SOME TENSORIAL AND HADAMARD PRODUCTS INTEGRAL INEQUALITIES FOR CONTINUOUS FIELDS OF OPERATORS IN HILBERT SPACES VIA A CARTWRIGHT-FIELD RESULT

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Abstract. Let H be a Hilbert space and Ω a locally compact Hausdorff space endowed with a Radon measure μ with $\int_\Omega 1d\mu(t)=1$. In this paper we show among others that, if $(A_\tau)_{\tau\in\Omega}$ and $(B_\tau)_{\tau\in\Omega}$ are continuous fields of positive operators in $B(H)$ such that $\operatorname{Sp}(A_\tau)$, $\operatorname{Sp}(B_\tau)\subseteq[m,M]\subset(0,\infty)$ for each $\tau\in\Omega$, then for all $\nu\in[0,1]$ we have the tensorial inequality

$$\begin{aligned}
&\frac{1}{M}\nu(1-\nu)\left\{\frac{1}{2}\left[\int_\Omega A_\tau^2 d\mu(\tau)\otimes 1+1\otimes\int_\Omega B_\tau^2 d\mu(\tau)\right]\right.\\
&\left.-\int_\Omega A_\tau d\mu(\tau)\otimes\int_\Omega B_\tau d\mu(\tau)\right\}\\
&\leq(1-\nu)\int_\Omega A_\tau d\mu(\tau)\otimes 1+\nu 1\otimes\int_\Omega B_\tau d\mu(\tau)\\
&-\int_\Omega A_\tau^{1-\nu}d\mu(\tau)\otimes\int_\Omega B_\tau^{\nu}d\mu(\tau)\\
&\leq\frac{1}{m}\nu(1-\nu)\left\{\frac{1}{2}\left[\int_\Omega A_\tau^2 d\mu(\tau)\otimes 1+1\otimes\int_\Omega B_\tau^2 d\mu(\tau)\right]\right.\\
&\left.-\int_\Omega A_\tau d\mu(\tau)\otimes\int_\Omega B_\tau d\mu(\tau)\right\}.
\end{aligned}$$

We also have the following inequalities for the Hadamard product

$$\begin{aligned}
&\frac{1}{M}\nu(1-\nu)\left[\int_\Omega\frac{A_\tau^2+B_\tau^2}{2}d\mu(\tau)\circ 1-\int_\Omega A_\tau d\mu(\tau)\circ\int_\Omega B_\tau d\mu(\tau)\right]\\
&\leq\int_\Omega\left[(1-\nu)A_\tau+\nu B_\tau\right]d\mu(\tau)\circ 1-\int_\Omega A_\tau^{1-\nu}d\mu(\tau)\circ\int_\Omega B_\tau^{\nu}d\mu(\tau)\\
&\leq\frac{1}{m}\nu(1-\nu)\left[\int_\Omega\frac{A_\tau^2+B_\tau^2}{2}d\mu(\tau)\circ 1-\int_\Omega A_\tau d\mu(\tau)\circ\int_\Omega B_\tau d\mu(\tau)\right]
\end{aligned}$$

for all $\nu\in[0,1]$.

1. Introduction

We have the following inequality that provides a refinement and a reverse for the celebrated Young's inequality

$$\frac{1}{2}\nu(1-\nu)\frac{(b-a)^2}{\max\{a,b\}}\leq(1-\nu)a+\nu b-a^{1-\nu}b^{\nu}\leq\frac{1}{2}\nu(1-\nu)\frac{(b-a)^2}{\min\{a,b\}}\tag{1.1}$$

for any $a,b>0$ and $\nu\in[0,1]$.

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This result was obtained in 1978 by Cartwright and Field [4] who established a more general result for n variables and gave an application for a probability measure supported on a finite interval.

Since $\max\{a,b\}\min\{a,b\}=ab$ for $a,b>0$, then by (1.1) we get

$$\frac{1}{2}\nu(1-\nu)\min\{a,b\}\frac{(b-a)^2}{ab}\leq(1-\nu)a+\nu b-a^{1-\nu}b^{\nu}$$
$$\leq\frac{1}{2}\nu(1-\nu)\max\{a,b\}\frac{(b-a)^2}{ab},$$

namely

$$0\leq\frac{1}{2}\nu(1-\nu)\min\{a,b\}\left(\frac{a}{b}+\frac{b}{a}-2\right)\leq(1-\nu)a+\nu b-a^{1-\nu}b^{\nu} \tag{1.2}$$
$$\leq\frac{1}{2}\nu(1-\nu)\max\{a,b\}\left(\frac{a}{b}+\frac{b}{a}-2\right),$$

for any $a,b>0$ and $\nu\in[0,1]$.

Let $I_1,...,I_k$ be intervals from $\mathbb{R}$ and let $f:I_1\times...\times I_k\to\mathbb{R}$ be an essentially bounded real function defined on the product of the intervals. Let $A=(A_1,...,A_n)$ be a k-tuple of bounded selfadjoint operators on Hilbert spaces $H_1,...,H_k$ such that the spectrum of A_i is contained in I_i for $i=1,...,k$. We say that such a k-tuple is in the domain of f. If

$$A_i=\int_{I_i}\lambda_i dE_i(\lambda_i)$$

is the spectral resolution of A_i for $i=1,...,k$; by following [2], we define

$$f(A_1,...,A_k):=\int_{I_1}...\int_{I_k}f(\lambda_1,...,\lambda_1)\,dE_1(\lambda_1)\otimes...\otimes dE_k(\lambda_k) \tag{1.3}$$

as a bounded selfadjoint operator on the tensorial product $H_1\otimes...\otimes H_k$.

If the Hilbert spaces are of finite dimension, then the above integrals become finite sums, and we may consider the functional calculus for arbitrary real functions. This construction [2] extends the definition of Korányi [5] for functions of two variables and have the property that

$$f(A_1,...,A_k)=f_1(A_1)\otimes...\otimes f_k(A_k),$$

whenever f can be separated as a product $f(t_1,...,t_k)=f_1(t_1)...f_k(t_k)$ of k functions each depending on only one variable.

It is know that, *if f is super-multiplicative (sub-multiplicative)* on $[0,\infty)$, namely

$$f(st)\geq(\leq)f(s)f(t)\text{ for all }s,t\in[0,\infty)$$

and if f is continuous on $[0,\infty)$, then [7, p. 173]

$$f(A\otimes B)\geq(\leq)f(A)\otimes f(B)\text{ for all }A,\ B\geq 0. \tag{1.4}$$

This follows by observing that, if

$$A=\int_{[0,\infty)}tdE(t)\text{ and }B=\int_{[0,\infty)}sdF(s)$$

are the spectral resolutions of A and B, then

$$f(A\otimes B)=\int_{[0,\infty)}\int_{[0,\infty)}f(st)\,dE(t)\otimes dF(s) \tag{1.5}$$

for the continuous function f on $[0,\infty)$.

Recall the *geometric operator mean* for the positive operators A, $B>0$

$$A\#_t B := A^{1/2}(A^{-1/2}BA^{-1/2})^t A^{1/2},$$

where $t\in[0,1]$ and

$$A\#B := A^{1/2}(A^{-1/2}BA^{-1/2})^{1/2} A^{1/2}.$$

By the definitions of $\#$ and $\otimes$ we have

$$A\#B = B\#A \text{ and } (A\#B)\otimes(B\#A) = (A\otimes B)\#(B\otimes A).$$

In 2007, S. Wada [9] obtained the following *Callebaut type inequalities* for tensorial product

$$\begin{aligned}(A\#B)\otimes(A\#B) &\le \frac{1}{2}\left[(A\#_\alpha B)\otimes(A\#_{1-\alpha}B)+(A\#_{1-\alpha}B)\otimes(A\#_\alpha B)\right]\\ &\le \frac{1}{2}(A\otimes B+B\otimes A)\end{aligned} \tag{1.6}$$

for A, $B>0$ and $\alpha\in[0,1]$.

Recall that the *Hadamard product* of A and B in $B(H)$ is defined to be the operator $A\circ B\in B(H)$ satisfying

$$\langle (A\circ B)e_j, e_j\rangle = \langle Ae_j, e_j\rangle\langle Be_j, e_j\rangle$$

for all $j\in\mathbb{N}$, where $\{e_j\}_{j\in\mathbb{N}}$ is an *orthonormal basis* for the separable Hilbert space H.

It is known that, see [6], we have the representation

$$A\circ B = \mathcal{U}^*(A\otimes B)\mathcal{U} \tag{1.7}$$

where $\mathcal{U}: H\to H\otimes H$ is the isometry defined by $\mathcal{U}e_j = e_j\otimes e_j$ for all $j\in\mathbb{N}$.

If f *is super-multiplicative (sub-multiplicative)* on $[0,\infty)$, then also [7, p. 173]

$$f(A\circ B)\ge(\le) f(A)\circ f(B) \text{ for all } A,\ B\ge 0. \tag{1.8}$$

We recall the following elementary inequalities for the Hadamard product

$$A^{1/2}\circ B^{1/2}\le\left(\frac{A+B}{2}\right)\circ 1 \text{ for } A,\ B\ge 0$$

and *Fiedler inequality*

$$A\circ A^{-1}\ge 1 \text{ for } A>0. \tag{1.9}$$

As extension of Kadison's Schwarz inequality on the Hadamard product, Ando [1] showed that

$$A\circ B\le\left(A^2\circ 1\right)^{1/2}\left(B^2\circ 1\right)^{1/2} \text{ for } A,\ B\ge 0$$

and Aujla and Vasudeva [3] gave an alternative upper bound

$$A\circ B\le\left(A^2\circ B^2\right)^{1/2} \text{ for } A,\ B\ge 0.$$

It has been shown in [8] that $\left(A^2\circ 1\right)^{1/2}\left(B^2\circ 1\right)^{1/2}$ and $\left(A^2\circ B^2\right)^{1/2}$ are incomparable for 2-square positive definite matrices A and B.

Let Ω be a locally compact Hausdorff space endowed with a Radon measure μ. A field $(A_t)_{t\in\Omega}$ of operators in $B(H)$ is called a continuous field of operators if the parametrization $t\longmapsto A_t$ is norm continuous on $B(H)$. If, in addition, the norm function $t\longmapsto\|A_t\|$ is Lebesgue integrable on Ω, we can form the Bochner integral

$\int_{\Omega} A_t d\mu(t)$, which is the unique operator in $B(H)$ such that $\varphi\left(\int_{\Omega} A_t d\mu(t)\right) = \int_{\Omega} \varphi(A_t)\, d\mu(t)$ for every bounded linear functional φ on $B(H)$. Assume also that, $\int_{\Omega} 1 d\mu(t) = 1$.

Motivated by the above results, in this paper we show among others that, if $(A_\tau)_{\tau\in\Omega}$ and $(B_\tau)_{\tau\in\Omega}$ are continuous fields of positive operators in $B(H)$ such that $\operatorname{Sp}(A_\tau), \operatorname{Sp}(B_\tau) \subseteq [m, M] \subset (0, \infty)$ for each $\tau \in \Omega$, then for all $\nu \in [0,1]$ we have the tensorial inequality

$$\begin{aligned}
&\frac{1}{M}\nu(1-\nu)\left\{\frac{1}{2}\left[\int_{\Omega} A_\tau^2 d\mu(\tau) \otimes 1 + 1 \otimes \int_{\Omega} B_\tau^2 d\mu(\tau)\right]\right.\\
&\left.-\int_{\Omega} A_\tau d\mu(\tau) \otimes \int_{\Omega} B_\tau d\mu(\tau)\right\}\\
&\leq (1-\nu)\int_{\Omega} A_\tau d\mu(\tau) \otimes 1 + \nu 1 \otimes \int_{\Omega} B_\tau d\mu(\tau)\\
&-\int_{\Omega} A_\tau^{1-\nu} d\mu(\tau) \otimes \int_{\Omega} B_\tau^{\nu} d\mu(\tau)\\
&\leq \frac{1}{m}\nu(1-\nu)\left\{\frac{1}{2}\left[\int_{\Omega} A_\tau^2 d\mu(\tau) \otimes 1 + 1 \otimes \int_{\Omega} B_\tau^2 d\mu(\tau)\right]\right.\\
&\left.-\int_{\Omega} A_\tau d\mu(\tau) \otimes \int_{\Omega} B_\tau d\mu(\tau)\right\}.
\end{aligned}$$

We also have the following inequalities for the Hadamard product

$$\begin{aligned}
&\frac{1}{M}\nu(1-\nu)\left[\int_{\Omega} \frac{A_\tau^2 + B_\tau^2}{2} d\mu(\tau) \circ 1 - \int_{\Omega} A_\tau d\mu(\tau) \circ \int_{\Omega} B_\tau d\mu(\tau)\right]\\
&\leq \int_{\Omega} \left[(1-\nu) A_\tau + \nu B_\tau\right] d\mu(\tau) \circ 1 - \int_{\Omega} A_\tau^{1-\nu} d\mu(\tau) \circ \int_{\Omega} B_\tau^{\nu} d\mu(\tau)\\
&\leq \frac{1}{m}\nu(1-\nu)\left[\int_{\Omega} \frac{A_\tau^2 + B_\tau^2}{2} d\mu(\tau) \circ 1 - \int_{\Omega} A_\tau d\mu(\tau) \circ \int_{\Omega} B_\tau d\mu(\tau)\right]
\end{aligned}$$

for all $\nu \in [0,1]$.

2. Main Results

The following result is of interest in itself:

Lemma 1. *Assume that the operators A, B satisfy the conditions $0 < m \leq A$, $B \leq M$, then*

$$\begin{aligned}
0 &\leq \frac{1}{M}\nu(1-\nu)\left(\frac{A^2 \otimes 1 + 1 \otimes B^2}{2} - A \otimes B\right)\\
&\leq (1-\nu) A \otimes 1 + \nu 1 \otimes B - A^{1-\nu} \otimes B^{\nu}\\
&\leq \frac{1}{m}\nu(1-\nu)\left(\frac{A^2 \otimes 1 + 1 \otimes B^2}{2} - A \otimes B\right)
\end{aligned} \tag{2.1}$$

and

$$
\begin{aligned}
0 &\leq m\nu(1-\nu)\left(\frac{A\otimes B^{-1}+A^{-1}\otimes B}{2}-1\right) \\
&\leq (1-\nu)A\otimes 1+\nu 1\otimes B-A^{1-\nu}\otimes B^{\nu} \\
&\leq M\nu(1-\nu)\left(\frac{A\otimes B^{-1}+A^{-1}\otimes B}{2}-1\right)
\end{aligned} \tag{2.2}
$$

for all $\nu\in[0,1]$.

In particular, we have

$$
\begin{aligned}
0 &\leq \frac{1}{4M}\left(\frac{A^2\otimes 1+1\otimes B^2}{2}-A\otimes B\right) \\
&\leq \frac{1}{2}(A\otimes 1+1\otimes B)-A^{1/2}\otimes B^{1/2}\leq \frac{1}{4m}\left(\frac{A^2\otimes 1+1\otimes B^2}{2}-A\otimes B\right)
\end{aligned} \tag{2.3}
$$

and

$$
\begin{aligned}
0 &\leq \frac{1}{4}m\left(\frac{A\otimes B^{-1}+A^{-1}\otimes B}{2}-1\right) \\
&\leq \frac{1}{2}(A\otimes 1+1\otimes B)-A^{1/2}\otimes B^{1/2}\leq \frac{1}{4}M\left(\frac{A\otimes B^{-1}+A^{-1}\otimes B}{2}-1\right).
\end{aligned} \tag{2.4}
$$

Proof. Now if $t,\ s\in[m,M]\subset(0,\infty)$, then we have from (1.1) and (1.2) the following two inequalities

$$
\begin{aligned}
0 &\leq \frac{1}{2M}\nu(1-\nu)\left(t^2-2ts+s^2\right)\leq (1-\nu)t+\nu s-t^{1-\nu}s^{\nu} \\
&\leq \frac{1}{2m}\nu(1-\nu)\left(t^2-2ts+s^2\right)
\end{aligned} \tag{2.5}
$$

and

$$
\begin{aligned}
0 &\leq \frac{1}{2}m\nu(1-\nu)\left(\frac{t}{s}+\frac{s}{t}-2\right)\leq (1-\nu)t+\nu s-t^{1-\nu}s^{\nu} \\
&\leq \frac{1}{2}M\nu(1-\nu)\left(\frac{t}{s}+\frac{s}{t}-2\right)
\end{aligned} \tag{2.6}
$$

for $\nu\in[0,1]$.

If

$$A=\int_m^M t\,dE(t)\ \text{ and }\ B=\int_m^M s\,dF(s)$$

are the spectral resolutions of A and B, then by taking the double integral $\int_m^M\int_m^M$ over $dE(t)\otimes dF(s)$, we get

$$
\begin{aligned}
0 &\leq \frac{1}{2M}\nu(1-\nu)\int_m^M\int_m^M\left(t^2-2ts+s^2\right)dE(t)\otimes dF(s) \\
&\leq \int_m^M\int_m^M\left[(1-\nu)t+\nu s-t^{1-\nu}s^{\nu}\right]dE(t)\otimes dF(s) \\
&\leq \frac{1}{2m}\nu(1-\nu)\int_m^M\int_m^M\left(t^2-2ts+s^2\right)dE(t)\otimes dF(s)
\end{aligned} \tag{2.7}
$$

and

$$0 \leq \frac{1}{2} m\nu (1-\nu) \int_m^M \int_m^M \left(\frac{t}{s} + \frac{s}{t} - 2 \right) dE(t) \otimes dF(s) \tag{2.8}$$
$$\leq \int_m^M \int_m^M \left[(1-\nu) t + \nu s - t^{1-\nu} s^{\nu} \right] dE(t) \otimes dF(s)$$
$$\leq \frac{1}{2} M\nu (1-\nu) \int_m^M \int_m^M \left(\frac{t}{s} + \frac{s}{t} - 2 \right) dE(t) \otimes dF(s).$$

Observe that

$$\int_m^M \int_m^M \left(t^2 - 2ts + s^2 \right) dE(t) \otimes dF(s)$$
$$= \int_m^M \int_m^M t^2 dE(t) \otimes dF(s) + \int_m^M \int_m^M s^2 dE(t) \otimes dF(s)$$
$$- 2 \int_m^M \int_m^M ts dE(t) \otimes dF(s)$$
$$= A^2 \otimes 1 + 1 \otimes B^2 - 2A \otimes B,$$

$$\int_m^M \int_m^M \left[(1-\nu) t + \nu s - t^{1-\nu} s^{\nu} \right] dE(t) \otimes dF(s)$$
$$= (1-\nu) \int_m^M \int_m^M t dE(t) \otimes dF(s) + \nu \int_m^M \int_m^M s dE(t) \otimes dF(s)$$
$$- \int_m^M \int_m^M t^{1-\nu} s^{\nu} dE(t) \otimes dF(s)$$
$$= (1-\nu) A \otimes 1 + \nu 1 \otimes B - A^{1-\nu} \otimes B^{\nu}$$

and

$$\int_m^M \int_m^M \left(\frac{t}{s} + \frac{s}{t} - 2 \right) dE(t) \otimes dF(s)$$
$$= \int_m^M \int_m^M \left(ts^{-1} + t^{-1}s - 2 \right) dE(t) \otimes dF(s)$$
$$= \int_m^M \int_m^M ts^{-1} dE(t) \otimes dF(s) + \int_m^M \int_m^M t^{-1} s dE(t) \otimes dF(s)$$
$$- 2 \int_m^M \int_m^M dE(t) \otimes dF(s)$$
$$= A \otimes B^{-1} + A^{-1} \otimes B - 2.$$

Then by (2.7) and (2.8) we deduce the desired results (2.1) and (2.2). □

Corollary 1. *With the assumptions of Theorem 1, we have*

$$0 \leq \frac{1}{M} \nu (1-\nu) \left(\frac{A^2 + B^2}{2} \circ 1 - A \circ B \right) \tag{2.9}$$
$$\leq [(1-\nu) A + \nu B] \circ 1 - A^{1-\nu} \circ B^{\nu}$$
$$\leq \frac{1}{m} \nu (1-\nu) \left(\frac{A^2 + B^2}{2} \circ 1 - A \circ B \right)$$

and

$$
\begin{aligned}
0 &\leq m\nu(1-\nu)\left(\frac{A\circ B^{-1}+A^{-1}\circ B}{2}-1\right) \\
&\leq [(1-\nu)A+\nu B]\circ 1-A^{1-\nu}\circ B^{\nu} \\
&\leq M\nu(1-\nu)\left(\frac{A\circ B^{-1}+A^{-1}\circ B}{2}-1\right)
\end{aligned} \tag{2.10}
$$

for all $\nu\in[0,1]$.

In particular, we have

$$
\begin{aligned}
0 &\leq \frac{1}{4M}\left(\frac{A^2+B^2}{2}\circ 1-A\circ B\right) \\
&\leq \frac{1}{2}(A\circ 1+1\circ B)-A^{1/2}\circ B^{1/2}\leq \frac{1}{4m}\left(\frac{A^2+B^2}{2}\circ 1-A\circ B\right)
\end{aligned} \tag{2.11}
$$

and

$$
\begin{aligned}
0 &\leq \frac{1}{4}m\left(\frac{A\circ B^{-1}+A^{-1}\circ B}{2}-1\right) \\
&\leq \frac{1}{2}(A\circ 1+1\circ B)-A^{1/2}\circ B^{1/2}\leq \frac{1}{4}M\left(\frac{A\circ B^{-1}+A^{-1}\circ B}{2}-1\right).
\end{aligned} \tag{2.12}
$$

Proof. For X, Y we have the representation

$$X\circ Y=\mathcal{U}^*(X\otimes Y)\mathcal{U},$$

where $\mathcal{U}:H\to H\otimes H$ is the isometry defined by $\mathcal{U}e_j=e_j\otimes e_j$ for all $j\in\mathbb{N}$.

By (2.1) we derive

$$
\begin{aligned}
0 &\leq \frac{1}{2M}\nu(1-\nu)\mathcal{U}^*\left(A^2\otimes 1+1\otimes B^2-2A\otimes B\right)\mathcal{U} \\
&\leq \mathcal{U}^*\left[(1-\nu)A\otimes 1+\nu 1\otimes B-A^{1-\nu}\otimes B^{\nu}\right]\mathcal{U} \\
&\leq \frac{1}{2m}\nu(1-\nu)\mathcal{U}^*\left(A^2\otimes 1+1\otimes B^2-2A\otimes B\right)\mathcal{U},
\end{aligned}
$$

namely

$$
\begin{aligned}
0 &\leq \frac{1}{2M}\nu(1-\nu)\left[\mathcal{U}^*\left(A^2\otimes 1\right)\mathcal{U}+\mathcal{U}^*\left(1\otimes B^2\right)\mathcal{U}-2\mathcal{U}^*(A\otimes B)\mathcal{U}\right] \\
&\leq (1-\nu)\mathcal{U}^*(A\otimes 1)\mathcal{U}+\nu\mathcal{U}^*(1\otimes B)\mathcal{U}-\mathcal{U}^*\left(A^{1-\nu}\otimes B^{\nu}\right)\mathcal{U} \\
&\leq \frac{1}{2m}\nu(1-\nu)\left[\mathcal{U}^*\left(A^2\otimes 1\right)\mathcal{U}+\mathcal{U}^*\left(1\otimes B^2\right)\mathcal{U}-2\mathcal{U}^*(A\otimes B)\mathcal{U}\right],
\end{aligned}
$$

which is equivalent to (2.9). □

Remark 1. *If we take* $B=A$ *in Corollary 1, then we get for an operator* $0<m\leq A\leq M$, *that*

$$
\begin{aligned}
0 &\leq \frac{1}{M}\nu(1-\nu)\left(A^2\circ 1-A\circ A\right) \\
&\leq A\circ 1-A^{1-\nu}\circ A^{\nu}\leq \frac{1}{m}\nu(1-\nu)\left(A^2\circ 1-A\circ A\right)
\end{aligned} \tag{2.13}
$$

and

$$0 \le m\nu(1-\nu)\left(A\circ A^{-1}-1\right) \le A\circ 1 - A^{1-\nu}\circ A^{\nu} \le M\nu(1-\nu)\left(A\circ A^{-1}-1\right) \tag{2.14}$$

for all $\nu \in [0,1]$.

In particular, we have

$$0 \le \frac{1}{4M}\left(A^2\circ 1 - A\circ A\right) \le A\circ 1 - A^{1/2}\circ A^{1/2} \le \frac{1}{4m}\left(A^2\circ 1 - A\circ A\right) \tag{2.15}$$

and

$$0 \le \frac{1}{4}m\left(A\circ A^{-1}-1\right) \le A\circ 1 - A^{1/2}\circ A^{1/2} \le \frac{1}{4}M\left(A\circ A^{-1}-1\right). \tag{2.16}$$

In what follows we assume that $\int_\Omega 1 d\mu(t) = 1$.

Theorem 1. *Let* $(A_\tau)_{\tau\in\Omega}$ *and* $(B_\tau)_{\tau\in\Omega}$ *be continuous fields of positive operators in* $B(H)$ *such that* $\operatorname{Sp}(A_\tau), \operatorname{Sp}(B_\tau) \subseteq [m,M] \subset (0,\infty)$ *for each* $\tau\in\Omega$. *Then for all* $\nu\in[0,1]$ *we have*

$$\begin{aligned}
&\frac{1}{M}\nu(1-\nu)\left\{\frac{1}{2}\left[\int_\Omega A_\tau^2 d\mu(\tau)\otimes 1 + 1\otimes\int_\Omega B_\tau^2 d\mu(\tau)\right]\right.\\
&\left.-\int_\Omega A_\tau d\mu(\tau)\otimes\int_\Omega B_\tau d\mu(\tau)\right\}\\
&\le (1-\nu)\int_\Omega A_\tau d\mu(\tau)\otimes 1 + \nu 1\otimes\int_\Omega B_\tau d\mu(\tau)\\
&-\int_\Omega A_\tau^{1-\nu} d\mu(\tau)\otimes\int_\Omega B_\tau^{\nu} d\mu(\tau)\\
&\le \frac{1}{m}\nu(1-\nu)\left\{\frac{1}{2}\left[\int_\Omega A_\tau^2 d\mu(\tau)\otimes 1 + 1\otimes\int_\Omega B_\tau^2 d\mu(\tau)\right]\right.\\
&\left.-\int_\Omega A_\tau d\mu(\tau)\otimes\int_\Omega B_\tau d\mu(\tau)\right\}
\end{aligned} \tag{2.17}$$

and

$$\begin{aligned}
&0 \le m\nu(1-\nu)\\
&\times\left(\frac{\int_\Omega A_\tau d\mu(\tau)\otimes\int_\Omega B_\tau^{-1} d\mu(\tau) + \int_\Omega A_\tau^{-1} d\mu(\tau)\otimes\int_\Omega B_\tau d\mu(\tau)}{2} - 1\right)\\
&\le (1-\nu)\int_\Omega A_\tau d\mu(\tau)\otimes 1 + \nu 1\otimes\int_\Omega B_\tau d\mu(\tau)\\
&-\int_\Omega A_\tau^{1-\nu} d\mu(\tau)\otimes\int_\Omega B_\tau^{\nu} d\mu(\tau)\\
&\le M\nu(1-\nu)\\
&\times\left(\frac{\int_\Omega A_\tau d\mu(\tau)\otimes\int_\Omega B_\tau^{-1} d\mu(\tau) + \int_\Omega A_\tau^{-1} d\mu(\tau)\otimes\int_\Omega B_\tau d\mu(\tau)}{2} - 1\right).
\end{aligned} \tag{2.18}$$

Proof. From (2.1) we have

$$0 \leq \frac{1}{M}\nu(1-\nu)\left(\frac{A_\tau^2 \otimes 1 + 1 \otimes B_\gamma^2}{2} - A_\tau \otimes B_\gamma\right) \tag{2.19}$$
$$\leq (1-\nu) A_\tau \otimes 1 + \nu 1 \otimes B_\gamma - A_\tau^{1-\nu} \otimes B_\gamma^\nu$$
$$\leq \frac{1}{m}\nu(1-\nu)\left(\frac{A^2 \otimes 1 + 1 \otimes B^2}{2} - A_\tau \otimes B_\gamma\right)$$

for all τ, $\gamma \in \Omega$. If we take the integral $\int_\Omega$ over $d\mu(\tau)$, then we get

$$0 \leq \frac{1}{M}\nu(1-\nu)\int_\Omega \left(\frac{A_\tau^2 \otimes 1 + 1 \otimes B_\gamma^2}{2} - A_\tau \otimes B_\gamma\right) d\mu(\tau) \tag{2.20}$$
$$\leq \int_\Omega \left[(1-\nu) A_\tau \otimes 1 + \nu 1 \otimes B_\gamma - A_\tau^{1-\nu} \otimes B_\gamma^\nu\right] d\mu(\tau)$$
$$\leq \frac{1}{m}\nu(1-\nu)\int_\Omega \left(\frac{A^2 \otimes 1 + 1 \otimes B^2}{2} - A_\tau \otimes B_\gamma\right) d\mu(\tau).$$

Using the properties of the Bochner's integral and the tensorial product we have

$$\int_\Omega \left(\frac{A_\tau^2 \otimes 1 + 1 \otimes B_\gamma^2}{2} - A_\tau \otimes B_\gamma\right) d\mu(\tau)$$
$$= \frac{1}{2}\left[\int_\Omega A_\tau^2 \otimes 1 + 1 \otimes B_\gamma^2\right] - \int_\Omega A_\tau d\mu(\tau) \otimes B_\gamma,$$

and

$$\int_\Omega \left[(1-\nu) A_\tau \otimes 1 + \nu 1 \otimes B_\gamma - A_\tau^{1-\nu} \otimes B_\gamma^\nu\right] d\mu(\tau)$$
$$= (1-\nu)\int_\Omega A_\tau d\mu(\tau) \otimes 1 + \nu 1 \otimes B_\gamma - \int_\Omega A_\tau^{1-\nu} d\mu(\tau) \otimes B_\gamma^\nu$$

for all $\gamma \in \Omega$.

From (2.20) we get

$$0 \leq \frac{1}{M}\nu(1-\nu) \tag{2.21}$$
$$\times\left(\frac{1}{2}\left[\int_\Omega A_\tau^2 \otimes 1 + 1 \otimes B_\gamma^2\right] - \int_\Omega A_\tau d\mu(\tau) \otimes B_\gamma\right)$$
$$\leq (1-\nu)\int_\Omega A_\tau d\mu(\tau) \otimes 1 + \nu 1 \otimes B_\gamma - \int_\Omega A_\tau^{1-\nu} d\mu(\tau) \otimes B_\gamma^\nu$$
$$\leq \frac{1}{m}\nu(1-\nu)$$
$$\times\left(\frac{1}{2}\left[\int_\Omega A_\tau^2 \otimes 1 + 1 \otimes B_\gamma^2\right] - \int_\Omega A_\tau d\mu(\tau) \otimes B_\gamma\right)$$

for all $\gamma \in \Omega$.

If we take the integral $\int_\Omega$ over $d\mu(\gamma)$, then we get

$$
\begin{aligned}
(2.22)\qquad 0 &\leq \frac{1}{M}\nu(1-\nu) \\
&\times \int_\Omega \left(\frac{1}{2}\left[\int_\Omega A_\tau^2 \otimes 1 d\mu(\tau) + 1 \otimes B_\gamma^2\right] - \int_\Omega A_\tau d\mu(\tau) \otimes B_\gamma\right) d\mu(\gamma) \\
&\leq \int_\Omega \left[(1-\nu)\int_\Omega A_\tau d\mu(\tau) \otimes 1 + \nu 1 \otimes B_\gamma \right.\\
&\left. - \int_\Omega A_\tau^{1-\nu} d\mu(\tau) \otimes B_\gamma^\nu\right] d\mu(\gamma) \\
&\leq \frac{1}{m}\nu(1-\nu) \\
&\times \int_\Omega \left(\frac{1}{2}\left[\int_\Omega A_\tau^2 \otimes 1 d\mu(\tau) + 1 \otimes B_\gamma^2\right] - \int_\Omega A_\tau d\mu(\tau) \otimes B_\gamma\right) d\mu(\gamma).
\end{aligned}
$$

Since

$$
\begin{aligned}
&\int_\Omega \left(\frac{1}{2}\left[\int_\Omega A_\tau^2 d\mu(\tau) \otimes 1 + 1 \otimes B_\gamma^2\right] - \int_\Omega A_\tau d\mu(\tau) \otimes B_\gamma\right) d\mu(\gamma) \\
&= \frac{1}{2}\left[\int_\Omega A_\tau^2 d\mu(\tau) \otimes 1 + 1 \otimes \int_\Omega B_\gamma^2 d\mu(\gamma)\right] - \int_\Omega A_\tau d\mu(\tau) \otimes \int_\Omega B_\gamma d\mu(\gamma)
\end{aligned}
$$

and

$$
\begin{aligned}
&\int_\Omega \left[(1-\nu)\int_\Omega A_\tau d\mu(\tau) \otimes 1 + \nu 1 \otimes B_\gamma \right.\\
&\left. - \int_\Omega A_\tau^{1-\nu} d\mu(\tau) \otimes B_\gamma^\nu\right] d\mu(\gamma) \\
&= (1-\nu)\int_\Omega A_\tau d\mu(\tau) \otimes 1 + \nu 1 \otimes \int_\Omega B_\gamma d\mu(\gamma) \\
&- \int_\Omega A_\tau^{1-\nu} d\mu(\tau) \otimes \int_\Omega B_\gamma^\nu d\mu(\gamma),
\end{aligned}
$$

hence by (2.22) we deduce (2.17).

The inequality (2.18) follows in a similar way from (2.2). □

Remark 2. *If we take $\nu = 1/2$ in Theorem 1, then we get*

$$
\begin{aligned}
(2.23)\qquad &\frac{1}{4M}\left\{\frac{1}{2}\left[\int_\Omega A_\tau^2 d\mu(\tau) \otimes 1 + 1 \otimes \int_\Omega B_\tau^2 d\mu(\tau)\right] \right.\\
&\left. - \int_\Omega A_\tau d\mu(\tau) \otimes \int_\Omega B_\tau d\mu(\tau)\right\} \\
&\leq \frac{1}{2}\left[\int_\Omega A_\tau d\mu(\tau) \otimes 1 + 1 \otimes \int_\Omega B_\tau d\mu(\tau)\right] \\
&- \int_\Omega A_\tau^{1/2} d\mu(\tau) \otimes \int_\Omega B_\tau^{1/2} d\mu(\tau) \\
&\leq \frac{1}{4m}\left\{\frac{1}{2}\left[\int_\Omega A_\tau^2 d\mu(\tau) \otimes 1 + 1 \otimes \int_\Omega B_\tau^2 d\mu(\tau)\right] \right.\\
&\left. - \int_\Omega A_\tau d\mu(\tau) \otimes \int_\Omega B_\tau d\mu(\tau)\right\}
\end{aligned}
$$

and

$$
\begin{aligned}
(2.24)\qquad & 0 \leq \frac{1}{4}m \\
& \times\left(\frac{\int_\Omega A_\tau d\mu(\tau)\otimes\int_\Omega B_\tau^{-1}d\mu(\tau)+\int_\Omega A_\tau^{-1}d\mu(\tau)\otimes\int_\Omega B_\tau d\mu(\tau)}{2}-1\right) \\
& \leq \frac{1}{2}\left[\int_\Omega A_\tau d\mu(\tau)\otimes 1+1\otimes\int_\Omega B_\tau d\mu(\tau)\right] \\
& -\int_\Omega A_\tau^{1/2}d\mu(\tau)\otimes\int_\Omega B_\tau^{1/2}d\mu(\tau) \\
& \leq \frac{1}{4}M \\
& \times\left(\frac{\int_\Omega A_\tau d\mu(\tau)\otimes\int_\Omega B_\tau^{-1}d\mu(\tau)+\int_\Omega A_\tau^{-1}d\mu(\tau)\otimes\int_\Omega B_\tau d\mu(\tau)}{2}-1\right).
\end{aligned}
$$

We have the following result for Hadamard product:

Corollary 2. *With the assumptions of Theorem 1 we have*

$$
\begin{aligned}
(2.25)\qquad & \frac{1}{M}\nu(1-\nu)\left[\int_\Omega \frac{A_\tau^2+B_\tau^2}{2}d\mu(\tau)\circ 1-\int_\Omega A_\tau d\mu(\tau)\circ\int_\Omega B_\tau d\mu(\tau)\right] \\
& \leq \int_\Omega\left[(1-\nu)A_\tau+\nu B_\tau\right]d\mu(\tau)\circ 1-\int_\Omega A_\tau^{1-\nu}d\mu(\tau)\circ\int_\Omega B_\tau^{\nu}d\mu(\tau) \\
& \leq \frac{1}{m}\nu(1-\nu)\left[\int_\Omega \frac{A_\tau^2+B_\tau^2}{2}d\mu(\tau)\circ 1-\int_\Omega A_\tau d\mu(\tau)\circ\int_\Omega B_\tau d\mu(\tau)\right]
\end{aligned}
$$

and

$$
\begin{aligned}
(2.26)\qquad & 0 \leq m\nu(1-\nu) \\
& \times\left(\frac{\int_\Omega A_\tau d\mu(\tau)\circ\int_\Omega B_\tau^{-1}d\mu(\tau)+\int_\Omega A_\tau^{-1}d\mu(\tau)\circ\int_\Omega B_\tau d\mu(\tau)}{2}-1\right) \\
& \leq \int_\Omega\left[(1-\nu)A_\tau+\nu B_\tau\right]d\mu(\tau)\circ 1-\int_\Omega A_\tau^{1-\nu}d\mu(\tau)\circ\int_\Omega B_\tau^{\nu}d\mu(\tau) \\
& \leq M\nu(1-\nu) \\
& \times\left(\frac{\int_\Omega A_\tau d\mu(\tau)\circ\int_\Omega B_\tau^{-1}d\mu(\tau)+\int_\Omega A_\tau^{-1}d\mu(\tau)\circ\int_\Omega B_\tau d\mu(\tau)}{2}-1\right).
\end{aligned}
$$

For $\nu = 1/2$ *we get*

$$
\begin{aligned}
(2.27)\qquad & \frac{1}{4M}\left[\int_\Omega \frac{A_\tau^2+B_\tau^2}{2}d\mu(\tau)\circ 1-\int_\Omega A_\tau d\mu(\tau)\circ\int_\Omega B_\tau d\mu(\tau)\right] \\
& \leq \int_\Omega \frac{A_\tau+B_\tau}{2}d\mu(\tau)\circ 1-\int_\Omega A_\tau^{1/2}d\mu(\tau)\circ\int_\Omega B_\tau^{1/2}d\mu(\tau) \\
& \leq \frac{1}{4}\left[\int_\Omega \frac{A_\tau^2+B_\tau^2}{2}d\mu(\tau)\circ 1-\int_\Omega A_\tau d\mu(\tau)\circ\int_\Omega B_\tau d\mu(\tau)\right]
\end{aligned}
$$

and

$$\begin{aligned}
(2.28)\qquad 0 &\leq \frac{1}{4}m \\
&\times\left(\frac{\int_\Omega A_\tau d\mu(\tau)\circ\int_\Omega B_\tau^{-1}d\mu(\tau)+\int_\Omega A_\tau^{-1}d\mu(\tau)\circ\int_\Omega B_\tau d\mu(\tau)}{2}-1\right) \\
&\leq \int_\Omega \frac{A_\tau+B_\tau}{2}d\mu(\tau)\circ 1-\int_\Omega A_\tau^{1/2}d\mu(\tau)\circ\int_\Omega B_\tau^{1/2}d\mu(\tau) \\
&\leq \frac{1}{4}M \\
&\times\left(\frac{\int_\Omega A_\tau d\mu(\tau)\circ\int_\Omega B_\tau^{-1}d\mu(\tau)+\int_\Omega A_\tau^{-1}d\mu(\tau)\circ\int_\Omega B_\tau d\mu(\tau)}{2}-1\right).
\end{aligned}$$

Remark 3. *Let $(A_\tau)_{\tau\in\Omega}$ be a continuous field of positive operators in $B(H)$ such that $\operatorname{Sp}(A_\tau)\subseteq[m,M]\subset(0,\infty)$ for each $\tau\in\Omega$. Then for all $\nu\in[0,1]$ we have, by taking $B_\tau=A_\tau$, $\tau\in\Omega$ in Corollary 2 that*

$$\begin{aligned}
(2.29)\qquad &\frac{1}{M}\nu(1-\nu)\left[\int_\Omega A_\tau^2 d\mu(\tau)\circ 1-\int_\Omega A_\tau d\mu(\tau)\circ\int_\Omega A_\tau d\mu(\tau)\right] \\
&\leq \int_\Omega A_\tau d\mu(\tau)\circ 1-\int_\Omega A_\tau^{1-\nu}d\mu(\tau)\circ\int_\Omega A_\tau^{\nu}d\mu(\tau) \\
&\leq \frac{1}{m}\nu(1-\nu)\left[\int_\Omega A_\tau^2 d\mu(\tau)\circ 1-\int_\Omega A_\tau d\mu(\tau)\circ\int_\Omega A_\tau d\mu(\tau)\right]
\end{aligned}$$

and

$$\begin{aligned}
(2.30)\qquad 0 &\leq m\nu(1-\nu)\left(\int_\Omega A_\tau d\mu(\tau)\circ\int_\Omega A_\tau^{-1}d\mu(\tau)-1\right) \\
&\leq \int_\Omega A_\tau d\mu(\tau)\circ 1-\int_\Omega A_\tau^{1-\nu}d\mu(\tau)\circ\int_\Omega A_\tau^{\nu}d\mu(\tau) \\
&\leq M\nu(1-\nu)\left(\int_\Omega A_\tau d\mu(\tau)\circ\int_\Omega A_\tau^{-1}d\mu(\tau)-1\right).
\end{aligned}$$

For $\nu=1/2$ we obtain

$$\begin{aligned}
(2.31)\qquad &\frac{1}{4M}\left[\int_\Omega A_\tau^2 d\mu(\tau)\circ 1-\int_\Omega A_\tau d\mu(\tau)\circ\int_\Omega A_\tau d\mu(\tau)\right] \\
&\leq \int_\Omega A_\tau d\mu(\tau)\circ 1-\int_\Omega A_\tau^{1/2}d\mu(\tau)\circ\int_\Omega A_\tau^{1/2}d\mu(\tau) \\
&\leq \frac{1}{4m}\left[\int_\Omega A_\tau^2 d\mu(\tau)\circ 1-\int_\Omega A_\tau d\mu(\tau)\circ\int_\Omega A_\tau d\mu(\tau)\right]
\end{aligned}$$

and

$$\begin{aligned}
(2.32)\qquad 0 &\leq \frac{1}{4}m\left(\int_\Omega A_\tau d\mu(\tau)\circ\int_\Omega A_\tau^{-1}d\mu(\tau)-1\right) \\
&\leq \int_\Omega A_\tau d\mu(\tau)\circ 1-\int_\Omega A_\tau^{1/2}d\mu(\tau)\circ\int_\Omega A_\tau^{1/2}d\mu(\tau) \\
&\leq \frac{1}{4}M\left(\int_\Omega A_\tau d\mu(\tau)\circ\int_\Omega A_\tau^{-1}d\mu(\tau)-1\right).
\end{aligned}$$

3. Some Related Results

We also have:

Lemma 2. *Assume that the finite intervals $I,\ J\subset(0,\infty)$ satisfy the condition $I/J:=\{x/y,\ x\in I,\ y\in J\}\subseteq[\gamma,\Gamma]\subset(0,\infty)$. If $A,\ B$ are selfadjoint operators with $\operatorname{Sp}(A)\subset I$ and $\operatorname{Sp}(B)\subset J$, then*

$$\begin{aligned}
(3.1)\qquad 0 &\le \frac{1}{2}\nu(1-\nu)\,c(\gamma,\Gamma)\\
&\le \frac{1}{\max\{\Gamma,1\}}\nu(1-\nu)\left(\frac{A^2\otimes B^{-2}+1}{2}-A\otimes B^{-1}\right)\\
&\le 1-\nu+\nu A\otimes B^{-1}-A^{\nu}\otimes B^{-\nu}\\
&\le \frac{1}{\min\{\gamma,1\}}\nu(1-\nu)\left(\frac{A^2\otimes B^{-2}+1}{2}-A\otimes B^{-1}\right)\\
&\le \frac{1}{2}\nu(1-\nu)\,C(\gamma,\Gamma),
\end{aligned}$$

where

$$c(\gamma,\Gamma):=\begin{cases}(\Gamma-1)^2 & \text{if } \Gamma<1,\\ 0 & \text{if } \gamma\le 1\le\Gamma,\\ \frac{(\gamma-1)^2}{\Gamma} & \text{if } 1<\gamma\end{cases}$$

and

$$C(\gamma,\Gamma):=\begin{cases}\frac{(\gamma-1)^2}{\gamma} & \text{if } \Gamma<1,\\ \frac{1}{\gamma}\max\left\{(\gamma-1)^2,(\Gamma-1)^2\right\} & \text{if } \gamma\le 1\le\Gamma,\\ (\Gamma-1)^2 & \text{if } 1<\gamma.\end{cases}$$

We also have

$$\begin{aligned}
(3.2)\qquad 0 &\le \frac{1}{2}\nu(1-\nu)\,c(\gamma,\Gamma)(1\otimes B)\\
&\le \frac{1}{\max\{\Gamma,1\}}\nu(1-\nu)\left(\frac{A^2\otimes B^{-1}+1\otimes B}{2}-A\otimes 1\right)\\
&\le (1-\nu)(1\otimes B)+\nu A\otimes 1-A^{\nu}\otimes B^{1-\nu}\\
&\le \frac{1}{\min\{\gamma,1\}}\nu(1-\nu)\left(\frac{A^2\otimes B^{-1}+1\otimes B}{2}-A\otimes 1\right)\\
&\le \frac{1}{2}\nu(1-\nu)\,C(\gamma,\Gamma)(1\otimes B).
\end{aligned}$$

Proof. If we write the inequality (1.1) for $a=1$ and $b=x$ we get

$$(3.3)\qquad \frac{1}{2}\nu(1-\nu)\frac{(x-1)^2}{\max\{x,1\}}\le 1-\nu+\nu x-x^{\nu}\le\frac{1}{2}\nu(1-\nu)\frac{(x-1)^2}{\min\{x,1\}}$$

for any $x>0$ and for any $\nu\in[0,1]$.

If $x\in[\gamma,\Gamma]\subset(0,\infty)$, then $\max\{x,1\}\leq\max\{\Gamma,1\}$ and $\min\{\gamma,1\}\leq\min\{x,1\}$ and by (3.3) we get

$$0\leq\frac{1}{2}\nu(1-\nu)\frac{\min_{x\in[\gamma,\Gamma]}(x-1)^2}{\max\{\Gamma,1\}}\leq\frac{1}{2}\nu(1-\nu)\frac{(x-1)^2}{\max\{\Gamma,1\}}\tag{3.4}$$
$$\leq 1-\nu+\nu x-x^{\nu}\leq\frac{1}{2}\nu(1-\nu)\frac{(x-1)^2}{\min\{\gamma,1\}}$$
$$\leq\frac{1}{2}\nu(1-\nu)\frac{\max_{x\in[\gamma,\Gamma]}(x-1)^2}{\min\{\gamma,1\}}$$

for any $x\in[\gamma,\Gamma]$ and for any $\nu\in[0,1]$.

Observe that

$$\min_{x\in[\gamma,\Gamma]}(x-1)^2=\begin{cases}(\Gamma-1)^2 \text{ if } \Gamma<1,\\ 0 \text{ if } \gamma\leq 1\leq\Gamma,\\ (\gamma-1)^2 \text{ if } 1<\gamma\end{cases}$$

and

$$\max_{x\in[\gamma,\Gamma]}(x-1)^2=\begin{cases}(\gamma-1)^2 \text{ if } \Gamma<1,\\ \max\left\{(\gamma-1)^2,(\Gamma-1)^2\right\} \text{ if } \gamma\leq 1\leq\Gamma,\\ (\Gamma-1)^2 \text{ if } 1<\gamma.\end{cases}$$

Then

$$\frac{\min_{x\in[\gamma,\Gamma]}(x-1)^2}{\max\{\Gamma,1\}}=c(\gamma,\Gamma)$$

and

$$\frac{\max_{x\in[\gamma,\Gamma]}(x-1)^2}{\min\{\gamma,1\}}=C(\gamma,\Gamma).$$

Using the inequality (3.4) we have

$$0\leq\frac{1}{2}\nu(1-\nu)c(\gamma,\Gamma)\leq\frac{1}{2}\nu(1-\nu)\frac{(x-1)^2}{\max\{\Gamma,1\}}\tag{3.5}$$
$$\leq 1-\nu+\nu x-x^{\nu}\leq\frac{1}{2}\nu(1-\nu)\frac{(x-1)^2}{\min\{\gamma,1\}}$$
$$\leq\frac{1}{2}\nu(1-\nu)C(\gamma,\Gamma)$$

for any $x\in[\gamma,\Gamma]$ and for any $\nu\in[0,1]$.

Let t, $s>0$ such that $\frac{t}{s}\in[\gamma,\Gamma]$, then by (3.5) we get for $x=\frac{t}{s}$ that

$$0\leq\frac{1}{2}\nu(1-\nu)c(\gamma,\Gamma)\leq\frac{1}{2\max\{\Gamma,1\}}\nu(1-\nu)\left(t^2s^{-2}+1-2ts^{-1}\right)\tag{3.6}$$
$$\leq 1-\nu+\nu ts^{-1}-t^{\nu}s^{-\nu}$$
$$\leq\frac{1}{2\min\{\gamma,1\}}\nu(1-\nu)\left(t^2s^{-2}+1-2ts^{-1}\right)\leq\frac{1}{2}\nu(1-\nu)C(\gamma,\Gamma).$$

If

$$A=\int_I tdE(t) \text{ and } B=\int_J sdF(s)$$

are the spectral resolutions of A and B, then by taking the double integral $\int_I \int_J$ over $dE(t) \otimes dF(s)$, we get

$$\begin{aligned}
0 &\leq \frac{1}{2}\nu(1-\nu)\, c(\gamma,\Gamma) \int_I \int_J dE(t) \otimes dF(s) \\
&\leq \frac{1}{2\max\{\Gamma,1\}}\nu(1-\nu) \int_I \int_J \left(t^2 s^{-2} + 1 - 2ts^{-1}\right) dE(t) \otimes dF(s) \\
&\leq \int_I \int_J \left[1 - \nu + \nu t s^{-1} - t^{\nu} s^{-\nu}\right] dE(t) \otimes dF(s) \\
&\leq \frac{1}{2\min\{\gamma,1\}}\nu(1-\nu) \int_I \int_J \left(t^2 s^{-2} + 1 - 2ts^{-1}\right) dE(t) \otimes dF(s) \\
&\leq \frac{1}{2}\nu(1-\nu)\, C(\gamma,\Gamma) \int_I \int_J dE(t) \otimes dF(s),
\end{aligned}$$

which is equivalent to (3.1).

Now, if we multiply by s in the inequality (3.6), then we get

$$\begin{aligned}
0 &\leq \frac{1}{2}\nu(1-\nu)\, c(\gamma,\Gamma)\, s \leq \frac{1}{2\max\{\Gamma,1\}}\nu(1-\nu)\left(t^2 s^{-1} + s - 2t\right) \\
&\leq (1-\nu)\, s + \nu t - t^{\nu} s^{1-\nu} \\
&\leq \frac{1}{2\min\{\gamma,1\}}\nu(1-\nu)\left(t^2 s^{-1} + s - 2t\right) \\
&\leq \frac{1}{2}\nu(1-\nu)\, C(\gamma,\Gamma)\, s,
\end{aligned}$$

for $t \in I$, $s \in J$, which, by a similar argument as above, gives the desired tensorial inequality (3.2). □

Remark 4. *We observe that if* $0 < \gamma_1 \leq A \leq \Gamma_1$ *and* $0 < \gamma_2 \leq B \leq \Gamma_2$ *then we can take* $\gamma = \frac{\gamma_1}{\Gamma_2}$ *and* $\Gamma = \frac{\Gamma_1}{\gamma_2}$ *in the above inequalities (3.1) and (3.2). If* $\gamma_2 = \gamma_1 = m$ *and* $\Gamma_2 = \Gamma_1 = M$ *then we can take* $\gamma = \frac{m}{M} \leq 1$ *and* $\Gamma = \frac{M}{m} \geq 1$ *in (3.1) and (3.2).*

Corollary 3. *With the assumptions of Lemma 2, we have the following inequalities for the Hadamard product*

$$\begin{aligned}
0 &\leq \frac{1}{2}\nu(1-\nu)\, c(\gamma,\Gamma) \\
&\leq \frac{1}{\max\{\Gamma,1\}}\nu(1-\nu)\left(\frac{A^2 \circ B^{-2} + 1}{2} - A \circ B^{-1}\right) \\
&\leq 1 - \nu + \nu A \circ B^{-1} - A^{\nu} \circ B^{-\nu} \\
&\leq \frac{1}{\min\{\gamma,1\}}\nu(1-\nu)\left(\frac{A^2 \circ B^{-2} + 1}{2} - A \circ B^{-1}\right) \\
&\leq \frac{1}{2}\nu(1-\nu)\, C(\gamma,\Gamma),
\end{aligned} \tag{3.7}$$

and

$$0 \le \frac{1}{2}\nu(1-\nu)c(\gamma,\Gamma)(1\circ B) \tag{3.8}$$
$$\le \frac{1}{\max\{\Gamma,1\}}\nu(1-\nu)\left(\frac{A^2\circ B^{-1}+1\circ B}{2}-A\circ 1\right)$$
$$\le [(1-\nu)B+\nu A]\circ 1 - A^\nu \circ B^{1-\nu}$$
$$\le \frac{1}{\min\{\gamma,1\}}\nu(1-\nu)\left(\frac{A^2\circ B^{-1}+1\circ B}{2}-A\circ 1\right)$$
$$\le \frac{1}{2}\nu(1-\nu)C(\gamma,\Gamma)(1\circ B)$$

for all $\nu \in [0,1]$.

Remark 5. *We observe that, if* $0 < m \le A$, $B \le M$, *then by Corollary 3 we get*

$$0 \le \frac{m}{M}\nu(1-\nu)\left(\frac{A^2\circ B^{-2}+1}{2}-A\circ B^{-1}\right) \tag{3.9}$$
$$\le 1-\nu+\nu A\circ B^{-1} - A^\nu\circ B^{-\nu}$$
$$\le \frac{M}{m}\nu(1-\nu)\left(\frac{A^2\circ B^{-2}+1}{2}-A\circ B^{-1}\right)$$
$$\le \frac{1}{2}\nu(1-\nu)\frac{M}{m}\left(\frac{M}{m}-1\right)^2,$$

and

$$0 \le \frac{m}{M}\nu(1-\nu)\left(\frac{A^2\circ B^{-1}+1\circ B}{2}-A\circ 1\right) \tag{3.10}$$
$$\le [(1-\nu)B+\nu A]\circ 1 - A^\nu\circ B^{1-\nu}$$

$$\le \frac{M}{m}\nu(1-\nu)\left(\frac{A^2\circ B^{-1}+1\circ B}{2}-A\circ 1\right)$$
$$\le \frac{1}{2}\nu(1-\nu)\frac{M}{m}\left(\frac{M}{m}-1\right)^2(1\circ B).$$

In particular, for $\nu = 1/2$, *we derive*

$$0 \le \frac{m}{4M}\left(\frac{A^2\circ B^{-2}+1}{2}-A\circ B^{-1}\right) \tag{3.11}$$
$$\le \frac{1}{2}\left(1+A\circ B^{-1}\right) - A^{1/2}\circ B^{-1/2}$$
$$\le \frac{M}{4m}\left(\frac{A^2\circ B^{-2}+1}{2}-A\circ B^{-1}\right) \le \frac{M}{8m}\left(\frac{M}{m}-1\right)^2,$$

and

$$0 \le \frac{m}{4M}\left(\frac{A^2 \circ B^{-1} + 1 \circ B}{2} - A \circ 1\right) \tag{3.12}$$
$$\le \frac{A+B}{2} \circ 1 - A^{\nu} \circ B^{1-\nu}$$
$$\le \frac{M}{4m}\left(\frac{A^2 \circ B^{-1} + 1 \circ B}{2} - A \circ 1\right) \le \frac{1}{8}\frac{M}{m}\left(\frac{M}{m} - 1\right)^2 (1 \circ B).$$

Moreover, if $0 < m \le A \le M$, then by taking $B = A$ in (3.9)-(3.12), we get

$$0 \le \frac{m}{M}\nu(1-\nu)\left(\frac{A^2 \circ A^{-2} + 1}{2} - A \circ A^{-1}\right) \tag{3.13}$$
$$\le 1 - \nu + \nu A \circ A^{-1} - A^{\nu} \circ A^{-\nu}$$
$$\le \frac{M}{m}\nu(1-\nu)\left(\frac{A^2 \circ A^{-2} + 1}{2} - A \circ A^{-1}\right) \le \frac{1}{2}\nu(1-\nu)\frac{M}{m}\left(\frac{M}{m} - 1\right)^2,$$

and

$$0 \le \frac{m}{2M}\nu(1-\nu)\left(A^2 \circ B^{-1} - A \circ 1\right) \le A \circ 1 - A^{\nu} \circ A^{1-\nu} \tag{3.14}$$
$$\le \frac{M}{2m}\nu(1-\nu)\left(A^2 \circ B^{-1} - A \circ 1\right)$$
$$\le \frac{1}{2}\nu(1-\nu)\frac{M}{m}\left(\frac{M}{m} - 1\right)^2 (1 \circ A).$$

In particular, for $\nu = 1/2$, we derive

$$0 \le \frac{m}{4M}\left(\frac{A^2 \circ A^{-2} + 1}{2} - A \circ A^{-1}\right) \le \frac{1 + A \circ A^{-1}}{2} - A^{1/2} \circ A^{-1/2} \tag{3.15}$$
$$\le \frac{M}{4m}\left(\frac{A^2 \circ A^{-2} + 1}{2} - A \circ A^{-1}\right) \le \frac{M}{8m}\left(\frac{M}{m} - 1\right)^2,$$

and

$$0 \le \frac{m}{8M}\left(A^2 \circ A^{-1} - A \circ 1\right) \le A \circ 1 - A^{1/2} \circ A^{-1/2} \tag{3.16}$$
$$\le \frac{M}{8m}\left(A^2 \circ A^{-1} - A \circ 1\right) \le \frac{1}{8}\frac{M}{m}\left(\frac{M}{m} - 1\right)^2 (1 \circ A).$$

Let $0 < \gamma_1 \le A \le \Gamma_1$ and $0 < \gamma_2 \le B \le \Gamma_2$ then we can take $\gamma = \frac{\gamma_1}{\Gamma_2}$ and $\Gamma = \frac{\Gamma_1}{\gamma_2}$ and consider

$$c(\gamma_1, \gamma_2, \Gamma_1, \Gamma_2) := \begin{cases} \left(\frac{\Gamma_1}{\gamma_2} - 1\right)^2 \text{ if } \frac{\Gamma_1}{\gamma_2} < 1, \\ 0 \text{ if } \frac{\gamma_1}{\Gamma_2} \le 1 \le \frac{\Gamma_1}{\gamma_2}, \\ \frac{\gamma_2}{\Gamma_1}\left(\frac{\gamma_1}{\Gamma_2} - 1\right)^2 \text{ if } 1 < \frac{\gamma_1}{\Gamma_2} \end{cases}$$

and

$$C\left(\gamma_1,\gamma_2,\Gamma_1,\Gamma_2\right):=\begin{cases}\frac{\Gamma_2}{\gamma_1}\left(\frac{\gamma_1}{\Gamma_2}-1\right)^2 \text{ if } \frac{\Gamma_1}{\gamma_2}<1,\\ \frac{\Gamma_2}{\gamma_1}\max\left\{\left(\frac{\gamma_1}{\Gamma_2}-1\right)^2,\left(\frac{\Gamma_1}{\gamma_2}-1\right)^2\right\}\\ \text{if } \frac{\gamma_1}{\Gamma_2}\leq 1\leq\frac{\Gamma_1}{\gamma_2},\\ \left(\frac{\Gamma_1}{\gamma_2}-1\right)^2 \text{ if } 1<\frac{\gamma_1}{\Gamma_2}.\end{cases}$$

We also have

Theorem 2. *Let* $(A_\tau)_{\tau\in\Omega}$ *and* $(B_\tau)_{\tau\in\Omega}$ *be continuous fields of positive operators in* $B(H)$ *such that* $\operatorname{Sp}(A_\tau)\subseteq[\gamma_1,\Gamma_1]$, $\operatorname{Sp}(B_\tau)\subseteq[\gamma_1,\Gamma_1]\subset(0,\infty)$ *for each* $\tau\in\Omega$. *Then for all* $\nu\in[0,1]$ *we have*

$$\begin{aligned}(3.17)\quad 0&\leq\frac{1}{2}\nu(1-\nu)c(\gamma_1,\gamma_2,\Gamma_1,\Gamma_2)\\ &\leq\frac{1}{\max\left\{\frac{\Gamma_1}{\gamma_2},1\right\}}\nu(1-\nu)\\ &\times\left(\frac{\int_\Omega A_\tau^2d\mu(\tau)\otimes\int_\Omega B_\tau^{-2}d\mu(\tau)+1}{2}-\int_\Omega A_\tau d\mu(\tau)\otimes\int_\Omega B_\tau^{-1}d\mu(\tau)\right)\\ &\leq 1-\nu+\nu\int_\Omega A_\tau d\mu(\tau)\otimes\int_\Omega B_\tau^{-1}d\mu(\tau)\\ &-\int_\Omega A_\tau^{\nu}d\mu(\tau)\otimes\int_\Omega B_\tau^{-\nu}d\mu(\tau)\\ &\leq\frac{1}{\min\left\{\frac{\gamma_1}{\Gamma_2},1\right\}}\\ &\times\left(\frac{\int_\Omega A_\tau^2d\mu(\tau)\otimes\int_\Omega B_\tau^{-2}d\mu(\tau)+1}{2}-\int_\Omega A_\tau d\mu(\tau)\otimes\int_\Omega B_\tau^{-1}d\mu(\tau)\right)\\ &\leq\frac{1}{2}\nu(1-\nu)C(\gamma_1,\gamma_2,\Gamma_1,\Gamma_2)\end{aligned}$$

and

$$\begin{aligned}(3.18)\quad 0&\leq\frac{1}{2}\nu(1-\nu)c(\gamma_1,\gamma_2,\Gamma_1,\Gamma_2)\left(1\otimes\int_\Omega B_\tau d\mu(\tau)\right)\\ &\leq\frac{1}{\max\left\{\frac{\Gamma_1}{\gamma_2},1\right\}}\nu(1-\nu)\\ &\times\left(\frac{\int_\Omega A_\tau^2d\mu(\tau)\otimes\int_\Omega B_\tau^{-1}d\mu(\tau)+1\otimes\int_\Omega B_\tau d\mu(\tau)}{2}-\int_\Omega A_\tau d\mu(\tau)\otimes 1\right)\\ &\leq(1-\nu)1\otimes\int_\Omega B_\tau d\mu(\tau)+\nu\int_\Omega A_\tau d\mu(\tau)\otimes 1\\ &-\int_\Omega A_\tau^{\nu}d\mu(\tau)\otimes\int_\Omega B_\tau^{1-\nu}d\mu(\tau)\end{aligned}$$

$$\leq \frac{1}{\min\left\{\frac{\gamma_1}{\Gamma_2}, 1\right\}} \nu(1-\nu)$$
$$\times \left(\frac{\int_\Omega A_\tau^2 d\mu(\tau) \otimes \int_\Omega B_\tau^{-1} d\mu(\tau) + 1 \otimes \int_\Omega B_\tau d\mu(\tau)}{2} - \int_\Omega A_\tau d\mu(\tau) \otimes 1 \right)$$
$$\leq \frac{1}{2}\nu(1-\nu) C(\gamma_1, \gamma_2, \Gamma_1, \Gamma_2) \left(1 \otimes \int_\Omega B_\tau d\mu(\tau) \right).$$

Proof. From (3.1) we have

$$\begin{aligned} 0 &\leq \frac{1}{2}\nu(1-\nu) c(\gamma_1, \gamma_2, \Gamma_1, \Gamma_2) \\ &\leq \frac{1}{\max\left\{\frac{\Gamma_1}{\gamma_2}, 1\right\}} \nu(1-\nu) \left(\frac{A_\tau^2 \otimes B_\gamma^{-2} + 1}{2} - A_\tau \otimes B_\gamma^{-1} \right) \\ &\leq 1 - \nu + \nu A_v \otimes B^{-1} - A_v^\nu \otimes B^{-\nu} \\ &\leq \frac{1}{\min\left\{\frac{\gamma_1}{\Gamma_2}, 1\right\}} \nu(1-\nu) \left(\frac{A_\tau^2 \otimes B_\gamma^{-2} + 1}{2} - A_\tau \otimes B_\gamma^{-1} \right) \\ &\leq \frac{1}{2}\nu(1-\nu) C(\gamma_1, \gamma_2, \Gamma_1, \Gamma_2), \end{aligned} \tag{3.19}$$

for all τ, $\gamma \in \Omega$.

Now, by employing a similar argument to the one in the proof of Theorem 1, we deduce the desired result (3.17). □

Corollary 4. *With the assumptions of Theorem 2, we have the Hadamard product inequalities*

$$\begin{aligned} 0 &\leq \frac{1}{2}\nu(1-\nu) c(\gamma_1, \gamma_2, \Gamma_1, \Gamma_2) \\ &\leq \frac{1}{\max\left\{\frac{\Gamma_1}{\gamma_2}, 1\right\}} \nu(1-\nu) \\ &\times \left(\frac{\int_\Omega A_\tau^2 d\mu(\tau) \circ \int_\Omega B_\tau^{-2} d\mu(\tau) + 1}{2} - \int_\Omega A_\tau d\mu(\tau) \circ \int_\Omega B_\tau^{-1} d\mu(\tau) \right) \\ &\leq 1 - \nu + \nu \int_\Omega A_\tau d\mu(\tau) \circ \int_\Omega B_\tau^{-1} d\mu(\tau) \\ &- \int_\Omega A_\tau^\nu d\mu(\tau) \circ \int_\Omega B_\tau^{-\nu} d\mu(\tau) \\ &\leq \frac{1}{\min\left\{\frac{\gamma_1}{\Gamma_2}, 1\right\}} \\ &\times \left(\frac{\int_\Omega A_\tau^2 d\mu(\tau) \circ \int_\Omega B_\tau^{-2} d\mu(\tau) + 1}{2} - \int_\Omega A_\tau d\mu(\tau) \circ \int_\Omega B_\tau^{-1} d\mu(\tau) \right) \\ &\leq \frac{1}{2}\nu(1-\nu) C(\gamma_1, \gamma_2, \Gamma_1, \Gamma_2) \end{aligned} \tag{3.20}$$

and

$$
\begin{aligned}
(3.21)\quad 0 &\leq \frac{1}{2}\nu(1-\nu)\,c(\gamma_1,\gamma_2,\Gamma_1,\Gamma_2)\left(1\circ\int_\Omega B_\tau d\mu(\tau)\right)\\
&\leq \frac{1}{\max\left\{\frac{\Gamma_1}{\gamma_2},1\right\}}\nu(1-\nu)\\
&\times\left(\frac{\int_\Omega A_\tau^2 d\mu(\tau)\circ\int_\Omega B_\tau^{-1}d\mu(\tau)+1\circ\int_\Omega B_\tau d\mu(\tau)}{2}-\int_\Omega A_\tau d\mu(\tau)\circ 1\right)\\
&\leq \int_\Omega((1-\nu)B_\tau+\nu A_\tau)\,d\mu(\tau)\circ 1-\int_\Omega A_\tau^\nu d\mu(\tau)\circ\int_\Omega B_\tau^{1-\nu}d\mu(\tau)\\
&\leq \frac{1}{\min\left\{\frac{\gamma_1}{\Gamma_2},1\right\}}\nu(1-\nu)\\
&\times\left(\frac{\int_\Omega A_\tau^2 d\mu(\tau)\circ\int_\Omega B_\tau^{-1}d\mu(\tau)+1\circ\int_\Omega B_\tau d\mu(\tau)}{2}-\int_\Omega A_\tau d\mu(\tau)\circ 1\right)\\
&\leq \frac{1}{2}\nu(1-\nu)\,C(\gamma_1,\gamma_2,\Gamma_1,\Gamma_2)\left(1\circ\int_\Omega B_\tau d\mu(\tau)\right).
\end{aligned}
$$

Remark 6. *Let $(A_\tau)_{\tau\in\Omega}$ and $(B_\tau)_{\tau\in\Omega}$ be continuous fields of positive operators in $B(H)$ such that $\operatorname{Sp}(A_\tau)$, $\operatorname{Sp}(B_\tau)\subseteq[m,M]\subset(0,\infty)$ for each $\tau\in\Omega$. Then for all $\nu\in[0,1]$ we have*

$$
\begin{aligned}
(3.22)\quad 0 &\leq \frac{m}{M}\nu(1-\nu)\\
&\times\left(\frac{\int_\Omega A_\tau^2 d\mu(\tau)\circ\int_\Omega B_\tau^{-2}d\mu(\tau)+1}{2}-\int_\Omega A_\tau d\mu(\tau)\circ\int_\Omega B_\tau^{-1}d\mu(\tau)\right)\\
&\leq 1-\nu+\nu\int_\Omega A_\tau d\mu(\tau)\circ\int_\Omega B_\tau^{-1}d\mu(\tau)\\
&-\int_\Omega A_\tau^\nu d\mu(\tau)\circ\int_\Omega B_\tau^{-\nu}d\mu(\tau)\\
&\leq \frac{M}{m}\nu(1-\nu)\\
&\times\left(\frac{\int_\Omega A_\tau^2 d\mu(\tau)\circ\int_\Omega B_\tau^{-2}d\mu(\tau)+1}{2}-\int_\Omega A_\tau d\mu(\tau)\circ\int_\Omega B_\tau^{-1}d\mu(\tau)\right)\\
&\leq \frac{1}{2}\nu(1-\nu)\frac{M}{m}\left(\frac{M}{m}-1\right)^2
\end{aligned}
$$

and

$$
\begin{aligned}
(3.23)\quad 0 &\leq \frac{m}{M}\nu(1-\nu)\\
&\times\left(\frac{\int_\Omega A_\tau^2 d\mu(\tau)\circ\int_\Omega B_\tau^{-1}d\mu(\tau)+1\circ\int_\Omega B_\tau d\mu(\tau)}{2}-\int_\Omega A_\tau d\mu(\tau)\circ 1\right)\\
&\leq \int_\Omega((1-\nu)B_\tau+\nu A_\tau)\,d\mu(\tau)\circ 1-\int_\Omega A_\tau^\nu d\mu(\tau)\circ\int_\Omega B_\tau^{1-\nu}d\mu(\tau)
\end{aligned}
$$

$$
\begin{aligned}
&\leq \frac{M}{m}\nu(1-\nu) \\
&\times\left(\frac{\int_\Omega A_\tau^2 d\mu(\tau)\circ\int_\Omega B_\tau^{-1}d\mu(\tau)+1\circ\int_\Omega B_\tau d\mu(\tau)}{2}-\int_\Omega A_\tau d\mu(\tau)\circ 1\right) \\
&\leq \frac{1}{2}\nu(1-\nu)\frac{M}{m}\left(\frac{M}{m}-1\right)^2\left(1\circ\int_\Omega B_\tau d\mu(\tau)\right).
\end{aligned}
$$

For $B_\tau = A_\tau$, $\tau\in\Omega$ *we derive*

$$
\begin{aligned}
0 &\leq \frac{m}{M}\nu(1-\nu) \\
&\times\left(\frac{\int_\Omega A_\tau^2 d\mu(\tau)\circ\int_\Omega A_\tau^{-2}d\mu(\tau)+1}{2}-\int_\Omega A_\tau d\mu(\tau)\circ\int_\Omega A_\tau^{-1}d\mu(\tau)\right) \\
&\leq 1-\nu+\nu\int_\Omega A_\tau d\mu(\tau)\circ\int_\Omega A_\tau^{-1}d\mu(\tau) \\
&-\int_\Omega A_\tau^{\nu}d\mu(\tau)\circ\int_\Omega A_\tau^{-\nu}d\mu(\tau) \\
&\leq \frac{M}{m}\nu(1-\nu) \\
&\times\left(\frac{\int_\Omega A_\tau^2 d\mu(\tau)\circ\int_\Omega A_\tau^{-2}d\mu(\tau)+1}{2}-\int_\Omega A_\tau d\mu(\tau)\circ\int_\Omega A_\tau^{-1}d\mu(\tau)\right) \\
&\leq \frac{1}{2}\nu(1-\nu)\frac{M}{m}\left(\frac{M}{m}-1\right)^2
\end{aligned}
\tag{3.24}
$$

and

$$
\begin{aligned}
0 &\leq \frac{m}{M}\nu(1-\nu) \\
&\times\left(\frac{\int_\Omega A_\tau^2 d\mu(\tau)\circ\int_\Omega A_\tau^{-1}d\mu(\tau)+1\circ\int_\Omega A_\tau d\mu(\tau)}{2}-\int_\Omega A_\tau d\mu(\tau)\circ 1\right) \\
&\leq \int_\Omega A_\tau d\mu(\tau)\circ 1-\int_\Omega A_\tau^{\nu}d\mu(\tau)\circ\int_\Omega A_\tau^{1-\nu}d\mu(\tau) \\
&\leq \frac{M}{m}\nu(1-\nu) \\
&\times\left(\frac{\int_\Omega A_\tau^2 d\mu(\tau)\circ\int_\Omega A_\tau^{-1}d\mu(\tau)+1\circ\int_\Omega A_\tau d\mu(\tau)}{2}-\int_\Omega A_\tau d\mu(\tau)\circ 1\right) \\
&\leq \frac{1}{2}\nu(1-\nu)\frac{M}{m}\left(\frac{M}{m}-1\right)^2\left(1\circ\int_\Omega A_\tau d\mu(\tau)\right).
\end{aligned}
\tag{3.25}
$$

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