+uA\otimes 1\right]^{-2m}ds
\end{aligned} \tag{3.1}
$$

for all $A,\ B>0$.

For $m=1$ we obtain

$$
\begin{aligned}
\ln(A)\otimes 1 &= 1\otimes\ln B+A\otimes B^{-1}-1 \\
& -(A\otimes 1-1\otimes B)^2 \\
& \times\int_0^1(1-s)\left[(1-u)1\otimes B+uA\otimes 1\right]^{-2}ds.
\end{aligned} \tag{3.2}
$$

We observe that

$$
f^{(2m)}(t)=-\frac{(2m-1)!}{t^{2m}},\ m\geq 1,\ t>0.
$$

If $0<\gamma\leq t\leq\Gamma$, then

$$
-\frac{(2m-1)!}{\gamma^{2m}}\leq f^{(2m)}(t)\leq-\frac{(2m-1)!}{\Gamma^{2m}}
$$

and by (2.15) we get for $0<\gamma\leq A,\ B\leq\Gamma$ that

$$
\begin{aligned}
& \frac{1}{2m\gamma^{2m}}(A\otimes 1-1\otimes B)^{2m} \\
& \geq 1\otimes\ln B-\ln(A)\otimes 1 \\
& +\sum_{k=1}^{2m-1}\frac{(-1)^{k-1}}{k}(A\otimes 1-1\otimes B)^k\left(1\otimes B^{-k}\right) \\
& \geq\frac{1}{2m\Gamma^{2m}}(A\otimes 1-1\otimes B)^{2m}
\end{aligned} \tag{3.3}
$$

for $m\geq 1$.

For $m=1$, we derive

$$\begin{aligned}
&\frac{1}{2\gamma^2}\left(A\otimes 1-1\otimes B\right)^2 \\
&\geq 1\otimes \ln B-\ln\left(A\right)\otimes 1+A\otimes B^{-1}-1 \\
&\geq \frac{1}{2\Gamma^2}\left(A\otimes 1-1\otimes B\right)^2
\end{aligned} \tag{3.4}$$

for $0<\gamma\leq A,\ B\leq \Gamma$.

The function $f^{(2m)}\left(t\right)=-\frac{(2m-1)!}{t^{2m}}$ is concave on $(0,\infty)$ and by (2.21) we get for $A,\ B>0$ that

$$\begin{aligned}
&1\otimes \ln B-\ln\left(A\right)\otimes 1 \\
&+\sum_{k=1}^{2m-1}\frac{\left(-1\right)^{k-1}}{k}\left(A\otimes 1-1\otimes B\right)^k\left(1\otimes B^{-k}\right) \\
&\leq \frac{1}{2m}\left(A\otimes 1-1\otimes B\right)^{2m}\left[\frac{2m\left(1\otimes B^{-2m}\right)+A^{-2m}\otimes 1}{2m+1}\right].
\end{aligned} \tag{3.5}$$

For $m=1$ we derive

$$\begin{aligned}
&1\otimes \ln B-\ln\left(A\right)\otimes 1+A\otimes B^{-1}-1 \\
&\leq \frac{1}{2}\left(A\otimes 1-1\otimes B\right)^2\left[\frac{2\left(1\otimes B^{-2}\right)+A^{-2}\otimes 1}{3}\right].
\end{aligned} \tag{3.6}$$

Consider the exponential function $f\left(t\right)=\exp\left(t\right)$, then by (2.16) we get

$$\begin{aligned}
\exp\left(A\right)\otimes 1&=\sum_{k=0}^{2m-1}\frac{1}{k!}\left(A\otimes 1-1\otimes B\right)^k\left(1\otimes \exp\left(B\right)\right) \\
&+\frac{1}{(2m-1)!}\left(A\otimes 1-1\otimes B\right)^{2m} \\
&\times\int_0^1\left(1-u\right)^{2m-1}\exp\left(\left(1-u\right)1\otimes B+uA\otimes 1\right)du
\end{aligned} \tag{3.7}$$

for any selfadjoint operators $A,\ B$.

If $\gamma\leq A,\ B\leq\Gamma$, then by (2.15) we get

$$\begin{aligned}
&\frac{1}{(2m)!}\exp\left(\gamma\right)\left(A\otimes 1-1\otimes B\right)^{2m} \\
&\leq \exp\left(A\right)\otimes 1-\sum_{k=0}^{2m-1}\frac{1}{k!}\left(A\otimes 1-1\otimes B\right)^k\left(1\otimes \exp\left(B\right)\right) \\
&\leq \frac{1}{(2m)!}\exp\left(\Gamma\right)\left(A\otimes 1-1\otimes B\right)^{2m}.
\end{aligned} \tag{3.8}$$

From (2.21) we also get

$$\begin{aligned}
&\exp\left(A\right)\otimes 1-\sum_{k=0}^{2m-1}\frac{1}{k!}\left(A\otimes 1-1\otimes B\right)^k\left(1\otimes \exp\left(B\right)\right) \\
&\leq \frac{1}{(2m)!}\left(A\otimes 1-1\otimes B\right)^{2m}\left[\frac{2m\left(1\otimes \exp\left(B\right)\right)+\exp\left(A\right)\otimes 1}{2m+1}\right],
\end{aligned} \tag{3.9}$$

for any selfadjoint operators A, B.

If C, $D>0$ and if we take in (3.7) $A=\ln C$, $B=\ln D$, then we get

$$C\otimes 1=\sum_{k=0}^{2m-1}\frac{1}{k!}\left(\ln C\otimes 1-1\otimes\ln D\right)^{k}\left(1\otimes D\right)+\frac{1}{(2m-1)!}\left(\ln C\otimes 1-1\otimes\ln D\right)^{2m}\times\int_{0}^{1}(1-u)^{2m-1}\exp\left((1-u)\,1\otimes\ln D+u\ln C\otimes 1\right)du \tag{3.10}$$

and for $m=1$,

$$C\otimes 1=(1\otimes D)+\left(\ln C\otimes 1-1\otimes\ln D\right)(1\otimes D)+\left(\ln C\otimes 1-1\otimes\ln D\right)^{2}\times\int_{0}^{1}(1-u)\exp\left((1-u)\,1\otimes\ln D+u\ln C\otimes 1\right)du. \tag{3.11}$$

If $0<\psi\le C$, $D\le\Psi$, then by (3.8) we derive

$$\frac{1}{(2m)!}\psi\left(\ln C\otimes 1-1\otimes\ln D\right)^{2m}\le C\otimes 1-\sum_{k=0}^{2m-1}\frac{1}{k!}\left(\ln C\otimes 1-1\otimes\ln D\right)^{k}(1\otimes D)\le\frac{1}{(2m)!}\Psi\left(\ln C\otimes 1-1\otimes\ln D\right)^{2m} \tag{3.12}$$

and for $m=1$,

$$\frac{1}{2}\psi\left(\ln C\otimes 1-1\otimes\ln D\right)^{2}\le C\otimes 1-1\otimes D-\left(\ln C\otimes 1-1\otimes\ln D\right)(1\otimes D)\le\frac{1}{2}\Psi\left(\ln C\otimes 1-1\otimes\ln D\right)^{2}. \tag{3.13}$$

From (3.9) we also get

$$C\otimes 1-\sum_{k=0}^{2m-1}\frac{1}{k!}\left(\ln C\otimes 1-1\otimes\ln D\right)^{k}(1\otimes D)\le\frac{1}{(2m)!}\left(\ln C\otimes 1-1\otimes\ln D\right)^{2m}\left[\frac{2m\,(1\otimes D)+C\otimes 1}{2m+1}\right] \tag{3.14}$$

and for $m=1$,

$$C\otimes 1-1\otimes D-\left(\ln C\otimes 1-1\otimes\ln D\right)(1\otimes D)\le\frac{1}{2}\left(\ln C\otimes 1-1\otimes\ln D\right)^{2}\left[\frac{2\,(1\otimes D)+C\otimes 1}{3}\right] \tag{3.15}$$

for all C, $D>0$.

If we take in (3.13) $(1 \otimes D)^{-1/2}$ both sides and use the commutativity of $C \otimes 1$ with $1 \otimes D$ we derive from (3.13) that

$$\begin{aligned} &\frac{1}{2}\psi \left(\ln C \otimes 1 - 1 \otimes \ln D\right)^2 \left(1 \otimes D^{-1}\right) \\ &\leq 1 \otimes \ln D - 1 - \ln C \otimes 1 + C \otimes D^{-1} \\ &\leq \frac{1}{2}\Psi \left(\ln C \otimes 1 - 1 \otimes \ln D\right)^2 \left(1 \otimes D^{-1}\right) \end{aligned} \tag{3.16}$$

provided that $0 < \psi \leq C,\ D \leq \Psi$.

References

[1] T. Ando, Concavity of certain maps on positive definite matrices and applications to Hadamard products, *Lin. Alg. & Appl.* **26** (1979), 203-241.

[2] H. Araki and F. Hansen, Jensen's operator inequality for functions of several variables, *Proc. Amer. Math. Soc.* **128** (2000), No. 7, 2075-2084.

[3] J. S. Aujila and H. L. Vasudeva, Inequalities involving Hadamard product and operator means, *Math. Japon.* **42** (1995), 265-272.

[4] S. S. Dragomir, Some tensorial Hermite-Hadamard type inequalities for convex functions of selfadjoint operators in Hilbert spaces, Preprint *RGMIA Res. Rep. Coll.* **25** (2022), Art. 90, 14 pp. [Online `https://rgmia.org/papers/v25/v25a90.pdf`]

[5] A. Korányi. On some classes of analytic functions of several variables. *Trans. Amer. Math. Soc.,* **101** (1961), 520–554.

[6] A. Ebadian, I. Nikoufar and M. E. Gordji, Perspectives of matrix convex functions, *Proc. Natl. Acad. Sci. USA*, **108** (2011), no. 18, 7313–7314.

[7] J. I. Fujii, The Marcus-Khan theorem for Hilbert space operators. *Math. Jpn.* **41** (1995), 531-535

[8] T. Furuta, J. Mićić Hot, J. Pečarić and Y. Seo, *Mond-Pečarić Method in Operator Inequalities. Inequalities for Bounded Selfadjoint Operators on a Hilbert Space*, Element, Zagreb, 2005.

[9] K. Kitamura and Y. Seo, Operator inequalities on Hadamard product associated with Kadison's Schwarz inequalities, *Scient. Math.* **1** (1998), No. 2, 237-241.

[10] A. Ostrowski, Über die Absolutabweichung einer differentienbaren Funktionen von ihren Integralmittelwert, *Comment. Math. Hel,* **10** (1938), 226-227.

[11] S. Wada, On some refinement of the Cauchy-Schwarz Inequality, *Lin. Alg. & Appl.* **420** (2007), 433-440.

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