TENSORIAL AND HADAMARD PRODUCTS INTEGRAL INEQUALITIES FOR CONTINUOUS FIELDS OF OPERATORS IN HILBERT SPACES VIA KANTOROVICH RATIO

SILVESTRU SEVER DRAGOMIR[1,2]

Abstract. Let H be a Hilbert space and Ω a locally compact Hausdorff space endowed with a Radon measure μ with $\int_{\Omega} 1 d\mu(t) = 1$. In this paper we show among others that, if $(A_\tau)_{\tau\in\Omega}$ and $(B_\tau)_{\tau\in\Omega}$ are continuous fields of positive operators in $B(H)$ such that $\operatorname{Sp}(A_\tau), \operatorname{Sp}(B_\tau) \subseteq [m, M] \subset (0,\infty)$ for each $\tau \in \Omega$, then for all $\nu \in [0,1]$ we have the tensorial inequality

$$\begin{aligned}
&\left(\int_\Omega A_\tau^{1-\nu} d\mu(\tau)\right) \otimes \left(\int_\Omega B_\tau^{\nu} d\mu(\tau)\right)\\
&\leq (1-\nu)\left(\int_\Omega A_\tau d\mu(\tau)\right) \otimes 1 + \nu 1 \otimes \left(\int_\Omega B_\tau d\mu(\tau)\right)\\
&\leq K^R\left(\frac{M}{m}\right)\left(\int_\Omega A_\tau^{1-\nu} d\mu(\tau)\right) \otimes \left(\int_\Omega B_\tau^{\nu} d\mu(\tau)\right),
\end{aligned}$$

where $R = \max\{1-\nu, \nu\}$ and $K(\cdot)$ is *Kantorovich's ratio*. We also have the following inequalities for the Hadamard product

$$\begin{aligned}
&\left(\int_\Omega A_\tau^{1-\nu} d\mu(\tau)\right) \circ \left(\int_\Omega B_\tau^{\nu} d\mu(\tau)\right)\\
&\leq \int_\Omega [(1-\nu) A_\tau + \nu B_\tau] d\mu(\tau) \circ 1\\
&\leq K^R\left(\frac{M}{m}\right)\left(\int_\Omega A_\tau^{1-\nu} d\mu(\tau)\right) \circ \left(\int_\Omega B_\tau^{\nu} d\mu(\tau)\right)
\end{aligned}$$

for all $\nu \in [0,1]$.

1. Introduction

The famous *Young inequality* for scalars says that if $a, b > 0$ and $\nu \in [0,1]$, then

$$a^{1-\nu} b^{\nu} \leq (1-\nu) a + \nu b \tag{1.1}$$

with equality if and only if $a = b$. The inequality (1.1) is also called ν-weighted arithmetic-geometric mean inequality.

We recall that *Specht's ratio* is defined by

$$S(h) := \begin{cases} \dfrac{h^{\frac{1}{h-1}}}{e \ln\left(h^{\frac{1}{h-1}}\right)} \text{ if } h \in (0,1) \cup (1,\infty) \\ \\ 1 \text{ if } h = 1. \end{cases}$$

It is well known that $\lim_{h\to 1} S(h) = 1$, $S(h) = S\left(\frac{1}{h}\right) > 1$ for $h > 0$, $h \neq 1$. The function is decreasing on $(0,1)$ and increasing on $(1,\infty)$.

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Tominaga [14] had proved a multiplicative reverse Young inequality with the Specht's ratio [13] as follows:

$$(1.2) \qquad (1-\nu)\,a+\nu b\leq S\left(\frac{a}{b}\right)a^{1-\nu}b^{\nu}$$

for $a,\,b>0$ and $\nu\in[0,1]$.

He also obtained the following additive reverse

$$(1.3) \qquad (1-\nu)\,a+\nu b-a^{1-\nu}b^{\nu}\leq L\,(a,b)\ln S\left(\frac{a}{b}\right)$$

for $a,\,b>0$ and $\nu\in[0,1]$, where $L\,(\cdot,\cdot)$ is the *logarithmic mean* defined by

$$L\,(a,b):=\left\{\begin{array}{l}\frac{b-a}{\ln b-\ln a}\text{ for }b\neq a,\\ \\ a\text{ if }b=a.\end{array}\right.$$

If $0<m\leq a,\,b\leq M$, then also [14]

$$(1.4) \qquad \left(a^{1-\nu}b^{\nu}\leq\right)(1-\nu)\,a+\nu b\leq S\left(\frac{M}{m}\right)a^{1-\nu}b^{\nu}$$

and

$$(1.5) \qquad (0\leq)\,(1-\nu)\,a+\nu b-a^{1-\nu}b^{\nu}\leq aL\left(1,\frac{M}{m}\right)\ln S\left(\frac{M}{m}\right)$$

for $\nu\in[0,1]$.

We consider the *Kantorovich's ratio* defined by

$$(1.6) \qquad K\,(h):=\frac{(h+1)^{2}}{4h},\ h>0.$$

The function K is decreasing on $(0,1)$ and increasing on $[1,\infty)$, $K\,(h)\geq1$ for any $h>0$ and $K\,(h)=K\left(\frac{1}{h}\right)$ for any $h>0$.

The following multiplicative refinement and reverse of Young inequality in terms of Kantorovich's ratio holds

$$(1.7) \qquad K^{r}\left(\frac{a}{b}\right)a^{1-\nu}b^{\nu}\leq(1-\nu)\,a+\nu b\leq K^{R}\left(\frac{a}{b}\right)a^{1-\nu}b^{\nu},$$

where $a,\,b>0$, $\nu\in[0,1]$, $r=\min\{1-\nu,\nu\}$ and $R=\max\{1-\nu,\nu\}$.

The first inequality in (1.7) was obtained by Zuo et al. in [16] while the second by Liao et al. [12].

We can give a simple direct proof for (1.7) as follows.

Recall the following result obtained by the author in 2006 [4] that provides a refinement and a reverse for the weighted Jensen's discrete inequality:

$$(1.8) \qquad \begin{aligned}0&\leq n\min_{j\in\{1,2,\dots,n\}}\{p_{j}\}\left[\frac{1}{n}\sum_{j=1}^{n}\Phi\,(x_{j})-\Phi\left(\frac{1}{n}\sum_{j=1}^{n}x_{j}\right)\right]\\&\leq\frac{1}{P_{n}}\sum_{j=1}^{n}p_{j}\Phi\,(x_{j})-\Phi\left(\frac{1}{P_{n}}\sum_{j=1}^{n}p_{j}x_{j}\right)\\&\leq n\max_{j\in\{1,2,\dots,n\}}\{p_{j}\}\left[\frac{1}{n}\sum_{j=1}^{n}\Phi\,(x_{j})-\Phi\left(\frac{1}{n}\sum_{j=1}^{n}x_{j}\right)\right],\end{aligned}$$

where $\Phi : C \to \mathbb{R}$ is a convex function defined on convex subset C of the linear space X, $\{x_j\}_{j\in\{1,2,...,n\}}$ are vectors in C and $\{p_j\}_{j\in\{1,2,...,n\}}$ are nonnegative numbers with $P_n = \sum_{j=1}^{n} p_j > 0$.

For $n = 2$, we deduce from (1.8) that

$$0 \le 2\min\{\nu, 1-\nu\}\left[\frac{\Phi(x)+\Phi(y)}{2} - \Phi\left(\frac{x+y}{2}\right)\right] \tag{1.9}$$
$$\le \nu\Phi(x) + (1-\nu)\Phi(y) - \Phi[\nu x + (1-\nu)y]$$
$$\le 2\max\{\nu, 1-\nu\}\left[\frac{\Phi(x)+\Phi(y)}{2} - \Phi\left(\frac{x+y}{2}\right)\right]$$

for any x, $y \in \mathbb{R}$ and $\nu \in [0,1]$.

Now, if we write the inequality (1.9) for the convex function $\Phi(x) = -\ln x$, and for the positive numbers a and b we get (1.7).

Let $I_1, ..., I_k$ be intervals from $\mathbb{R}$ and let $f : I_1 \times ... \times I_k \to \mathbb{R}$ be an essentially bounded real function defined on the product of the intervals. Let $A = (A_1, ..., A_n)$ be a k-tuple of bounded selfadjoint operators on Hilbert spaces $H_1, ..., H_k$ such that the spectrum of A_i is contained in I_i for $i = 1, ..., k$. We say that such a k-tuple is in the domain of f. If

$$A_i = \int_{I_i} \lambda_i dE_i(\lambda_i)$$

is the spectral resolution of A_i for $i = 1, ..., k$; by following [2], we define

$$f(A_1, ..., A_k) := \int_{I_1} ... \int_{I_k} f(\lambda_1, ..., \lambda_k)\, dE_1(\lambda_1) \otimes ... \otimes dE_k(\lambda_k) \tag{1.10}$$

as a bounded selfadjoint operator on the tensorial product $H_1 \otimes ... \otimes H_k$.

If the Hilbert spaces are of finite dimension, then the above integrals become finite sums, and we may consider the functional calculus for arbitrary real functions. This construction [2] extends the definition of Korányi [11] for functions of two variables and have the property that

$$f(A_1, ..., A_k) = f_1(A_1) \otimes ... \otimes f_k(A_k),$$

whenever f can be separated as a product $f(t_1, ..., t_k) = f_1(t_1)...f_k(t_k)$ of k functions each depending on only one variable.

It is know that, *if f is super-multiplicative (sub-multiplicative)* on $[0, \infty)$, namely

$$f(st) \ge (\le) f(s) f(t) \text{ for all } s, t \in [0, \infty)$$

and if f is continuous on $[0, \infty)$, then [6, p. 173]

$$f(A \otimes B) \ge (\le) f(A) \otimes f(B) \text{ for all } A,\ B \ge 0. \tag{1.11}$$

This follows by observing that, if

$$A = \int_{[0,\infty)} t dE(t) \text{ and } B = \int_{[0,\infty)} s dF(s)$$

are the spectral resolutions of A and B, then

$$f(A \otimes B) = \int_{[0,\infty)} \int_{[0,\infty)} f(st)\, dE(t) \otimes dF(s) \tag{1.12}$$

for the continuous function f on $[0, \infty)$.

Recall the *geometric operator mean* for the positive operators $A,\ B>0$

$$A\#_t B := A^{1/2}(A^{-1/2}BA^{-1/2})^t A^{1/2}$$

where $t\in[0,1]$ and

$$A\#B := A^{1/2}(A^{-1/2}BA^{-1/2})^{1/2}A^{1/2}.$$

By the definitions of $\#$ and $\otimes$ we have

$$A\#B = B\#A \text{ and } (A\#B)\otimes(B\#A) = (A\otimes B)\#(B\otimes A).$$

In 2007, S. Wada [15] obtained the following *Callebaut type inequalities* for tensorial product

$$(A\#B)\otimes(A\#B) \le \frac{1}{2}\left[(A\#_\alpha B)\otimes(A\#_{1-\alpha}B)+(A\#_{1-\alpha}B)\otimes(A\#_\alpha B)\right] \le \frac{1}{2}(A\otimes B+B\otimes A) \tag{1.13}$$

for $A,\ B>0$ and $\alpha\in[0,1]$.

Recall that the *Hadamard product* of A and B in $B(H)$ is defined to be the operator $A\circ B\in B(H)$ satisfying

$$\langle (A\circ B)e_j,e_j\rangle = \langle Ae_j,e_j\rangle\langle Be_j,e_j\rangle$$

for all $j\in\mathbb{N}$, where $\{e_j\}_{j\in\mathbb{N}}$ is an *orthonormal basis* for the separable Hilbert space H.

It is known that, see [5], we have the representation

$$A\circ B = \mathcal{U}^*(A\otimes B)\mathcal{U} \tag{1.14}$$

where $\mathcal{U}:H\to H\otimes H$ is the isometry defined by $\mathcal{U}e_j = e_j\otimes e_j$ for all $j\in\mathbb{N}$.

If f *is super-multiplicative (sub-multiplicative)* on $[0,\infty)$, then also [6, p. 173]

$$f(A\circ B)\ge(\le) f(A)\circ f(B) \text{ for all } A,\ B\ge 0. \tag{1.15}$$

We recall the following elementary inequalities for the Hadamard product

$$A^{1/2}\circ B^{1/2}\le\left(\frac{A+B}{2}\right)\circ 1 \text{ for } A,\ B\ge 0$$

and Fiedler inequality

$$A\circ A^{-1}\ge 1 \text{ for } A>0.$$

As extension of Kadison's Schwarz inequality on the Hadamard product, Ando [1] showed that

$$A\circ B\le\left(A^2\circ 1\right)^{1/2}\left(B^2\circ 1\right)^{1/2} \text{ for } A,\ B\ge 0$$

and Aujla and Vasudeva [3] gave an alternative upper bound

$$A\circ B\le\left(A^2\circ B^2\right)^{1/2} \text{ for } A,\ B\ge 0.$$

It has been shown in [10] that $\left(A^2\circ 1\right)^{1/2}\left(B^2\circ 1\right)^{1/2}$ and $\left(A^2\circ B^2\right)^{1/2}$ are incomparable for 2-square positive definite matrices A and B.

Let Ω be a locally compact Hausdorff space endowed with a Radon measure μ. A field $(A_t)_{t\in\Omega}$ of operators in $B(H)$ is called a continuous field of operators if the parametrization $t\longmapsto A_t$ is norm continuous on $B(H)$. If, in addition, the norm function $t\longmapsto\|A_t\|$ is Lebesgue integrable on Ω, we can form the Bochner integral $\int_\Omega A_t d\mu(t)$, which is the unique operator in $B(H)$ such that $\varphi\left(\int_\Omega A_t d\mu(t)\right) =$

$\int_\Omega \varphi(A_t)\, d\mu(t)$ for every bounded linear functional φ on $B(H)$. Assume also that, $\int_\Omega 1 d\mu(t) = 1$.

Motivated by the above results, in this paper we show among others that, if $(A_\tau)_{\tau\in\Omega}$ and $(B_\tau)_{\tau\in\Omega}$ are continuous fields of positive operators in $B(H)$ such that $\operatorname{Sp}(A_\tau)$, $\operatorname{Sp}(B_\tau) \subseteq [m, M] \subset (0, \infty)$ for each $\tau \in \Omega$, then for all $\nu \in [0, 1]$ we have the tensorial inequality

$$\begin{aligned}
&\left(\int_\Omega A_\tau^{1-\nu} d\mu(\tau)\right) \otimes \left(\int_\Omega B_\tau^{\nu} d\mu(\tau)\right) \\
&\le (1-\nu)\left(\int_\Omega A_\tau d\mu(\tau)\right) \otimes 1 + \nu 1 \otimes \left(\int_\Omega B_\tau d\mu(\tau)\right) \\
&\le K^R\left(\frac{M}{m}\right)\left(\int_\Omega A_\tau^{1-\nu} d\mu(\tau)\right) \otimes \left(\int_\Omega B_\tau^{\nu} d\mu(\tau)\right),
\end{aligned}$$

where $R = \max\{1-\nu, \nu\}$ and $K(\cdot)$ is *Kantorovich's ratio.* We also have the following inequalities for the Hadamard product

$$\begin{aligned}
\left(\int_\Omega A_\tau^{1-\nu} d\mu(\tau)\right) \circ \left(\int_\Omega B_\tau^{\nu} d\mu(\tau)\right) &\le \int_\Omega \left[(1-\nu) A_\tau + \nu B_\tau\right] d\mu(\tau) \circ 1 \\
&\le K^R\left(\frac{M}{m}\right)\left(\int_\Omega A_\tau^{1-\nu} d\mu(\tau)\right) \circ \left(\int_\Omega B_\tau^{\nu} d\mu(\tau)\right)
\end{aligned}$$

for all $\nu \in [0, 1]$.

2. Main Results

We have the following result for the tensorial product:

Lemma 1. *Assume that A and B are selfadjoint operators with $0 \le m \le A$, $B \le M$ for some constants $m < M$, then for all $\nu \in [0, 1]$*

$$A^{1-\nu} \otimes B^\nu \le (1-\nu) A \otimes 1 + \nu 1 \otimes B \le K^R\left(\frac{M}{m}\right) A^{1-\nu} \otimes B^\nu \tag{2.1}$$

and, in particular

$$A^{1-\nu} \otimes A^\nu \le (1-\nu) A \otimes 1 + \nu 1 \otimes A \le K^R\left(\frac{M}{m}\right) A^{1-\nu} \otimes A^\nu, \tag{2.2}$$

where $R = \max\{1-\nu, \nu\}$.

For $\nu = 1/2$ we derive that

$$A^{1/2} \otimes B^{1/2} \le \frac{1}{2}(A \otimes 1 + 1 \otimes B) \le \frac{M+m}{2\sqrt{mM}} A^{1/2} \otimes B^{1/2} \tag{2.3}$$

and, in particular

$$A^{1/2} \otimes A^{1/2} \le \frac{1}{2}(A \otimes 1 + 1 \otimes A) \le \frac{M+m}{2\sqrt{mM}} A^{1/2} \otimes A^{1/2}. \tag{2.4}$$

Proof. Let $t, s \in [m, M] \subset (0, \infty)$, then $\frac{m}{M} \le \frac{t}{s} \le \frac{M}{m}$ with $\frac{m}{M} < 1 < \frac{M}{m}$. If $\frac{t}{s} \in \left[\frac{m}{M}, 1\right)$ then $K\left(\frac{t}{s}\right) \le K\left(\frac{m}{M}\right) = K\left(\frac{M}{m}\right)$. If $\frac{t}{s} \in \left(1, \frac{M}{m}\right]$ then also $K\left(\frac{t}{s}\right) \le K\left(\frac{M}{m}\right)$. Therefore for any t, $s \in [m, M]$ we have from (1.7) that

$$t^{1-\nu} s^\nu \le (1-\nu) t + \nu s \le K^R\left(\frac{M}{m}\right) t^{1-\nu} s^\nu, \tag{2.5}$$

where $R = \max\{1-\nu, \nu\}$.

If

$$A=\int_m^M t dE(t) \text{ and } B=\int_m^M s dF(s)$$

are the spectral resolutions of A and B, then by taking the integral $\int_m^M \int_m^M$ over $dE(t)\otimes dF(s)$ in (2.5), we derive that

$$\begin{aligned}\int_m^M \int_m^M [(1-\nu)t+\nu s]\,dE(t)\otimes dF(s)\\ \leq K^R\left(\frac{M}{m}\right)\int_m^M \int_m^M t^{1-\nu}s^{\nu}dE(t)\otimes dF(s).\end{aligned}\tag{2.6}$$

Observe, by (1.10), that

$$\begin{aligned}&\int_m^M \int_m^M [(1-\nu)t+\nu s]\,dE(t)\otimes dF(s)\\ &=(1-\nu)\int_m^M \int_m^M t dE(t)\otimes dF(s)+\nu\int_m^M \int_m^M s dE(t)\otimes dF(s)\\ &=(1-\nu)A\otimes 1+\nu 1\otimes B\end{aligned}$$

and

$$\int_m^M \int_m^M t^{1-\nu}s^{\nu}dE(t)\otimes dF(s)=A^{1-\nu}\otimes B^{\nu}$$

and by (2.6) we derive (2.1). □

Corollary 1. *With the assumptions of Theorem 1 we have*

$$A^{1-\nu}\circ B^{\nu}\leq[(1-\nu)A+\nu B]\circ 1\leq K^R\left(\frac{M}{m}\right)A^{1-\nu}\circ B^{\nu}\tag{2.7}$$

and, in particular

$$A^{1-\nu}\circ A^{\nu}\leq A\circ 1\leq K^R\left(\frac{M}{m}\right)A^{1-\nu}\circ A^{\nu}.\tag{2.8}$$

For $\nu=1/2$ we derive that

$$A^{1/2}\circ B^{1/2}\leq\frac{A+B}{2}\circ 1\leq\frac{M+m}{2\sqrt{mM}}A^{1/2}\circ B^{1/2}\tag{2.9}$$

and, in particular

$$A^{1/2}\circ A^{1/2}\leq A\circ 1\leq\frac{M+m}{2\sqrt{mM}}A^{1/2}\circ A^{1/2}.\tag{2.10}$$

Proof. We have the representation

$$X\circ Y=\mathcal{U}^*(X\otimes Y)\mathcal{U}$$

where $\mathcal{U}:H\to H\otimes H$ is the isometry defined by $\mathcal{U}e_j=e_j\otimes e_j$ for all $j\in\mathbb{N}$.

If we take $\mathcal{U}^*$ at the left and $\mathcal{U}$ at the right in (2.1), then we get

$$\begin{aligned}\mathcal{U}^*\left(A^{1-\nu}\otimes B^{\nu}\right)\mathcal{U}&\leq(1-\nu)\mathcal{U}^*(A\otimes 1)\mathcal{U}+\nu\mathcal{U}^*(1\otimes B)\mathcal{U}\\ &\leq K^R\left(\frac{M}{m}\right)\mathcal{U}^*\left(A^{1-\nu}\otimes B^{\nu}\right)\mathcal{U},\end{aligned}$$

which gives

$$A^{1-\nu} \circ B^{\nu} \leq (1-\nu) A \circ 1 + \nu 1 \circ B \leq K^{R}\left(\frac{M}{m}\right) A^{1-\nu} \circ B^{\nu}$$

that is equivalent to (2.7). □

In what follows we assume that $\int_{\Omega} 1 d\mu(t) = 1$.

Theorem 1. *Let* $(A_\tau)_{\tau\in\Omega}$ *and* $(B_\tau)_{\tau\in\Omega}$ *be continuous fields of positive operators in* $B(H)$ *such that* $\operatorname{Sp}(A_\tau), \operatorname{Sp}(B_\tau) \subseteq [m, M] \subset (0, \infty)$ *for each* $\tau \in \Omega$. *Then for all* $\nu \in [0,1]$ *we have*

$$\begin{aligned}(2.11)\qquad &\left(\int_{\Omega} A_\tau^{1-\nu} d\mu(\tau)\right) \otimes \left(\int_{\Omega} B_\tau^{\nu} d\mu(\tau)\right)\\ &\leq (1-\nu)\left(\int_{\Omega} A_\tau d\mu(\tau)\right) \otimes 1 + \nu 1 \otimes \left(\int_{\Omega} B_\tau d\mu(\tau)\right)\\ &\leq K^{R}\left(\frac{M}{m}\right)\left(\int_{\Omega} A_\tau^{1-\nu} d\mu(\tau)\right) \otimes \left(\int_{\Omega} B_\tau^{\nu} d\mu(\tau)\right).\end{aligned}$$

In particular,

$$\begin{aligned}(2.12)\qquad &\left(\int_{\Omega} A_\tau^{1-\nu} d\mu(\tau)\right) \otimes \left(\int_{\Omega} A_\tau^{\nu} d\mu(\tau)\right)\\ &\leq (1-\nu)\left(\int_{\Omega} A_\tau d\mu(\tau)\right) \otimes 1 + \nu 1 \otimes \left(\int_{\Omega} A_\tau d\mu(\tau)\right)\\ &\leq K^{R}\left(\frac{M}{m}\right)\left(\int_{\Omega} A_\tau^{1-\nu} d\mu(\tau)\right) \otimes \left(\int_{\Omega} A_\tau^{\nu} d\mu(\tau)\right).\end{aligned}$$

Proof. From (2.1) we get

$$(2.13)\qquad A_\tau^{1-\nu} \otimes B_\gamma^{\nu} \leq (1-\nu) A_\tau \otimes 1 + \nu 1 \otimes B_\gamma \leq K^{R}\left(\frac{M}{m}\right) A_\tau^{1-\nu} \otimes B_\gamma^{\nu}$$

for all τ, $\gamma \in \Omega$. If we take the integral $\int_{\Omega}$ over $d\mu(\tau)$, then we get

$$\begin{aligned}(2.14)\qquad \int_{\Omega}\left(A_\tau^{1-\nu} \otimes B_\gamma^{\nu}\right) d\mu(\tau) &\leq \int_{\Omega}\left[(1-\nu) A_\tau \otimes 1 + \nu 1 \otimes B_\gamma\right] d\mu(\tau)\\ &\leq K^{R}\left(\frac{M}{m}\right) \int_{\Omega}\left(A_\tau^{1-\nu} \otimes B_\gamma^{\nu}\right) d\mu(\tau).\end{aligned}$$

Using the properties of the Bochner's integral and the tensorial product we have

$$\int_{\Omega}\left(A_\tau^{1-\nu} \otimes B_\gamma^{\nu}\right) d\mu(\tau) = \left(\int_{\Omega} A_\tau^{1-\nu} d\mu(\tau)\right) \otimes B_\gamma^{\nu}$$

and

$$\begin{aligned}&\int_{\Omega}\left[(1-\nu) A_\tau \otimes 1 + \nu 1 \otimes B_\gamma\right] d\mu(\tau)\\ &= (1-\nu)\left(\int_{\Omega} A_\tau d\mu(\tau)\right) \otimes 1 + \nu 1 \otimes B_\gamma\end{aligned}$$

for all $\gamma \in \Omega$.

From (2.14) we then get

$$\begin{aligned}
&\left(\int_{\Omega} A_{\tau}^{1-\nu} d\mu(\tau)\right) \otimes B_{\gamma}^{\nu} \\
&\leq (1-\nu)\left(\int_{\Omega} A_{\tau} d\mu(\tau)\right) \otimes 1+\nu 1 \otimes B_{\gamma} \\
&\leq K^{R}\left(\frac{M}{m}\right)\left(\int_{\Omega} A_{\tau}^{1-\nu} d\mu(\tau)\right) \otimes B_{\gamma}^{\nu}
\end{aligned} \tag{2.15}$$

for all $\gamma \in \Omega$.

If we take the integral $\int_{\Omega}$ over $d\mu(\gamma)$, then we get

$$\begin{aligned}
&\int_{\Omega}\left[\left(\int_{\Omega} A_{\tau}^{1-\nu} d\mu(\tau)\right) \otimes B_{\gamma}^{\nu}\right] d\mu(\gamma) \\
&\leq \int_{\Omega}\left[(1-\nu)\left(\int_{\Omega} A_{\tau} d\mu(\tau)\right) \otimes 1+\nu 1 \otimes B_{\gamma}\right] d\mu(\gamma) \\
&\leq K^{R}\left(\frac{M}{m}\right) \int_{\Omega}\left[\left(\int_{\Omega} A_{\tau}^{1-\nu} d\mu(\tau)\right) \otimes B_{\gamma}^{\nu}\right] d\mu(\gamma) .
\end{aligned} \tag{2.16}$$

Since

$$\begin{aligned}
&\int_{\Omega}\left[\left(\int_{\Omega} A_{\tau}^{1-\nu} d\mu(\tau)\right) \otimes B_{\gamma}^{\nu}\right] d\mu(\gamma) \\
&=\left(\int_{\Omega} A_{\tau}^{1-\nu} d\mu(\tau)\right) \otimes\left(\int_{\Omega} B_{\gamma}^{\nu} d\mu(\gamma)\right)
\end{aligned}$$

and

$$\begin{aligned}
&\int_{\Omega}\left[(1-\nu)\left(\int_{\Omega} A_{\tau} d\mu(\tau)\right) \otimes 1+\nu 1 \otimes B_{\gamma}\right] d\mu(\gamma) \\
&=(1-\nu)\left(\int_{\Omega} A_{\tau} d\mu(\tau)\right) \otimes 1+\nu 1 \otimes\left(\int_{\Omega} B_{\gamma} d\mu(\gamma)\right),
\end{aligned}$$

hence by (2.16) we derive (2.11). □

Remark 1. *If we take in (2.11) $\nu=1/2$, then we get*

$$\begin{aligned}
&\left(\int_{\Omega} A_{\tau}^{1/2} d\mu(\tau)\right) \otimes\left(\int_{\Omega} B_{\tau}^{1/2} d\mu(\tau)\right) \\
&\leq \frac{1}{2}\left[\left(\int_{\Omega} A_{\tau} d\mu(\tau)\right) \otimes 1+1 \otimes\left(\int_{\Omega} B_{\tau} d\mu(\tau)\right)\right] \\
&\leq \frac{M+m}{2\sqrt{mM}}\left(\int_{\Omega} A_{\tau}^{1/2} d\mu(\tau)\right) \otimes\left(\int_{\Omega} B_{\tau}^{1/2} d\mu(\tau)\right).
\end{aligned} \tag{2.17}$$

In particular,

$$\begin{aligned}
&\left(\int_{\Omega} A_{\tau}^{1/2} d\mu(\tau)\right) \otimes\left(\int_{\Omega} A_{\tau}^{1/2} d\mu(\tau)\right) \\
&\leq \frac{1}{2}\left[\left(\int_{\Omega} A_{\tau} d\mu(\tau)\right) \otimes 1+1 \otimes\left(\int_{\Omega} A_{\tau} d\mu(\tau)\right)\right] \\
&\leq \frac{M+m}{2\sqrt{mM}}\left(\int_{\Omega} A_{\tau}^{1/2} d\mu(\tau)\right) \otimes\left(\int_{\Omega} A_{\tau}^{1/2} d\mu(\tau)\right).
\end{aligned} \tag{2.18}$$

We have the following result for the Hadamard product:

Corollary 2. *With the assumptions of Theorem 1, we have*

$$\begin{aligned}(2.19)\qquad & \left(\int_{\Omega} A_{\tau}^{1-\nu} d\mu(\tau)\right) \circ \left(\int_{\Omega} B_{\tau}^{\nu} d\mu(\tau)\right) \\ & \leq \int_{\Omega} \left((1-\nu) A_{\tau} + \nu B_{\tau}\right) d\mu(\tau) \circ 1 \\ & \leq K^{R}\left(\frac{M}{m}\right) \left(\int_{\Omega} A_{\tau}^{1-\nu} d\mu(\tau)\right) \circ \left(\int_{\Omega} B_{\tau}^{\nu} d\mu(\tau)\right).\end{aligned}$$

In particular,

$$\begin{aligned}(2.20)\qquad & \left(\int_{\Omega} A_{\tau}^{1-\nu} d\mu(\tau)\right) \circ \left(\int_{\Omega} A_{\tau}^{\nu} d\mu(\tau)\right) \\ & \leq \int_{\Omega} A_{\tau} d\mu(\tau) \circ 1 \\ & \leq K^{R}\left(\frac{M}{m}\right) \left(\int_{\Omega} A_{\tau}^{1-\nu} d\mu(\tau)\right) \circ \left(\int_{\Omega} B_{\tau}^{\nu} d\mu(\gamma)\right).\end{aligned}$$

Proof. If we use the identity (1.14) and apply $\mathcal{U}^*$ to the left and $\mathcal{U}$ to the right of inequality (2.1), we get

$$\begin{aligned}(2.21)\qquad & \mathcal{U}^*\left[\left(\int_{\Omega} A_{\tau}^{1-\nu} d\mu(\tau)\right) \otimes \left(\int_{\Omega} B_{\gamma}^{\nu} d\mu(\gamma)\right)\right]\mathcal{U} \\ & \leq \mathcal{U}^*\left[(1-\nu)\left(\int_{\Omega} A_{\tau} d\mu(\tau)\right) \otimes 1 + \nu 1 \otimes \left(\int_{\Omega} B_{\gamma} d\mu(\gamma)\right)\right]\mathcal{U} \\ & \leq K^{R}\left(\frac{M}{m}\right) \mathcal{U}^*\left[\left(\int_{\Omega} A_{\tau}^{1-\nu} d\mu(\tau)\right) \otimes \left(\int_{\Omega} B_{\gamma}^{\nu} d\mu(\gamma)\right)\right]\mathcal{U}.\end{aligned}$$

Since

$$\begin{aligned} & \mathcal{U}^*\left[\left(\int_{\Omega} A_{\tau}^{1-\nu} d\mu(\tau)\right) \otimes \left(\int_{\Omega} B_{\gamma}^{\nu} d\mu(\gamma)\right)\right]\mathcal{U} \\ & = \left(\int_{\Omega} A_{\tau}^{1-\nu} d\mu(\tau)\right) \circ \left(\int_{\Omega} B_{\gamma}^{\nu} d\mu(\gamma)\right)\end{aligned}$$

and

$$\begin{aligned} & \mathcal{U}^*\left[(1-\nu)\left(\int_{\Omega} A_{\tau} d\mu(\tau)\right) \otimes 1 + \nu 1 \otimes \left(\int_{\Omega} B_{\gamma} d\mu(\gamma)\right)\right]\mathcal{U} \\ & = (1-\nu)\mathcal{U}^*\left[\left(\int_{\Omega} A_{\tau} d\mu(\tau)\right) \otimes 1\right]\mathcal{U} \\ & + \nu\mathcal{U}^*\left[1 \otimes \left(\int_{\Omega} B_{\gamma} d\mu(\gamma)\right)\right]\mathcal{U},\end{aligned}$$

hence by (2.21), we derive (2.19). □

Remark 2. *If we take $\nu = 1/2$ in (2.19), then we get*

$$\left(\int_\Omega A_\tau^{1/2} d\mu(\tau)\right) \circ \left(\int_\Omega B_\tau^{1/2} d\mu(\tau)\right) \tag{2.22}$$
$$\leq \int_\Omega \left(\frac{A_\tau + B_\tau}{2}\right) d\mu(\tau) \circ 1$$
$$\leq \frac{M+m}{2\sqrt{mM}} \left(\int_\Omega A_\tau^{1/2} d\mu(\tau)\right) \circ \left(\int_\Omega B_\tau^{1/2} d\mu(\tau)\right).$$

In particular,

$$\left(\int_\Omega A_\tau^{1/2} d\mu(\tau)\right) \circ \left(\int_\Omega A_\tau^{1/2} d\mu(\tau)\right) \tag{2.23}$$
$$\leq \int_\Omega A_\tau d\mu(\tau) \circ 1$$
$$\leq \frac{M+m}{2\sqrt{mM}} \left(\int_\Omega A_\tau^{1/2} d\mu(\tau)\right) \circ \left(\int_\Omega A_\tau^{1/2} d\mu(\gamma)\right).$$

3. Related Results

We have:

Lemma 2. *Let I and J be two intervals and f, g defined and continuous on an interval containing $I \cup J$. Assume that*

$$0 < \gamma_1 \leq \frac{f(t)}{g(t)} \leq \Gamma_1 \text{ for } t \in I \tag{3.1}$$

and

$$0 < \gamma_2 \leq \frac{f(s)}{g(s)} \leq \Gamma_2 \text{ for } s \in J. \tag{3.2}$$

Define

$$U(\gamma_1, \Gamma_1, \gamma_2, \Gamma_2) := \begin{cases} K\left(\frac{\Gamma_1}{\gamma_2}\right) \text{ if } 1 \leq \frac{\gamma_1}{\Gamma_2}, \\ \\ \max\left\{K\left(\frac{\Gamma_1}{\gamma_2}\right), K\left(\frac{\gamma_1}{\Gamma_2}\right)\right\} \\ \text{if } \frac{\gamma_1}{\Gamma_2} < 1 < \frac{\Gamma_1}{\gamma_2}, \\ \\ K\left(\frac{\gamma_1}{\Gamma_2}\right) \text{ if } \frac{\Gamma_1}{\gamma_2} \leq 1, \end{cases}$$

and

$$u(\gamma_1, \Gamma_1, \gamma_2, \Gamma_2) = \begin{cases} K\left(\frac{\gamma_1}{\Gamma_2}\right) \text{ if } 1 \leq \frac{\gamma_1}{\Gamma_2}, \\ \\ 1 \text{ if } \frac{\gamma_1}{\Gamma_2} < 1 < \frac{\Gamma_1}{\gamma_2}, \\ \\ K\left(\frac{\Gamma_1}{\gamma_2}\right) \text{ if } \frac{\Gamma_1}{\gamma_2} \leq 1. \end{cases}$$

If A and B are selfadjoint operators with $\operatorname{Sp}(A) \subset I$ *and* $\operatorname{Sp}(B) \subset J$, *then*

$$\begin{aligned} u^r(\gamma_1, \Gamma_1, \gamma_2, \Gamma_2) & \left[f^{1-\nu}(A) g^\nu(A)\right] \otimes \left[f^\nu(B) g^{1-\nu}(B)\right] \\ & \leq (1-\nu) f(A) \otimes g(B) + \nu g(A) \otimes f(B) \\ & \leq U^R(\gamma_1, \Gamma_1, \gamma_2, \Gamma_2) \left[f^{1-\nu}(A) g^\nu(A)\right] \otimes \left[f^\nu(B) g^{1-\nu}(B)\right] \end{aligned} \tag{3.3}$$

for $\nu \in [0,1]$, *where* $r = \min\{1-\nu, \nu\}$ *and* $R = \max\{1-\nu, \nu\}$.

In particular,

$$\begin{aligned} u^{1/2}(\gamma_1, \Gamma_1, \gamma_2, \Gamma_2) & \left[f^{1/2}(A) g^{1/2}(A)\right] \otimes \left[f^{1/2}(B) g^{1/2}(B)\right] \\ & \leq \frac{1}{2}\left[f(A) \otimes g(B) + g(A) \otimes f(B)\right] \\ & \leq U^{1/2}(\gamma_1, \Gamma_1, \gamma_2, \Gamma_2) \left[f^{1/2}(A) g^{1/2}(A)\right] \otimes \left[f^{1/2}(B) g^{1/2}(B)\right]. \end{aligned} \tag{3.4}$$

Proof. If $a \in [\gamma_1, \Gamma_1] \subset (0, \infty)$ and $b \in [\gamma_2, \Gamma_2] \subset (0, \infty)$, then

$$\frac{a}{b} \in \left[\frac{\gamma_1}{\Gamma_2}, \frac{\Gamma_1}{\gamma_2}\right] \subset (0, \infty).$$

The function K is decreasing on $(0,1)$ and increasing on $[1, \infty)$, then we observe that

$$\max_{\tau \in \left[\frac{\gamma_1}{\Gamma_2}, \frac{\Gamma_1}{\gamma_2}\right]} K(\tau) = U(\gamma_1, \Gamma_1, \gamma_2, \Gamma_2)$$

and

$$\min_{\tau \in \left[\frac{\gamma_1}{\Gamma_2}, \frac{\Gamma_1}{\gamma_2}\right]} K(\tau) = u(\gamma_1, \Gamma_1, \gamma_2, \Gamma_2).$$

By (1.7) we then get

$$\begin{aligned} & u^r(\gamma_1, \Gamma_1, \gamma_2, \Gamma_2) a^{1-\nu} b^\nu \\ & \leq K^r\left(\frac{a}{b}\right) a^{1-\nu} b^\nu \leq (1-\nu) a + \nu b \\ & \leq K^R\left(\frac{a}{b}\right) a^{1-\nu} b^\nu \leq U^R(\gamma_1, \Gamma_1, \gamma_2, \Gamma_2) a^{1-\nu} b^\nu, \end{aligned} \tag{3.5}$$

where $r = \min\{1-\nu, \nu\}$ and $R = \max\{1-\nu, \nu\}$.

Now, if we take

$$a = \frac{f(t)}{g(t)},\ t \in I \text{ and } b = \frac{f(s)}{g(s)},\ s \in J$$

in (3.5), then we get

$$\begin{aligned} & u^r(\gamma_1, \Gamma_1, \gamma_2, \Gamma_2) \left(\frac{f(t)}{g(t)}\right)^{1-\nu} \left(\frac{f(s)}{g(s)}\right)^\nu \\ & \leq (1-\nu) \frac{f(t)}{g(t)} + \nu \frac{f(s)}{g(s)} \\ & \leq U^R(\gamma_1, \Gamma_1, \gamma_2, \Gamma_2) \left(\frac{f(t)}{g(t)}\right)^{1-\nu} \left(\frac{f(s)}{g(s)}\right)^\nu, \end{aligned} \tag{3.6}$$

for $t \in I$ and $s \in J$.

This is equivalent to

$$u^{r}\left(\gamma_{1},\Gamma_{1},\gamma_{2},\Gamma_{2}\right)f^{1-\nu}\left(t\right)g^{\nu}\left(t\right)f^{\nu}\left(s\right)g^{1-\nu}\left(s\right) \tag{3.7}$$
$$\leq\left(1-\nu\right)f\left(t\right)g\left(s\right)+\nu g\left(t\right)f\left(s\right)$$
$$\leq U^{R}\left(\gamma_{1},\Gamma_{1},\gamma_{2},\Gamma_{2}\right)f^{1-\nu}\left(t\right)g^{\nu}\left(t\right)f^{\nu}\left(s\right)g^{1-\nu}\left(s\right),$$

for $t\in I$ and $s\in J$.

If

$$A=\int_{I}tdE\left(t\right)\text{ and }B=\int_{J}sdF\left(s\right)$$

are the spectral resolutions of A and B, then by taking the integral $\int_{I}\int_{J}$ over $dE\left(t\right)\otimes dF\left(s\right)$ in (3.7), we derive that

$$u^{r}\left(\gamma_{1},\Gamma_{1},\gamma_{2},\Gamma_{2}\right)\int_{I}\int_{J}f^{1-\nu}\left(t\right)g^{\nu}\left(t\right)f^{\nu}\left(s\right)g^{1-\nu}\left(s\right)dE\left(t\right)\otimes dF\left(s\right) \tag{3.8}$$
$$\leq\int_{I}\int_{J}\left[\left(1-\nu\right)f\left(t\right)g\left(s\right)+\nu g\left(t\right)f\left(s\right)\right]dE\left(t\right)\otimes dF\left(s\right)$$
$$\leq U^{R}\left(\gamma_{1},\Gamma_{1},\gamma_{2},\Gamma_{2}\right)\int_{I}\int_{J}f^{1-\nu}\left(t\right)g^{\nu}\left(t\right)f^{\nu}\left(s\right)g^{1-\nu}\left(s\right)dE\left(t\right)\otimes dF\left(s\right).$$

By utilizing (1.10) we get

$$\int_{I}\int_{J}f^{1-\nu}\left(t\right)g^{\nu}\left(t\right)f^{\nu}\left(s\right)g^{1-\nu}\left(s\right)dE\left(t\right)\otimes dF\left(s\right)$$
$$=\left[f^{1-\nu}\left(A\right)g^{\nu}\left(A\right)\right]\otimes\left[f^{\nu}\left(B\right)g^{1-\nu}\left(B\right)\right]$$

and

$$\int_{I}\int_{J}\left[\left(1-\nu\right)f\left(t\right)g\left(s\right)+\nu g\left(t\right)f\left(s\right)\right]dE\left(t\right)\otimes dF\left(s\right)$$
$$=\left(1-\nu\right)\int_{I}\int_{J}f\left(t\right)g\left(s\right)dE\left(t\right)\otimes dF\left(s\right)+\nu\int_{I}\int_{J}g\left(t\right)f\left(s\right)dE\left(t\right)\otimes dF\left(s\right)$$
$$=\left(1-\nu\right)f\left(A\right)\otimes g\left(B\right)+\nu g\left(A\right)\otimes f\left(B\right).$$

Therefore, by (3.8) we obtain the desired result (3.3). □

Corollary 3. *With the assumptions of Lemma 2,*

$$u^{r}\left(\gamma_{1},\Gamma_{1},\gamma_{2},\Gamma_{2}\right)\left[f^{1-\nu}\left(A\right)g^{\nu}\left(A\right)\right]\circ\left[f^{\nu}\left(B\right)g^{1-\nu}\left(B\right)\right] \tag{3.9}$$
$$\leq\left(1-\nu\right)f\left(A\right)\circ g\left(B\right)+\nu g\left(A\right)\circ f\left(B\right)$$
$$\leq U^{R}\left(\gamma_{1},\Gamma_{1},\gamma_{2},\Gamma_{2}\right)\left[f^{1-\nu}\left(A\right)g^{\nu}\left(A\right)\right]\circ\left[f^{\nu}\left(B\right)g^{1-\nu}\left(B\right)\right]$$

for $\nu\in[0,1]$.

In particular,

$$u^{1/2}\left(\gamma_{1},\Gamma_{1},\gamma_{2},\Gamma_{2}\right)\left[f^{1/2}\left(A\right)g^{1/2}\left(A\right)\right]\circ\left[f^{1/2}\left(B\right)g^{1/2}\left(B\right)\right] \tag{3.10}$$
$$\leq\frac{1}{2}\left[f\left(A\right)\circ g\left(B\right)+g\left(A\right)\circ f\left(B\right)\right]$$
$$\leq U^{1/2}\left(\gamma_{1},\Gamma_{1},\gamma_{2},\Gamma_{2}\right)\left[f^{1/2}\left(A\right)g^{1/2}\left(A\right)\right]\circ\left[f^{1/2}\left(B\right)g^{1/2}\left(B\right)\right].$$

Proof. We have the representation

$$X \circ Y = \mathcal{U}^* (X \otimes Y) \mathcal{U},$$

where $\mathcal{U} : H \to H \otimes H$ is the isometry defined by $\mathcal{U} e_j = e_j \otimes e_j$ for all $j \in \mathbb{N}$.

If we take $\mathcal{U}^*$ at the left and $\mathcal{U}$ at the right in (3.3), then we get

$$\begin{aligned} & u^r (\gamma_1, \Gamma_1, \gamma_2, \Gamma_2) \mathcal{U}^* \left(\left[f^{1-\nu} (A) g^\nu (A) \right] \otimes \left[f^\nu (B) g^{1-\nu} (B) \right] \right) \mathcal{U} \\ & \leq \mathcal{U}^* \left[(1-\nu) f (A) \otimes g (B) + \nu g (A) \otimes f (B) \right] \mathcal{U} \\ & \leq U^R (\gamma_1, \Gamma_1, \gamma_2, \Gamma_2) \mathcal{U}^* \left(\left[f^{1-\nu} (A) g^\nu (A) \right] \otimes \left[f^\nu (B) g^{1-\nu} (B) \right] \right) \mathcal{U}, \end{aligned}$$

namely

$$\begin{aligned} & u^r (\gamma_1, \Gamma_1, \gamma_2, \Gamma_2) \mathcal{U}^* \left(\left[f^{1-\nu} (A) g^\nu (A) \right] \otimes \left[f^\nu (B) g^{1-\nu} (B) \right] \right) \mathcal{U} \\ & \leq (1-\nu) \mathcal{U}^* \left[f (A) \otimes g (B) \right] \mathcal{U} + \nu \mathcal{U}^* \left[g (A) \otimes f (B) \right] \mathcal{U} \\ & \leq U^R (\gamma_1, \Gamma_1, \gamma_2, \Gamma_2) \mathcal{U}^* \left(\left[f^{1-\nu} (A) g^\nu (A) \right] \otimes \left[f^\nu (B) g^{1-\nu} (B) \right] \right) \mathcal{U}, \end{aligned}$$

which is equivalent to

$$\begin{aligned} & u^r (\gamma_1, \Gamma_1, \gamma_2, \Gamma_2) \mathcal{U}^* \left(\left[f^{1-\nu} (A) g^\nu (A) \right] \circ \left[f^\nu (B) g^{1-\nu} (B) \right] \right) \mathcal{U} \\ & \leq (1-\nu) \mathcal{U}^* \left[f (A) \circ g (B) \right] \mathcal{U} + \nu \mathcal{U}^* \left[g (A) \circ f (B) \right] \mathcal{U} \\ & \leq U^R (\gamma_1, \Gamma_1, \gamma_2, \Gamma_2) \mathcal{U}^* \left(\left[f^{1-\nu} (A) g^\nu (A) \right] \circ \left[f^\nu (B) g^{1-\nu} (B) \right] \right) \mathcal{U} \end{aligned}$$

and the inequality (3.9) is obtained. □

Corollary 4. *Assume that f, g are continuous on I and*

$$0 < \gamma \leq \frac{f(t)}{g(t)} \leq \Gamma \text{ for } t \in I.$$

If A and B are selfadjoint operators with $\operatorname{Sp}(A), \operatorname{Sp}(B) \subset I$, then

$$\begin{aligned} & \left[f^{1-\nu} (A) g^\nu (A) \right] \otimes \left[f^\nu (B) g^{1-\nu} (B) \right] \\ & \leq (1-\nu) f (A) \otimes g (B) + \nu g (A) \otimes f (B) \\ & \leq \left[\frac{(\gamma + \Gamma)^2}{4 \gamma \Gamma} \right]^R \left[f^{1-\nu} (A) g^\nu (A) \right] \otimes \left[f^\nu (B) g^{1-\nu} (B) \right]. \end{aligned} \tag{3.11}$$

In particular,

$$\begin{aligned} & \left[f^{1/2} (A) g^{1/2} (A) \right] \otimes \left[f^{1/2} (B) g^{1/2} (B) \right] \\ & \leq \frac{1}{2} \left[f (A) \otimes g (B) + g (A) \otimes f (B) \right] \\ & \leq \frac{\gamma + \Gamma}{2 \sqrt{\gamma \Gamma}} \left[f^{1/2} (A) g^{1/2} (A) \right] \otimes \left[f^{1/2} (B) g^{1/2} (B) \right]. \end{aligned} \tag{3.12}$$

We also have for $B = A$ that

$$\begin{aligned} & \left[f^{1-\nu} (A) g^\nu (A) \right] \otimes \left[f^\nu (A) g^{1-\nu} (A) \right] \\ & \leq (1-\nu) f (A) \otimes g (A) + \nu g (A) \otimes f (A) \\ & \leq \left[\frac{(\gamma + \Gamma)^2}{4 \gamma \Gamma} \right]^R \left[f^{1-\nu} (A) g^\nu (A) \right] \otimes \left[f^\nu (A) g^{1-\nu} (A) \right]. \end{aligned} \tag{3.13}$$

In particular,

$$\begin{aligned}
&\left[f^{1/2}(A)\,g^{1/2}(A)\right]\otimes\left[f^{1/2}(A)\,g^{1/2}(A)\right] \\
&\leq \frac{1}{2}\left[f(A)\otimes g(A)+g(A)\otimes f(A)\right] \\
&\leq \frac{\gamma+\Gamma}{2\sqrt{\gamma\Gamma}}\left[f^{1/2}(A)\,g^{1/2}(A)\right]\otimes\left[f^{1/2}(A)\,g^{1/2}(A)\right].
\end{aligned}\tag{3.14}$$

The proof follows by taking $\gamma_1=\gamma_2=\gamma$ and $\Gamma_1=\Gamma_2=\Gamma$ in Lemma 2.

Remark 3. *With the assumptions of Corollary 4 we have the following inequalities for the Hadamard product*

$$\begin{aligned}
&\left[f^{1-\nu}(A)\,g^{\nu}(A)\right]\circ\left[f^{\nu}(B)\,g^{1-\nu}(B)\right] \\
&\leq (1-\nu)\,f(A)\circ g(B)+\nu g(A)\circ f(B) \\
&\leq \left[\frac{(\gamma+\Gamma)^2}{4\gamma\Gamma}\right]^{R}\left[f^{1-\nu}(A)\,g^{\nu}(A)\right]\circ\left[f^{\nu}(B)\,g^{1-\nu}(B)\right].
\end{aligned}\tag{3.15}$$

In particular,

$$\begin{aligned}
&\left[f^{1/2}(A)\,g^{1/2}(A)\right]\circ\left[f^{1/2}(B)\,g^{1/2}(B)\right] \\
&\leq \frac{1}{2}\left[f(A)\circ g(B)+g(A)\circ f(B)\right] \\
&\leq \frac{\gamma+\Gamma}{2\sqrt{\gamma\Gamma}}\left[f^{1/2}(A)\,g^{1/2}(A)\right]\otimes\left[f^{1/2}(B)\,g^{1/2}(B)\right].
\end{aligned}\tag{3.16}$$

We also have for $B=A$ that

$$\begin{aligned}
&\left[f^{1-\nu}(A)\,g^{\nu}(A)\right]\circ\left[f^{\nu}(A)\,g^{1-\nu}(A)\right] \\
&\leq f(A)\circ g(A) \\
&\leq \left[\frac{(\gamma+\Gamma)^2}{4\gamma\Gamma}\right]^{R}\left[f^{1-\nu}(A)\,g^{\nu}(A)\right]\circ\left[f^{\nu}(A)\,g^{1-\nu}(A)\right].
\end{aligned}\tag{3.17}$$

In particular,

$$\begin{aligned}
&\left[f^{1/2}(A)\,g^{1/2}(A)\right]\circ\left[f^{1/2}(A)\,g^{1/2}(A)\right] \\
&\leq f(A)\circ g(A) \\
&\leq \frac{\gamma+\Gamma}{2\sqrt{\gamma\Gamma}}\left[f^{1/2}(A)\,g^{1/2}(A)\right]\circ\left[f^{1/2}(A)\,g^{1/2}(A)\right].
\end{aligned}\tag{3.18}$$

We also have the following result for two functions and two fields of operators:

Theorem 2. *Let $(A_\tau)_{\tau\in\Omega}$ and $(B_\tau)_{\tau\in\Omega}$be continuous fields of positive operators in $B(H)$ such that $\mathrm{Sp}(A_\tau)\subset I$, $\mathrm{Sp}(B_\tau)\subset J\subset(0,\infty)$ for each $\tau\in\Omega$. Assume that f, g are defined and continuous on an interval containing $I\cup J$ and satisfy the* *boundedness conditions (3.1) and (3.2), then for all $\nu \in [0,1]$ we have*

$$
\begin{aligned}
(3.19)\qquad & u^{r}\left(\gamma_1,\Gamma_1,\gamma_2,\Gamma_2\right)\\
&\times\left[\int_\Omega f^{1-\nu}\left(A_\tau\right)g^{\nu}\left(A_\tau\right)d\mu\left(\tau\right)\right]\otimes\left[\int_\Omega f^{\nu}\left(B_\tau\right)g^{1-\nu}\left(B_\tau\right)d\mu\left(\tau\right)\right]\\
&\le\left(1-\nu\right)\int_\Omega f\left(A_\tau\right)d\mu\left(\tau\right)\otimes\int_\Omega g\left(B_\tau\right)d\mu\left(\tau\right)\\
&+\nu\int_\Omega g\left(A_\tau\right)d\mu\left(\tau\right)\otimes\int_\Omega f\left(B_\tau\right)d\mu\left(\tau\right)\\
&\le U^{R}\left(\gamma_1,\Gamma_1,\gamma_2,\Gamma_2\right)\\
&\times\left[\int_\Omega f^{1-\nu}\left(A_\tau\right)g^{\nu}\left(A_\tau\right)d\mu\left(\tau\right)\right]\otimes\left[\int_\Omega f^{\nu}\left(B_\tau\right)g^{1-\nu}\left(B_\tau\right)d\mu\left(\tau\right)\right]
\end{aligned}
$$

for $\nu \in [0,1]$, where $r=\min\{1-\nu,\nu\}$ and $R=\max\{1-\nu,\nu\}$.

In particular, for $\nu = 1/2$

$$
\begin{aligned}
(3.20)\qquad & u^{1/2}\left(\gamma_1,\Gamma_1,\gamma_2,\Gamma_2\right)\\
&\times\left[\int_\Omega f^{1/2}\left(A_\tau\right)g^{1/2}\left(A_\tau\right)d\mu\left(\tau\right)\right]\otimes\left[\int_\Omega f^{1/2}\left(B_\tau\right)g^{1/2}\left(B_\tau\right)d\mu\left(\tau\right)\right]\\
&\le\frac{1}{2}\left[\int_\Omega f\left(A_\tau\right)d\mu\left(\tau\right)\otimes\int_\Omega g\left(B_\tau\right)d\mu\left(\tau\right)\right.\\
&\left.+\int_\Omega g\left(A_\tau\right)d\mu\left(\tau\right)\otimes\int_\Omega f\left(B_\tau\right)d\mu\left(\tau\right)\right]\\
&\le U^{1/2}\left(\gamma_1,\Gamma_1,\gamma_2,\Gamma_2\right)\\
&\times\left[\int_\Omega f^{1/2}\left(A_\tau\right)g^{1/2}\left(A_\tau\right)d\mu\left(\tau\right)\right]\otimes\left[\int_\Omega f^{1/2}\left(B_\tau\right)g^{1/2}\left(B_\tau\right)d\mu\left(\tau\right)\right].
\end{aligned}
$$

Proof. From (3.3) we get

$$
\begin{aligned}
& u^{r}\left(\gamma_1,\Gamma_1,\gamma_2,\Gamma_2\right)\left[f^{1-\nu}\left(A_\tau\right)g^{\nu}\left(A_\tau\right)\right]\otimes\left[f^{\nu}\left(B_\gamma\right)g^{1-\nu}\left(B_\gamma\right)\right]\\
&\le\left(1-\nu\right)f\left(A_\tau\right)\otimes g\left(B_\gamma\right)+\nu g\left(A_\tau\right)\otimes f\left(B_\gamma\right)\\
&\le U^{R}\left(\gamma_1,\Gamma_1,\gamma_2,\Gamma_2\right)\left[f^{1-\nu}\left(A_\tau\right)g^{\nu}\left(A_\tau\right)\right]\otimes\left[f^{\nu}\left(B_\gamma\right)g^{1-\nu}\left(B_\gamma\right)\right]
\end{aligned}
$$

for all τ, $\gamma \in \Omega$.

If we take the integral $\int_\Omega$ over $d\mu(\tau)$, then the integral $\int_\Omega$ over $d\mu(\gamma)$ and using the properties of the tensorial product versus the integral $\int_\Omega$, we deduce the desired result (3.19). □

Corollary 5. *Let $(A_\tau)_{\tau\in\Omega}$ be a continuous field of positive operators in $B(H)$ such that $\operatorname{Sp}(A_\tau) \subset I \subset (0,\infty)$ for each $\tau \in \Omega$. Assume that f, g are defined and continuous on an interval containing I and satisfy the boundedness conditions*

(3.1), then for all $\nu \in [0,1]$ *we have*

$$\begin{aligned}(3.21)\qquad & \left[\int_{\Omega} f^{1-\nu}(A_\tau)\, g^{\nu}(A_\tau)\, d\mu(\tau)\right] \otimes \left[\int_{\Omega} f^{\nu}(A_\tau)\, g^{1-\nu}(A_\tau)\, d\mu(\tau)\right] \\ & \leq (1-\nu) \int_{\Omega} f(A_\tau)\, d\mu(\tau) \otimes \int_{\Omega} g(A_\tau)\, d\mu(\tau) \\ & + \nu \int_{\Omega} g(A_\tau)\, d\mu(\tau) \otimes \int_{\Omega} f(A_\tau)\, d\mu(\tau) \\ & \leq \left[\frac{(\gamma+\Gamma)^2}{4\gamma\Gamma}\right]^R \\ & \times \left[\int_{\Omega} f^{1-\nu}(A_\tau)\, g^{\nu}(A_\tau)\, d\mu(\tau)\right] \otimes \left[\int_{\Omega} f^{\nu}(A_\tau)\, g^{1-\nu}(A_\tau)\, d\mu(\tau)\right]\end{aligned}$$

for $\nu \in [0,1]$, *where* $R = \max\{1-\nu, \nu\}$.

In particular, for $\nu = 1/2$

$$\begin{aligned}(3.22)\qquad & \left[\int_{\Omega} f^{1/2}(A_\tau)\, g^{1/2}(A_\tau)\, d\mu(\tau)\right] \otimes \left[\int_{\Omega} f^{1/2}(A_\tau)\, g^{1/2}(A_\tau)\, d\mu(\tau)\right] \\ & \leq \frac{1}{2}\left[\int_{\Omega} f(A_\tau)\, d\mu(\tau) \otimes \int_{\Omega} g(A_\tau)\, d\mu(\tau)\right. \\ & \left. + \int_{\Omega} g(A_\tau)\, d\mu(\tau) \otimes \int_{\Omega} f(A_\tau)\, d\mu(\tau)\right] \\ & \leq \frac{\gamma+\Gamma}{2\sqrt{\gamma\Gamma}} \\ & \times \left[\int_{\Omega} f^{1/2}(A_\tau)\, g^{1/2}(A_\tau)\, d\mu(\tau)\right] \otimes \left[\int_{\Omega} f^{1/2}(A_\tau)\, g^{1/2}(A_\tau)\, d\mu(\tau)\right].\end{aligned}$$

We also have the following results for the Hadamard product:

Corollary 6. *With the assumptions of Theorem 2, we have*

$$\begin{aligned}(3.23)\qquad & u^r(\gamma_1, \Gamma_1, \gamma_2, \Gamma_2) \\ & \times \left[\int_{\Omega} f^{1-\nu}(A_\tau)\, g^{\nu}(A_\tau)\, d\mu(\tau)\right] \circ \left[\int_{\Omega} f^{\nu}(B_\tau)\, g^{1-\nu}(B_\tau)\, d\mu(\tau)\right] \\ & \leq (1-\nu) \int_{\Omega} f(A_\tau)\, d\mu(\tau) \circ \int_{\Omega} g(B_\tau)\, d\mu(\tau) \\ & + \nu \int_{\Omega} g(A_\tau)\, d\mu(\tau) \circ \int_{\Omega} f(B_\tau)\, d\mu(\tau) \\ & \leq U^R(\gamma_1, \Gamma_1, \gamma_2, \Gamma_2) \\ & \times \left[\int_{\Omega} f^{1-\nu}(A_\tau)\, g^{\nu}(A_\tau)\, d\mu(\tau)\right] \circ \left[\int_{\Omega} f^{\nu}(B_\tau)\, g^{1-\nu}(B_\tau)\, d\mu(\tau)\right]\end{aligned}$$

for $\nu \in [0,1]$, *where* $r = \min\{1-\nu, \nu\}$ *and* $R = \max\{1-\nu, \nu\}$.

In particular, for $\nu = 1/2$

$$
\begin{aligned}
(3.24)\qquad & u^{1/2}(\gamma_1,\Gamma_1,\gamma_2,\Gamma_2) \\
& \times \left[\int_\Omega f^{1/2}(A_\tau)\, g^{1/2}(A_\tau)\, d\mu(\tau)\right] \circ \left[\int_\Omega f^{1/2}(B_\tau)\, g^{1/2}(B_\tau)\, d\mu(\tau)\right] \\
& \le \frac{1}{2}\left[\int_\Omega f(A_\tau)\, d\mu(\tau) \circ \int_\Omega g(B_\tau)\, d\mu(\tau)\right. \\
& \left. + \int_\Omega g(A_\tau)\, d\mu(\tau) \circ \int_\Omega f(B_\tau)\, d\mu(\tau)\right] \\
& \le U^{1/2}(\gamma_1,\Gamma_1,\gamma_2,\Gamma_2) \\
& \times \left[\int_\Omega f^{1/2}(A_\tau)\, g^{1/2}(A_\tau)\, d\mu(\tau)\right] \circ \left[\int_\Omega f^{1/2}(B_\tau)\, g^{1/2}(B_\tau)\, d\mu(\tau)\right].
\end{aligned}
$$

Remark 4. *From (3.23) we derive for* $\nu \in [0,1]$ *that*

$$
\begin{aligned}
(3.25)\qquad & \left[\int_\Omega f^{1-\nu}(A_\tau)\, g^{\nu}(A_\tau)\, d\mu(\tau)\right] \circ \left[\int_\Omega f^{\nu}(A_\tau)\, g^{1-\nu}(A_\tau)\, d\mu(\tau)\right] \\
& \le \int_\Omega f(A_\tau)\, d\mu(\tau) \circ \int_\Omega g(A_\tau)\, d\mu(\tau) \\
& \le \left[\frac{(\gamma+\Gamma)^2}{4\gamma\Gamma}\right]^R \\
& \times \left[\int_\Omega f^{1-\nu}(A_\tau)\, g^{\nu}(A_\tau)\, d\mu(\tau)\right] \circ \left[\int_\Omega f^{\nu}(A_\tau)\, g^{1-\nu}(A_\tau)\, d\mu(\tau)\right]
\end{aligned}
$$

and, in particular,

$$
\begin{aligned}
(3.26)\qquad & \left[\int_\Omega f^{1/2}(A_\tau)\, g^{1/2}(A_\tau)\, d\mu(\tau)\right] \circ \left[\int_\Omega f^{1/2}(A_\tau)\, g^{1/2}(A_\tau)\, d\mu(\tau)\right] \\
& \le \int_\Omega f(A_\tau)\, d\mu(\tau) \circ \int_\Omega g(A_\tau)\, d\mu(\tau) \\
& \le \frac{\gamma+\Gamma}{2\sqrt{\gamma\Gamma}} \\
& \times \left[\int_\Omega f^{1/2}(A_\tau)\, g^{1/2}(A_\tau)\, d\mu(\tau)\right] \circ \left[\int_\Omega f^{1/2}(A_\tau)\, g^{1/2}(A_\tau)\, d\mu(\tau)\right].
\end{aligned}
$$

4. Some Examples

Consider the functions $f(t) = t^p$ and $g(t) = t^q$ for $t > 0$ and $p, q \neq 0$. Then

$$\frac{f(t)}{g(t)} = t^{p-q}, \text{ for } t > 0.$$

Therefore

$$m^{p-q} \le \frac{f(t)}{g(t)} \le M^{p-q} \text{ for } t \in [m, M] \text{ and } p > q$$

and

$$M^{p-q} \le \frac{f(t)}{g(t)} \le m^{p-q} \text{ for } t \in [m, M] \text{ and } p < q.$$

Now, assume that $(A_\tau)_{\tau\in\Omega}$ and $(B_\tau)_{\tau\in\Omega}$ are continuous fields of positive operators in $B(H)$ such that $\operatorname{Sp}(A_\tau)$, $\operatorname{Sp}(B_\tau) \subset [m, M]$ for each $\tau \in \Omega$. From Theorem 2 we get for $p > q$ that

$$\begin{aligned}
&\int_\Omega A_\tau^{2(1-\nu)p+2\nu q} d\mu(\tau) \otimes \int_\Omega B_\tau^{2\nu p+2(1-\nu)q} d\mu(\tau) \\
&\leq (1-\nu) \int_\Omega A_\tau^{2p} d\mu(\tau) \otimes \int_\Omega B_\tau^{2q} d\mu(\tau) \\
&+ \nu \int_\Omega A_\tau^{2q} d\mu(\tau) \otimes \int_\Omega B_\tau^{2p} d\mu(\tau) \\
&\leq K^R\left(\left(\frac{M}{m}\right)^{2(p-q)}\right) \int_\Omega A_\tau^{2(1-\nu)p+2\nu q} d\mu(\tau) \otimes \int_\Omega B_\tau^{2\nu p+2(1-\nu)q} d\mu(\tau).
\end{aligned} \tag{4.1}$$

In particular, for $\nu = 1/2$,

$$\begin{aligned}
&\int_\Omega A_\tau^{p+q} d\mu(\tau) \otimes \int_\Omega B_\tau^{p+q} d\mu(\tau) \\
&\leq \frac{1}{2}\left(\int_\Omega A_\tau^{2p} d\mu(\tau) \otimes \int_\Omega B_\tau^{2q} d\mu(\tau) + \int_\Omega A_\tau^{2q} d\mu(\tau) \otimes \int_\Omega B_\tau^{2p} d\mu(\tau)\right) \\
&\leq K^{1/2}\left(\left(\frac{M}{m}\right)^{2(p-q)}\right) \int_\Omega A_\tau^{p+q} d\mu(\tau) \otimes \int_\Omega B_\tau^{p+q} d\mu(\tau).
\end{aligned} \tag{4.2}$$

We also have the inequalities for the Hadamard product

$$\begin{aligned}
&\int_\Omega A_\tau^{2(1-\nu)p+2\nu q} d\mu(\tau) \circ \int_\Omega B_\tau^{2\nu p+2(1-\nu)q} d\mu(\tau) \\
&\leq (1-\nu) \int_\Omega A_\tau^{2p} d\mu(\tau) \circ \int_\Omega B_\tau^{2q} d\mu(\tau) \\
&+ \nu \int_\Omega A_\tau^{2q} d\mu(\tau) \circ \int_\Omega B_\tau^{2p} d\mu(\tau) \\
&\leq K^R\left(\left(\frac{M}{m}\right)^{2(p-q)}\right) \int_\Omega A_\tau^{2(1-\nu)p+2\nu q} d\mu(\tau) \circ \int_\Omega B_\tau^{2\nu p+2(1-\nu)q} d\mu(\tau).
\end{aligned} \tag{4.3}$$

In particular, for $\nu = 1/2$,

$$\begin{aligned}
&\int_\Omega A_\tau^{p+q} d\mu(\tau) \circ \int_\Omega B_\tau^{p+q} d\mu(\tau) \\
&\leq \frac{1}{2}\left(\int_\Omega A_\tau^{2p} d\mu(\tau) \circ \int_\Omega B_\tau^{2q} d\mu(\tau) + \int_\Omega A_\tau^{2q} d\mu(\tau) \circ \int_\Omega B_\tau^{2p} d\mu(\tau)\right) \\
&\leq K^{1/2}\left(\left(\frac{M}{m}\right)^{2(p-q)}\right) \int_\Omega A_\tau^{p+q} d\mu(\tau) \circ \int_\Omega B_\tau^{p+q} d\mu(\tau).
\end{aligned} \tag{4.4}$$

Moreover, if we take $B_\tau = A_\tau$, $\tau \in \Omega$ in (4.3)-(4.4), then we get

$$\begin{aligned}(4.5)\qquad & \int_\Omega A_\tau^{2(1-\nu)p+2\nu q} d\mu(\tau) \circ \int_\Omega A_\tau^{2\nu p+2(1-\nu)q} d\mu(\tau) \\ & \le \int_\Omega A_\tau^{2p} d\mu(\tau) \circ \int_\Omega A_\tau^{2q} d\mu(\tau) \\ & \le K^R\left(\left(\frac{M}{m}\right)^{2(p-q)}\right) \int_\Omega A_\tau^{2(1-\nu)p+2\nu q} d\mu(\tau) \circ \int_\Omega A_\tau^{2\nu p+2(1-\nu)q} d\mu(\tau).\end{aligned}$$

In particular, for $\nu = 1/2$,

$$\begin{aligned}(4.6)\qquad & \int_\Omega A_\tau^{p+q} d\mu(\tau) \circ \int_\Omega A_\tau^{p+q} d\mu(\tau) \\ & \le \int_\Omega A_\tau^{2p} d\mu(\tau) \circ \int_\Omega A_\tau^{2q} d\mu(\tau) \\ & \le K^{1/2}\left(\left(\frac{M}{m}\right)^{2(p-q)}\right) \int_\Omega A_\tau^{p+q} d\mu(\tau) \circ \int_\Omega A_\tau^{p+q} d\mu(\tau).\end{aligned}$$

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[1]Mathematics, College of Engineering & Science, Victoria University, PO Box 14428, Melbourne City, MC 8001, Australia.
E-mail address: sever.dragomir@vu.edu.au
URL: http://rgmia.org/dragomir

[2]DST-NRF Centre of Excellence in the Mathematical, and Statistical Sciences, School of Computer Science, & Applied Mathematics, University of the Witwatersrand,, Private Bag 3, Johannesburg 2050, South Africa