REVERSES OF SOME INEQUALITIES FOR THE NORMALIZED DETERMINANTS OF SEQUENCES OF POSITIVE OPERATORS IN HILBERT SPACES

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Abstract. For positive invertible operators A on a Hilbert space H and a fixed unit vector $x \in H$, define the normalized determinant by $\Delta_x(A) := \exp \langle \ln Ax, x \rangle$. In this paper we prove among others that, if $0 < m \le A_j \le M$ and $p_j \ge 0$, $j \in \{1, ..., n\}$ with $\sum_{j=1}^n p_j = 1$, then

$$
\begin{aligned}
1 &\le \frac{\Delta_x\left(\sum_{j=1}^n p_j A_j\right)}{\prod_{j=1}^n \left[\Delta_x(A_j)\right]^{p_j}} \\
&\le \frac{\sum_{j=1}^n p_j \langle A_j x, x \rangle}{\prod_{j=1}^n \left[\Delta_x(A_j)\right]^{p_j}} \le \exp\left(\sum_{j=1}^n p_j \langle A_j x, x \rangle \sum_{j=1}^n p_j \left\langle A_j^{-1} x, x \right\rangle - 1\right) \\
&\le \begin{cases} \exp\left(\frac{M-m}{2}\left[\sum_{j=1}^n p_j \left\| A_j^{-1} x \right\|^2 - \left(\sum_{j=1}^n p_j \left\langle A_j^{-1} x, x \right\rangle\right)^2\right]^{1/2}\right), \\ \\ \exp\left(\frac{M-m}{2mM}\left[\sum_{j=1}^n p_j \| A_j x \|^2 - \left(\sum_{j=1}^n p_j \langle A_j x, x \rangle\right)^2\right]^{1/2}\right) \end{cases} \\
&\le \exp\left(\frac{(M-m)^2}{4mM}\right)
\end{aligned}
$$

for $x \in H$, $\|x\| = 1$.

1. Introduction

Let $B(H)$ be the space of all bounded linear operators on a Hilbert space H, and I stands for the identity operator on H. An operator A in $B(H)$ is said to be positive (in symbol: $A \ge 0$) if $\langle Ax, x \rangle \ge 0$ for all $x \in H$. In particular, $A > 0$ means that A is positive and invertible. For a pair A, B of selfadjoint operators the order relation $A \ge B$ means as usual that $A - B$ is positive.

In 1998, Fujii et al. [5], [6], introduced the *normalized determinant* $\Delta_x(A)$ for positive invertible operators A on a Hilbert space H and a fixed unit vector $x \in H$, namely $\|x\| = 1$, defined by $\Delta_x(A) := \exp \langle \ln Ax, x \rangle$ and discussed it as a continuous geometric mean and observed some inequalities around the determinant from this point of view.

Some of the fundamental properties of normalized determinant are as follows, [5].

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For each unit vector $x\in H$, see also [7], we have:

(i) *continuity*: the map $A\rightarrow\Delta_x(A)$ is norm continuous;
(ii) *bounds*: $\left\langle A^{-1}x,x\right\rangle^{-1}\leq\Delta_x(A)\leq\langle Ax,x\rangle$;
(iii) *continuous mean*: $\langle A^px,x\rangle^{1/p}\downarrow\Delta_x(A)$ for $p\downarrow 0$ and $\langle A^px,x\rangle^{1/p}\uparrow\Delta_x(A)$ for $p\uparrow 0$;
(iv) *power equality:* $\Delta_x(A^t)=\Delta_x(A)^t$ for all $t>0$;
(v) *homogeneity*: $\Delta_x(tA)=t\Delta_x(A)$ and $\Delta_x(tI)=t$ for all $t>0$;
(vi) *monotonicity*: $0<A\leq B$ implies $\Delta_x(A)\leq\Delta_x(B)$;
(vii) *multiplicativity*: $\Delta_x(AB)=\Delta_x(A)\Delta_x(B)$ for commuting A and B;
(viii) *Ky Fan type inequality*: $\Delta_x((1-\alpha)A+\alpha B)\geq\Delta_x(A)^{1-\alpha}\Delta_x(B)^{\alpha}$ for $0<\alpha<1$.

We define the logarithmic mean of two positive numbers a, b by

$$L(a,b):=\begin{cases}\frac{b-a}{\ln b-\ln a}\text{ if }b\neq a,\\ \\ a\text{ if }b=a.\end{cases}$$

In [5] the authors obtained the following additive reverse inequality for the operator A which satisfy the condition $0<mI\leq A\leq MI$, where m,M are positive numbers,

$$0\leq\langle Ax,x\rangle-\Delta_x(A)\leq L(m,M)\left[\ln L(m,M)+\frac{M\ln m-m\ln M}{M-m}-1\right]\tag{1.1}$$

for all $x\in H$, $\|x\|=1$.

We recall that *Specht's ratio* is defined by [11]

$$S(h):=\begin{cases}\frac{h^{\frac{1}{h-1}}}{e\ln\left(h^{\frac{1}{h-1}}\right)}\text{ if }h\in(0,1)\cup(1,\infty),\\ \\ 1\text{ if }h=1.\end{cases}\tag{1.2}$$

It is well known that $\lim_{h\rightarrow 1}S(h)=1$, $S(h)=S\left(\frac{1}{h}\right)>1$ for $h>0$, $h\neq 1$. The function is decreasing on $(0,1)$ and increasing on $(1,\infty)$.

In [6], the authors obtained the following multiplicative reverse inequality as well

$$1\leq\frac{\langle Ax,x\rangle}{\Delta_x(A)}\leq S\left(\frac{M}{m}\right)\tag{1.3}$$

for $0<mI\leq A\leq MI$ and $x\in H$, $\|x\|=1$.

The following result that provides an operator version for the Jensen inequality holds [10] (see also [4, p. 5]):

Theorem 1. *Assume that f is a convex function on the interval I, A a selfadjoint operator on the Hilbert space H and assume that* $\operatorname{Sp}(A)\subset\mathring{I}$, *the interior of I, then*

$$f(\langle Ax,x\rangle)\leq\langle f(A)x,x\rangle\tag{MP}$$

for each $x\in H$ with $\|x\|=1$.

As a special case of Theorem 1 we have the following Hölder-McCarthy inequality:

Theorem 2 (Hölder-McCarthy, 1967, [1])**.** *Let A be a selfadjoint positive operator on a Hilbert space H. Then*

(i) $\langle A^r x, x\rangle \geq \langle Ax, x\rangle^r$ *for all* $r > 1$ *and* $x \in H$ *with* $\|x\| = 1$;

(ii) $\langle A^r x, x\rangle \leq \langle Ax, x\rangle^r$ *for all* $0 < r < 1$ *and* $x \in H$ *with* $\|x\| = 1$;

(iii) If A is invertible, then $\langle A^r x, x\rangle \geq \langle Ax, x\rangle^r$ *for all* $r < 0$ *and* $x \in H$ *with* $\|x\| = 1$.

The following theorem is a multiple operator version of Theorem 1 (see for instance [4, p. 5]):

Theorem 3. *Assume that f is a convex function on the interval I. Let A_j be selfadjoint operators with* $\operatorname{Sp}(A_j) \subset \mathring{I}$, $j \in \{1, ..., n\}$ *and* $x_j \in H$, $j \in \{1, ..., n\}$ *with* $\sum_{j=1}^n \|x_j\|^2 = 1$, *then*

$$f\left(\sum_{j=1}^n \langle A_j x_j, x_j\rangle\right) \leq \sum_{j=1}^n \langle f(A_j) x_j, x_j\rangle . \tag{1.4}$$

The following particular case is of interest. Apparently it has not been stated before either in the monograph [4] or in the research papers cited therein.

Corollary 1. *Let A_j be selfadjoint operators with* $\operatorname{Sp}(A_j) \subset \mathring{I}$, $j \in \{1, ..., n\}$. *If* $p_j \geq 0$, $j \in \{1, ..., n\}$ *with* $\sum_{j=1}^n p_j = 1$, *then*

$$f\left(\sum_{j=1}^n p_j \langle A_j x, x\rangle\right) \leq \sum_{j=1}^n p_j \langle f(A_j) x, x\rangle , \tag{1.5}$$

for any $x \in H$ *with* $\|x\| = 1$.

Proof. Follows from Theorem 3 by choosing $x_j = \sqrt{p_j} \cdot x$, $j \in \{1, ..., n\}$ where $x \in H$ with $\|x\| = 1$. □

Motivated by the above results, in this paper we prove among others that, if $0 < m \leq A_j \leq M$ and $p_j \geq 0$, $j \in \{1, ..., n\}$ with $\sum_{j=1}^n p_j = 1$, then

$$\begin{aligned}
1 &\leq \frac{\Delta_x\left(\sum_{j=1}^n p_j A_j\right)}{\prod_{j=1}^n [\Delta_x(A_j)]^{p_j}} \\
&\leq \frac{\sum_{j=1}^n p_j \langle A_j x, x\rangle}{\prod_{j=1}^n [\Delta_x(A_j)]^{p_j}} \leq \exp\left(\sum_{j=1}^n p_j \langle A_j x, x\rangle \sum_{j=1}^n p_j \langle A_j^{-1} x, x\rangle - 1\right) \\
&\leq \begin{cases} \exp\left(\frac{M-m}{2}\left[\sum_{j=1}^n p_j \left\|A_j^{-1} x\right\|^2 - \left(\sum_{j=1}^n p_j \langle A_j^{-1} x, x\rangle\right)^2\right]^{1/2}\right), \\ \exp\left(\frac{M-m}{2mM}\left[\sum_{j=1}^n p_j \|A_j x\|^2 - \left(\sum_{j=1}^n p_j \langle A_j x, x\rangle\right)^2\right]^{1/2}\right) \end{cases} \\
&\leq \exp\left(\frac{(M-m)^2}{4mM}\right)
\end{aligned}$$

for $x \in H$, $\|x\| = 1$.

2. Main Results

We start with the following result:

Theorem 4. *Assume that $A_j > 0$ and $x_j \in H$, $j \in \{1, ..., n\}$ with $\sum_{j=1}^n \|x_j\|^2 = 1$, then*

$$1 \le \frac{\sum_{j=1}^n \langle A_j x_j, x_j\rangle}{\prod_{j=1}^n \Delta_{x_j}(A_j)} \le \exp\left(\sum_{j=1}^n \langle A_j x_j, x_j\rangle \sum_{j=1}^n \left\langle A_j^{-1} x_j, x_j\right\rangle - 1\right) \tag{2.1}$$

and

$$1 \le \frac{\prod_{j=1}^n \Delta_{x_j}(A_j)}{\left(\sum_{j=1}^n \left\langle A_j^{-1} x_j, x_j\right\rangle\right)^{-1}} \le \exp\left(\sum_{j=1}^n \langle A_j x_j, x_j\rangle \sum_{j=1}^n \left\langle A_j^{-1} x_j, x_j\right\rangle - 1\right). \tag{2.2}$$

Proof. Assume that f is a convex function on the interval I. In [2] we obtained the following :

If A_j are selfadjoint operators with $\operatorname{Sp}(A_j) \subset \mathring{I}$, $j \in \{1, ..., n\}$ and $x_j \in H$, $j \in \{1, ..., n\}$ with $\sum_{j=1}^n \|x_j\|^2 = 1$, then

$$\begin{aligned} 0 &\le \sum_{j=1}^n \langle f(A_j) x_j, x_j\rangle - f\left(\sum_{j=1}^n \langle A_j x_j, x_j\rangle\right) \\ &\le \sum_{j=1}^n \langle f'(A_j) A_j x_j, x_j\rangle - \sum_{j=1}^n \langle A_j x_j, x_j\rangle \sum_{j=1}^n \langle f'(A_j) x_j, x_j\rangle . \end{aligned}$$

If we write this inequality for the convex function $f(t) = -\ln t$, $t > 0$, then we get

$$\begin{aligned} 0 &\le \ln\left(\sum_{j=1}^n \langle A_j x_j, x_j\rangle\right) - \sum_{j=1}^n \langle \ln A_j x_j, x_j\rangle \\ &\le \sum_{j=1}^n \langle A_j x_j, x_j\rangle \sum_{j=1}^n \left\langle A_j^{-1} x_j, x_j\right\rangle - 1 \end{aligned} \tag{2.3}$$

for $A_j > 0$ and $x_j \in H$, $j \in \{1, ..., n\}$ with $\sum_{j=1}^n \|x_j\|^2 = 1$.

Now, if we take the exponential in (2.3), then we get

$$1 \le \frac{\sum_{j=1}^n \langle A_j x_j, x_j\rangle}{\prod_{j=1}^n \exp\langle \ln A_j x_j, x_j\rangle} \le \exp\left(\sum_{j=1}^n \langle A_j x_j, x_j\rangle \sum_{j=1}^n \left\langle A_j^{-1} x_j, x_j\right\rangle - 1\right)$$

for $A_j > 0$ and $x_j \in H$, $j \in \{1, ..., n\}$ with $\sum_{j=1}^n \|x_j\|^2 = 1$ and the inequality (2.1) is proved.

Now if we write (2.1) for A_j^{-1}, $j \in \{1, ..., n\}$, we get

$$1 \leq \frac{\sum_{j=1}^{n} \langle A_j^{-1} x_j, x_j \rangle}{\prod_{j=1}^{n} \Delta_{x_j}(A_j^{-1})} \leq \exp\left(\sum_{j=1}^{n} \langle A_j^{-1} x_j, x_j \rangle \sum_{j=1}^{n} \langle A_j x_j, x_j \rangle - 1\right). \tag{2.4}$$

Since

$$\prod_{j=1}^{n} \Delta_{x_j}(A_j^{-1}) = \prod_{j=1}^{n} [\Delta_{x_j}(A_j)]^{-1} = \left(\prod_{j=1}^{n} \Delta_{x_j}(A_j)\right)^{-1},$$

hence from (2.4) we derive (2.2). □

Corollary 2. *Assume that $A_j > 0$ and $p_j \geq 0$, $j \in \{1, ..., n\}$ with $\sum_{j=1}^{n} p_j = 1$, then*

$$1 \leq \frac{\sum_{j=1}^{n} p_j \langle A_j x, x \rangle}{\prod_{j=1}^{n} [\Delta_x(A_j)]^{p_j}} \leq \exp\left(\sum_{j=1}^{n} p_j \langle A_j x, x \rangle \sum_{j=1}^{n} p_j \langle A_j^{-1} x, x \rangle - 1\right) \tag{2.5}$$

and

$$1 \leq \frac{\prod_{j=1}^{n} [\Delta_x(A_j)]^{p_j}}{\left(\sum_{j=1}^{n} p_j \langle A_j^{-1} x, x \rangle\right)^{-1}} \leq \exp\left(\sum_{j=1}^{n} \langle A_j x_j, x_j \rangle \sum_{j=1}^{n} \langle A_j^{-1} x_j, x_j \rangle - 1\right), \tag{2.6}$$

for all $x \in H$, $\|x\| = 1$.

Remark 1. *The case of two operators A, $B > 0$ is as follows:*

$$\begin{aligned} 1 &\leq \frac{\langle ((1-t)A + tB)x, x \rangle}{[\Delta_x(A)]^{1-t}[\Delta_x(B)]^t} \\ &\leq \exp\left[\langle ((1-t)A + tB)x, x \rangle \left\langle \left((1-t)A^{-1} + tB^{-1}\right)x, x \right\rangle - 1\right] \end{aligned} \tag{2.7}$$

and

$$\begin{aligned} 1 &\leq \frac{[\Delta_x(A)]^{1-t}[\Delta_x(B)]^t}{\langle ((1-t)A^{-1} + tB^{-1})x, x \rangle^{-1}} \\ &\leq \exp\left[\langle ((1-t)A + tB)x, x \rangle \left\langle \left((1-t)A^{-1} + tB^{-1}\right)x, x \right\rangle - 1\right], \end{aligned} \tag{2.8}$$

for all $t \in [0, 1]$ and $x \in H$, $\|x\| = 1$.

For $B = A$ we get

$$1 \leq \frac{\langle Ax, x \rangle}{\Delta_x(A)} \leq \exp\left[\langle Ax, x \rangle \langle A^{-1} x, x \rangle - 1\right] \tag{2.9}$$

and

$$1 \leq \frac{\Delta_x(A)}{\langle (A^{-1})x, x \rangle^{-1}} \leq \exp\left[\langle Ax, x \rangle \langle A^{-1} x, x \rangle - 1\right] \tag{2.10}$$

for all $t \in [0, 1]$ and $x \in H$, $\|x\| = 1$.

Theorem 5. *Assume that* $0<m\leq A_j\leq M$ *for* $j\in\{1,...,n\}$ *and the given constants* $m<M$ *and* $x_j\in H$, $j\in\{1,...,n\}$ *with* $\sum_{j=1}^n\|x_j\|^2=1$, *then*

$$1\leq\frac{\sum_{j=1}^n\langle A_jx_j,x_j\rangle}{\prod_{j=1}^n\Delta_{x_j}(A_j)}\leq\exp\left(\sum_{j=1}^n\langle A_jx_j,x_j\rangle\sum_{j=1}^n\left\langle A_j^{-1}x_j,x_j\right\rangle-1\right)\tag{2.11}$$
$$\leq\begin{cases}\exp\left(\frac{M-m}{2}\left[\sum_{j=1}^n\left\|A_j^{-1}x_j\right\|^2-\left(\sum_{j=1}^n\left\langle A_j^{-1}x_j,x_j\right\rangle\right)^2\right]^{1/2}\right),\\ \exp\left(\frac{M-m}{2mM}\left[\sum_{j=1}^n\|A_jx_j\|^2-\left(\sum_{j=1}^n\langle A_jx_j,x_j\rangle\right)^2\right]^{1/2}\right),\end{cases}$$
$$\leq\exp\left(\frac{(M-m)^2}{4mM}\right)$$

and

$$1\leq\frac{\prod_{j=1}^n\Delta_{x_j}(A_j)}{\left(\sum_{j=1}^n\left\langle A_j^{-1}x_j,x_j\right\rangle\right)^{-1}}\leq\exp\left(\sum_{j=1}^n\langle A_jx_j,x_j\rangle\sum_{j=1}^n\left\langle A_j^{-1}x_j,x_j\right\rangle-1\right)\tag{2.12}$$
$$\leq\begin{cases}\exp\left(\frac{M-m}{2}\left[\sum_{j=1}^n\left\|A_j^{-1}x_j\right\|^2-\left(\sum_{j=1}^n\left\langle A_j^{-1}x_j,x_j\right\rangle\right)^2\right]^{1/2}\right),\\ \exp\left(\frac{M-m}{2mM}\left[\sum_{j=1}^n\|A_jx_j\|^2-\left(\sum_{j=1}^n\langle A_jx_j,x_j\rangle\right)^2\right]^{1/2}\right),\end{cases}$$
$$\leq\exp\left(\frac{(M-m)^2}{4mM}\right).$$

Proof. If A_j are selfadjoint operators with $\operatorname{Sp}(A_j)\subseteq[m,M]\subset\mathring{I}$, $j\in\{1,...,n\}$, then, see [2], the following more convenient reverse of Jensen's inequality holds:

$$(0\leq)\sum_{j=1}^n\langle f(A_j)x_j,x_j\rangle-f\left(\sum_{j=1}^n\langle A_jx_j,x_j\rangle\right)\tag{2.13}$$
$$\leq\sum_{j=1}^n\langle f'(A_j)A_jx_j,x_j\rangle-\sum_{j=1}^n\langle A_jx_j,x_j\rangle\sum_{j=1}^n\langle f'(A_j)x_j,x_j\rangle$$
$$\leq\begin{cases}\frac{1}{2}(M-m)\left[\sum_{j=1}^n\|f'(A_j)x_j\|^2-\left(\sum_{j=1}^n\langle f'(A_j)x_j,x_j\rangle\right)^2\right]^{1/2},\\ \frac{1}{2}(f'(M)-f'(m))\left[\sum_{j=1}^n\|A_jx_j\|^2-\left(\sum_{j=1}^n\langle A_jx_j,x_j\rangle\right)^2\right]^{1/2},\end{cases}$$
$$\leq\frac{1}{4}(M-m)(f'(M)-f'(m)),$$

If we write this inequality for the convex function $f(t) = -\ln t$, $t > 0$, then we get

$$0 \leq \ln\left(\sum_{j=1}^{n}\langle A_j x_j, x_j\rangle\right) - \sum_{j=1}^{n}\langle \ln A_j x_j, x_j\rangle \tag{2.14}$$
$$\leq \sum_{j=1}^{n}\langle A_j x_j, x_j\rangle \sum_{j=1}^{n}\left\langle A_j^{-1} x_j, x_j\right\rangle - 1$$
$$\leq \begin{cases} \frac{1}{2}(M-m)\left[\sum_{j=1}^{n}\left\|A_j^{-1}x_j\right\|^2 - \left(\sum_{j=1}^{n}\left\langle A_j^{-1}x_j, x_j\right\rangle\right)^2\right]^{1/2}, \\ \\ \frac{1}{2}\frac{M-m}{mM}\left[\sum_{j=1}^{n}\left\|A_j x_j\right\|^2 - \left(\sum_{j=1}^{n}\langle A_j x_j, x_j\rangle\right)^2\right]^{1/2}, \end{cases}$$
$$\leq \frac{(M-m)^2}{4mM}.$$

If we take the exponential in (2.14), we derive (2.11). □

Corollary 3. *Assume that* $0 < m \leq A_j \leq M$ *and* $p_j \geq 0$, $j \in \{1, ..., n\}$ *with* $\sum_{j=1}^{n} p_j = 1$, *then*

$$1 \leq \frac{\sum_{j=1}^{n} p_j \langle A_j x, x\rangle}{\prod_{j=1}^{n}\left[\Delta_x(A_j)\right]^{p_j}} \leq \exp\left(\sum_{j=1}^{n} p_j\langle A_j x, x\rangle \sum_{j=1}^{n} p_j\left\langle A_j^{-1}x, x\right\rangle - 1\right) \tag{2.15}$$
$$\leq \begin{cases} \exp\left(\frac{M-m}{2}\left[\sum_{j=1}^{n} p_j\left\|A_j^{-1}x\right\|^2 - \left(\sum_{j=1}^{n} p_j\left\langle A_j^{-1}x, x\right\rangle\right)^2\right]^{1/2}\right), \\ \\ \exp\left(\frac{M-m}{2mM}\left[\sum_{j=1}^{n} p_j\left\|A_j x\right\|^2 - \left(\sum_{j=1}^{n} p_j\langle A_j x, x\rangle\right)^2\right]^{1/2}\right) \end{cases}$$
$$\leq \exp\left(\frac{(M-m)^2}{4mM}\right)$$

and

$$1 \leq \frac{\prod_{j=1}^{n}\left[\Delta_x(A_j)\right]^{p_j}}{\left(\sum_{j=1}^{n} p_j\left\langle A_j^{-1}x, x\right\rangle\right)^{-1}} \leq \exp\left(\sum_{j=1}^{n}\langle A_j x_j, x_j\rangle \sum_{j=1}^{n}\left\langle A_j^{-1}x_j, x_j\right\rangle - 1\right) \tag{2.16}$$
$$\leq \begin{cases} \exp\left(\frac{M-m}{2}\left[\sum_{j=1}^{n} p_j\left\|A_j^{-1}x\right\|^2 - \left(\sum_{j=1}^{n} p_j\left\langle A_j^{-1}x, x\right\rangle\right)^2\right]^{1/2}\right), \\ \\ \exp\left(\frac{M-m}{2mM}\left[\sum_{j=1}^{n} p_j\left\|A_j x\right\|^2 - \left(\sum_{j=1}^{n} p_j\langle A_j x, x\rangle\right)^2\right]^{1/2}\right) \end{cases}$$
$$\leq \exp\left(\frac{(M-m)^2}{4mM}\right)$$

for all $x \in H,\ \|x\| = 1.$

Remark 2. *The case of two operators* $0 < m \leq A,\ B \leq M$ *is as follows:*

$$1 \leq \frac{\langle ((1-t)A + tB)x, x\rangle}{[\Delta_x(A)]^{1-t}[\Delta_x(B)]^t} \leq \exp\left[\langle ((1-t)A + tB)x, x\rangle \left\langle \left((1-t)A^{-1} + tB^{-1}\right)x, x\right\rangle - 1\right] \tag{2.17}$$

$$\leq \begin{cases} \exp\left(\frac{M-m}{2}\left[(1-t)\left\|A^{-1}x\right\|^2 + t\left\|B^{-1}x\right\|^2 - \left\langle \left((1-t)A^{-1} + tB^{-1}\right)x, x\right\rangle^2\right]^{1/2}\right), \\ \exp\left(\frac{M-m}{2mM}\left[(1-t)\|Ax\|^2 + t\|Bx\|^2 - \langle ((1-t)A + tB)x, x\rangle^2\right]^{1/2}\right) \end{cases}$$

$$\leq \exp\left(\frac{(M-m)^2}{4mM}\right)$$

and

$$1 \leq \frac{[\Delta_x(A)]^{1-t}[\Delta_x(B)]^t}{\langle ((1-t)A^{-1} + tB^{-1})x, x\rangle^{-1}} \leq \exp\left[\langle ((1-t)A + tB)x, x\rangle \left\langle \left((1-t)A^{-1} + tB^{-1}\right)x, x\right\rangle - 1\right] \tag{2.18}$$

$$\leq \begin{cases} \exp\left(\frac{M-m}{2}\left[(1-t)\left\|A^{-1}x\right\|^2 + t\left\|B^{-1}x\right\|^2 - \left\langle \left((1-t)A^{-1} + tB^{-1}\right)x, x\right\rangle^2\right]^{1/2}\right), \\ \exp\left(\frac{M-m}{2mM}\left[(1-t)\|Ax\|^2 + t\|Bx\|^2 - \langle ((1-t)A + tB)x, x\rangle^2\right]^{1/2}\right) \end{cases}$$

$$\leq \exp\left(\frac{(M-m)^2}{4mM}\right)$$

for all $t \in [0,1]$ *and* $x \in H,\ \|x\| = 1.$

For $B = A$ *we derive*

$$1 \leq \frac{\langle Ax, x\rangle}{\Delta_x(A)} \leq \exp\left[\langle Ax, x\rangle \left\langle A^{-1}x, x\right\rangle - 1\right] \leq \begin{cases} \exp\left(\frac{M-m}{2}\left[\left\|A^{-1}x\right\|^2 - \left\langle A^{-1}x, x\right\rangle^2\right]^{1/2}\right), \\ \exp\left(\frac{M-m}{2mM}\left[\|Ax\|^2 - \langle Ax, x\rangle^2\right]^{1/2}\right) \end{cases} \leq \exp\left(\frac{(M-m)^2}{4mM}\right) \tag{2.19}$$

and

$$1 \leq \frac{\Delta_x(A)}{\langle (A^{-1})x, x\rangle^{-1}} \leq \exp\left[\langle Ax, x\rangle \left\langle A^{-1}x, x\right\rangle - 1\right] \tag{2.20}$$

$$\leq \begin{cases} \exp\left(\frac{M-m}{2}\left[\left\|A^{-1}x\right\|^2 - \left\langle A^{-1}x, x\right\rangle^2\right]^{1/2}\right), \\ \\ \exp\left(\frac{M-m}{2mM}\left[\|Ax\|^2 - \langle Ax, x\rangle^2\right]^{1/2}\right) \end{cases}$$

$$\leq \exp\left(\frac{(M-m)^2}{4mM}\right)$$

for all $t \in [0,1]$ *and* $x \in H$, $\|x\| = 1$.

Theorem 6. *Assume that* $A_j > 0$ *and* $p_j \geq 0$, $j \in \{1, ..., n\}$ *with* $\sum_{j=1}^n p_j = 1$, *then we have the following reverse of Ky Fan's inequality*

$$1 \leq \frac{\Delta_x\left(\sum_{j=1}^n p_j A_j\right)}{\prod_{j=1}^n [\Delta_x(A_j)]^{p_j}} \leq \frac{\sum_{j=1}^n p_j \langle A_j x, x\rangle}{\prod_{j=1}^n [\Delta_x(A_j)]^{p_j}} \tag{2.21}$$

$$\leq \exp\left(\sum_{j=1}^n p_j \langle A_j x, x\rangle \sum_{j=1}^n p_j \left\langle A_j^{-1}x, x\right\rangle - 1\right)$$

for $x \in H$, $\|x\| = 1$.

Proof. Observe that, by Jensen's inequality for the concave function

$$\begin{aligned} &\sum_{j=1}^n p_j \langle A_j x, x\rangle \\ &= \left\langle \left(\sum_{j=1}^n p_j A_j\right) x, x\right\rangle = \exp\left(\ln\left\langle \left(\sum_{j=1}^n p_j A_j\right) x, x\right\rangle\right) \\ &\geq \exp\left\langle \ln\left(\sum_{j=1}^n p_j A_j\right) x, x\right\rangle = \Delta_x\left(\sum_{j=1}^n p_j A_j\right) \end{aligned}$$

for $x \in H$, $\|x\| = 1$.

By (2.5) we then get

$$\frac{\Delta_x\left(\sum_{j=1}^n p_j A_j\right)}{\prod_{j=1}^n [\Delta_x(A_j)]^{p_j}} \leq \frac{\sum_{j=1}^n p_j \langle A_j x, x\rangle}{\prod_{j=1}^n [\Delta_x(A_j)]^{p_j}}$$

$$\leq \exp\left(\sum_{j=1}^n p_j \langle A_j x, x\rangle \sum_{j=1}^n p_j \left\langle A_j^{-1}x, x\right\rangle - 1\right)$$

for $x \in H$, $\|x\| = 1$.

If we use Ky Fan's type inequality (viii) and a standard induction argument we also have

$$1 \leq \frac{\Delta_x \left(\sum_{j=1}^{n} p_j A_j\right)}{\prod_{j=1}^{n} \left[\Delta_x (A_j)\right]^{p_j}}$$

for $x \in H$, $\|x\| = 1$.

These prove the desired result (2.21). □

Corollary 4. *Assume that* $0 < m \leq A_j \leq M$ *and* $p_j \geq 0$, $j \in \{1, ..., n\}$ *with* $\sum_{j=1}^{n} p_j = 1$, *then*

$$(2.22)\quad 1 \leq \frac{\Delta_x \left(\sum_{j=1}^{n} p_j A_j\right)}{\prod_{j=1}^{n} \left[\Delta_x (A_j)\right]^{p_j}}$$

$$\leq \frac{\sum_{j=1}^{n} p_j \langle A_j x, x\rangle}{\prod_{j=1}^{n} \left[\Delta_x (A_j)\right]^{p_j}} \leq \exp\left(\sum_{j=1}^{n} p_j \langle A_j x, x\rangle \sum_{j=1}^{n} p_j \left\langle A_j^{-1} x, x\right\rangle - 1\right)$$

$$\leq \begin{cases} \exp\left(\frac{M-m}{2}\left[\sum_{j=1}^{n} p_j \left\|A_j^{-1} x\right\|^2 - \left(\sum_{j=1}^{n} p_j \left\langle A_j^{-1} x, x\right\rangle\right)^2\right]^{1/2}\right), \\ \exp\left(\frac{M-m}{2mM}\left[\sum_{j=1}^{n} p_j \|A_j x\|^2 - \left(\sum_{j=1}^{n} p_j \langle A_j x, x\rangle\right)^2\right]^{1/2}\right) \end{cases}$$

$$\leq \exp\left(\frac{(M-m)^2}{4mM}\right)$$

for $x \in H$, $\|x\| = 1$.

Remark 3. *The case of two operators* $0 < m \leq A, B \leq M$ *is as follows:*

$$(2.23)\quad 1 \leq \frac{\Delta_x ((1-t) A + tB)}{[\Delta_x (A)]^{1-t} [\Delta_x (B)]^t}$$

$$\leq \frac{\langle ((1-t) A + tB) x, x\rangle}{[\Delta_x (A)]^{1-t} [\Delta_x (B)]^t}$$

$$\leq \exp\left[\langle ((1-t) A + tB) x, x\rangle \left\langle \left((1-t) A^{-1} + tB^{-1}\right) x, x\right\rangle - 1\right]$$

$$\leq \begin{cases} \exp\left(\frac{M-m}{2}\left[(1-t)\left\|A^{-1}x\right\|^2 + t\left\|B^{-1}x\right\|^2 - \left\langle \left((1-t) A^{-1} + tB^{-1}\right) x, x\right\rangle^2\right]^{1/2}\right), \\ \exp\left(\frac{M-m}{2mM}\left[(1-t)\|Ax\|^2 + t\|Bx\|^2 - \langle ((1-t) A + tB) x, x\rangle^2\right]^{1/2}\right) \end{cases}$$

$$\leq \exp\left(\frac{(M-m)^2}{4mM}\right)$$

for $x \in H$, $\|x\| = 1$.

From (2.23) we have

$$\begin{aligned}\left[\Delta_x(A)\right]^{1-t}\left[\Delta_x(B)\right]^t &\le \Delta_x((1-t)A+tB) \le \langle((1-t)A+tB)x,x\rangle \\ &\le \exp\left(\frac{(M-m)^2}{4mM}\right)\left[\Delta_x(A)\right]^{1-t}\left[\Delta_x(B)\right]^t\end{aligned}$$

for all $t \in [0,1]$.

If we take the integral, then we get

$$\begin{aligned}\int_0^1 \left[\Delta_x(A)\right]^{1-t}\left[\Delta_x(B)\right]^t dt &\le \int_0^1 \Delta_x((1-t)A+tB)\,dt \\ &\le \int_0^1 \langle((1-t)A+tB)x,x\rangle\,dt \\ &\le \exp\left(\frac{(M-m)^2}{4mM}\right)\int_0^1 \left[\Delta_x(A)\right]^{1-t}\left[\Delta_x(B)\right]^t dt.\end{aligned}$$

Since

$$\int_0^1 \left[\Delta_x(A)\right]^{1-t}\left[\Delta_x(B)\right]^t dt = L(\Delta_x(A),\Delta_x(B))$$

and

$$\int_0^1 \langle((1-t)A+tB)x,x\rangle\,dt = \left\langle \frac{A+B}{2}x,x\right\rangle$$

hence we have the following integral inequalities of interest for $0 < m \le A,\ B \le M$,

$$\begin{aligned}L(\Delta_x(A),\Delta_x(B)) &\le \int_0^1 \Delta_x((1-t)A+tB)\,dt \\ &\le \left\langle \frac{A+B}{2}x,x\right\rangle \\ &\le \exp\left(\frac{(M-m)^2}{4mM}\right) L(\Delta_x(A),\Delta_x(B)),\end{aligned} \tag{2.24}$$

where $x \in H$, $\|x\| = 1$.

3. Related Results

Let A_j be selfadjoint operators with $\operatorname{Sp}(A_j) \subseteq [m,M]$ for $j \in \{1,...,n\}$ and for some scalars $m < M$. If f and g are continuous on $[m,M]$ and $\gamma := \min_{t\in[m,M]} f(t)$, $\Gamma := \max_{t\in[m,M]} f(t)$, $\delta := \min_{t\in[m,M]} g(t)$ and $\Delta := \max_{t\in[m,M]} g(t)$ then we have the following Gruss' type inequalities [3]

$$\left|\sum_{j=1}^n \langle f(A_j)g(A_j)x_j,x_j\rangle - \sum_{j=1}^n \langle f(A_j)x_j,x_j\rangle \sum_{j=1}^n \langle g(A_j)x_j,x_j\rangle\right| \tag{3.1}$$

$$\leq \frac{1}{4}(\Gamma-\gamma)(\Delta-\delta)$$

$$-\begin{cases} \left[\sum_{j=1}^{n}\langle \Gamma x_j - f(A_j)x_j, f(A_j)x_j - \gamma x_j\rangle \right. \\ \left. \times \sum_{j=1}^{n}\langle \Delta x_j - g(A_j)x_j, g(A_j)x - \delta x_j\rangle\right]^{\frac{1}{2}}, \\ \\ \left|\sum_{j=1}^{n}\langle f(A_j)x_j, x_j\rangle - \frac{\Gamma+\gamma}{2}\right| \left|\sum_{j=1}^{n}\langle g(A_j)x_j, x_j\rangle - \frac{\Delta+\delta}{2}\right| \end{cases}$$

$$\leq \frac{1}{4}(\Gamma-\gamma)(\Delta-\delta)$$

for each $x_j \in H, j \in \{1,...,n\}$ with $\sum_{j=1}^{n}\|x_j\|^2 = 1$.

Moreover if γ and δ are positive, then we also have

$$\left|\sum_{j=1}^{n}\langle f(A_j)g(A_j)x_j, x_j\rangle - \sum_{j=1}^{n}\langle f(A_j)x_j, x_j\rangle \sum_{j=1}^{n}\langle g(A_j)x_j, x_j\rangle\right| \tag{3.2}$$

$$\leq \begin{cases} \frac{(\Gamma-\gamma)(\Delta-\delta)}{4\sqrt{\Gamma\gamma\Delta\delta}}\sum_{j=1}^{n}\langle f(A_j)x_j, x_j\rangle \sum_{j=1}^{n}\langle g(A_j)x_j, x_j\rangle, \\ \\ \left(\sqrt{\Gamma}-\sqrt{\gamma}\right)\left(\sqrt{\Delta}-\sqrt{\delta}\right) \\ \times\left[\sum_{j=1}^{n}\langle f(A_j)x_j, x_j\rangle \sum_{j=1}^{n}\langle g(A_j)x_j, x_j\rangle\right]^{\frac{1}{2}} \end{cases}$$

for each $x_j \in H, j \in \{1,...,n\}$ with $\sum_{j=1}^{n}\|x_j\|^2 = 1$.

Now, if we take $f(t) = t^{-1}$ and $g(t) = t$, $t > 0$ and assume that A_j are selfadjoint operators with $\operatorname{Sp}(A_j) \subseteq [m, M] \subset (0, \infty)$ for $j \in \{1,...,n\}$, then by (3.1) and (3.2) we get

$$0 \leq \sum_{j=1}^{n}\left\langle A_j^{-1}x_j, x_j\right\rangle \sum_{j=1}^{n}\langle A_j x_j, x_j\rangle - 1 \tag{3.3}$$

$$\leq \frac{(M-m)^2}{4mM}$$

$$-\begin{cases} \left[\sum_{j=1}^{n}\left\langle\left(m^{-1}I - A_j^{-1}\right)x_j, \left(A_j^{-1} - M^{-1}I\right)x_j\right\rangle\right. \\ \left.\times\sum_{j=1}^{n}\langle (MI - A_j)x_j, (A_j - mI)x_j\rangle\right]^{\frac{1}{2}}, \\ \\ \left|\sum_{j=1}^{n}\left\langle A_j^{-1}x_j, x_j\right\rangle - \frac{m^{-1}+M^{-1}}{2}\right| \left|\sum_{j=1}^{n}\langle A_j x_j, x_j\rangle - \frac{m+M}{2}\right| \end{cases}$$

$$\leq \frac{(M-m)^2}{4mM}$$

and

$$0 \leq \sum_{j=1}^{n}\left\langle A_j^{-1}x_j, x_j\right\rangle \sum_{j=1}^{n}\langle A_j x_j, x_j\rangle - 1 \tag{3.4}$$

$$\leq \begin{cases} \frac{(M-m)^2}{4mM}\sum_{j=1}^{n}\left\langle A_j^{-1}x_j, x_j\right\rangle \sum_{j=1}^{n}\langle A_j x_j, x_j\rangle, \\ \\ \frac{\left(\sqrt{M}-\sqrt{m}\right)^2}{\sqrt{mM}}\left[\sum_{j=1}^{n}\left\langle A_j^{-1}x_j, x_j\right\rangle \sum_{j=1}^{n}\langle A_j x_j, x_j\rangle\right]^{\frac{1}{2}} \end{cases}$$

for each $x_j \in H, j \in \{1,...,n\}$ with $\sum_{j=1}^n \|x_j\|^2 = 1$.

If $p_j \geq 0$, $j \in \{1,...,n\}$ with $\sum_{j=1}^n p_j = 1$, then

$$0 \leq \sum_{j=1}^n p_j \langle A_j^{-1}x, x \rangle \sum_{j=1}^n p_j \langle A_j x, x \rangle - 1 \tag{3.5}$$
$$\leq \frac{(M-m)^2}{4mM}$$
$$- \begin{cases} \left[\sum_{j=1}^n p_j \left\langle \left(m^{-1}I - A_j^{-1}\right)x, \left(A_j^{-1} - M^{-1}I\right)x \right\rangle \right. \\ \left. \times \sum_{j=1}^n p_j \left\langle (MI - A_j)x, (A_j - mI)x \right\rangle \right]^{\frac{1}{2}}, \\ \\ \left| \sum_{j=1}^n p_j \left\langle A_j^{-1}x, x \right\rangle - \frac{m^{-1}+M^{-1}}{2} \right| \left| \sum_{j=1}^n p_j \langle A_j x, x \rangle - \frac{m+M}{2} \right| \end{cases}$$
$$\leq \frac{(M-m)^2}{4mM}$$

and

$$0 \leq \sum_{j=1}^n p_j \langle A_j^{-1}x, x \rangle \sum_{j=1}^n p_j \langle A_j x, x \rangle - 1 \tag{3.6}$$
$$\leq \begin{cases} \frac{(M-m)^2}{4mM} \sum_{j=1}^n p_j \left\langle A_j^{-1}x, x \right\rangle \sum_{j=1}^n p_j \langle A_j x, x \rangle, \\ \\ \frac{\left(\sqrt{M}-\sqrt{m}\right)^2}{\sqrt{mM}} \left[\sum_{j=1}^n p_j \left\langle A_j^{-1}x, x \right\rangle \sum_{j=1}^n p_j \langle A_j x, x \rangle \right]^{\frac{1}{2}} \end{cases}$$

for $x \in H$, $\|x\| = 1$.

Since

$$\sum_{j=1}^n \langle A_j x_j, x_j \rangle \leq M \text{ and} \sum_{j=1}^n \left\langle A_j^{-1} x_j, x_j \right\rangle \leq m^{-1}$$

then by second inequality in (3.4) we obtain

$$0 \leq \sum_{j=1}^n \left\langle A_j^{-1} x_j, x_j \right\rangle \sum_{j=1}^n \langle A_j x_j, x_j \rangle - 1 \leq \frac{\left(\sqrt{M} - \sqrt{m}\right)^2}{m}$$

for each $x_j \in H, j \in \{1,...,n\}$ with $\sum_{j=1}^n \|x_j\|^2 = 1$.

By the second branch in inequality (3.6) we also get

$$0 \leq \sum_{j=1}^n p_j \left\langle A_j^{-1}x, x \right\rangle \sum_{j=1}^n p_j \langle A_j x, x \rangle - 1 \leq \frac{\left(\sqrt{M} - \sqrt{m}\right)^2}{m}$$

for $p_j \geq 0$, $j \in \{1,...,n\}$ with $\sum_{j=1}^n p_j = 1$ and $x \in H$, $\|x\| = 1$.

We can state then the following result

Theorem 7. *Assume that $0 < m \le A_j \le M$ for $j \in \{1, ..., n\}$ and the given constants $m < M$ and $x_j \in H$, $j \in \{1, ..., n\}$ with $\sum_{j=1}^n \|x_j\|^2 = 1$, then*

$$1 \le \frac{\sum_{j=1}^n \langle A_j x_j, x_j\rangle}{\prod_{j=1}^n \Delta_{x_j}(A_j)} \le \exp\left(\sum_{j=1}^n \langle A_j x_j, x_j\rangle \sum_{j=1}^n \left\langle A_j^{-1} x_j, x_j\right\rangle - 1\right) \le \exp\left[\frac{\left(\sqrt{M}-\sqrt{m}\right)^2}{m}\right] \tag{3.7}$$

and

$$1 \le \frac{\prod_{j=1}^n \Delta_{x_j}(A_j)}{\left(\sum_{j=1}^n \left\langle A_j^{-1} x_j, x_j\right\rangle\right)^{-1}} \le \exp\left(\sum_{j=1}^n \langle A_j x_j, x_j\rangle \sum_{j=1}^n \left\langle A_j^{-1} x_j, x_j\right\rangle - 1\right) \le \exp\left[\frac{\left(\sqrt{M}-\sqrt{m}\right)^2}{m}\right]. \tag{3.8}$$

If $p_j \ge 0$, $j \in \{1, ..., n\}$ with $\sum_{j=1}^n p_j = 1$ and $x \in H$, $\|x\| = 1$, then for $0 < m \le A_j \le M$ for $j \in \{1, ..., n\}$,

$$1 \le \frac{\sum_{j=1}^n p_j \langle A_j x, x\rangle}{\prod_{j=1}^n [\Delta_x(A_j)]^{p_j}} \le \exp\left(\sum_{j=1}^n p_j \langle A_j x, x\rangle \sum_{j=1}^n p_j \left\langle A_j^{-1} x, x\right\rangle - 1\right) \le \exp\left[\frac{\left(\sqrt{M}-\sqrt{m}\right)^2}{m}\right] \tag{3.9}$$

and

$$1 \le \frac{\prod_{j=1}^n [\Delta_x(A_j)]^{p_j}}{\left(\sum_{j=1}^n p_j \left\langle A_j^{-1} x, x\right\rangle\right)^{-1}} \le \exp\left(\sum_{j=1}^n p_j \langle A_j x, x\rangle \sum_{j=1}^n p_j \left\langle A_j^{-1} x, x\right\rangle - 1\right) \le \exp\left[\frac{\left(\sqrt{M}-\sqrt{m}\right)^2}{m}\right] \tag{3.10}$$

Remark 4. *The case of two operators $0 < m \leq A,\ B \leq M$ is as follows:*

$$1 \leq \frac{\langle ((1-t)A + tB)x, x\rangle}{[\Delta_x(A)]^{1-t}[\Delta_x(B)]^t} \tag{3.11}$$
$$\leq \exp\left[\langle ((1-t)A + tB)x, x\rangle \left\langle \left((1-t)A^{-1} + tB^{-1}\right)x, x\right\rangle - 1\right]$$
$$\leq \frac{\left(\sqrt{M} - \sqrt{m}\right)^2}{m}$$

and

$$1 \leq \frac{[\Delta_x(A)]^{1-t}[\Delta_x(B)]^t}{\langle ((1-t)A^{-1} + tB^{-1})x, x\rangle^{-1}} \tag{3.12}$$
$$\leq \exp\left[\langle ((1-t)A + tB)x, x\rangle \left\langle \left((1-t)A^{-1} + tB^{-1}\right)x, x\right\rangle - 1\right]$$
$$\leq \exp\left[\frac{\left(\sqrt{M} - \sqrt{m}\right)^2}{m}\right]$$

for all $t \in [0,1]$ and $x \in H$, $\|x\| = 1$.

For $B = A$ we derive

$$1 \leq \frac{\langle Ax, x\rangle}{\Delta_x(A)} \leq \exp\left[\langle Ax, x\rangle \left\langle A^{-1}x, x\right\rangle - 1\right] \leq \exp\left[\frac{\left(\sqrt{M} - \sqrt{m}\right)^2}{m}\right] \tag{3.13}$$

and

$$1 \leq \frac{\Delta_x(A)}{\langle (A^{-1})x, x\rangle^{-1}} \leq \exp\left[\langle Ax, x\rangle \left\langle A^{-1}x, x\right\rangle - 1\right] \tag{3.14}$$
$$\leq \exp\left[\frac{\left(\sqrt{M} - \sqrt{m}\right)^2}{m}\right]$$

for $x \in H$, $\|x\| = 1$.

Finally, we also observe that the following reverse of Ky Fan's inequality holds

$$1 \leq \frac{\Delta_x\left(\sum_{j=1}^n p_j A_j\right)}{\prod_{j=1}^n [\Delta_x(A_j)]^{p_j}} \tag{3.15}$$
$$\leq \frac{\sum_{j=1}^n p_j \langle A_j x, x\rangle}{\prod_{j=1}^n [\Delta_x(A_j)]^{p_j}} \leq \exp\left(\sum_{j=1}^n p_j \langle A_j x, x\rangle \sum_{j=1}^n p_j \left\langle A_j^{-1}x, x\right\rangle - 1\right)$$
$$\leq \exp\left[\frac{\left(\sqrt{M} - \sqrt{m}\right)^2}{m}\right],$$

provided that $p_j \geq 0$, $j \in \{1, ..., n\}$ with $\sum_{j=1}^n p_j = 1$, $x \in H$, $\|x\| = 1$ and $0 < m \leq A_j \leq M$ for $j \in \{1, ..., n\}$.

We also have the integral inequalities of interest for $0 < m \leq A,\ B \leq M$,

$$L\left(\Delta_x\left(A\right), \Delta_x\left(B\right)\right) \leq \int_0^1 \Delta_x\left(\left(1-t\right)A + tB\right)dt \leq \left\langle \frac{A+B}{2}x, x\right\rangle \leq \exp\left[\frac{\left(\sqrt{M}-\sqrt{m}\right)^2}{m}\right] L\left(\Delta_x\left(A\right), \Delta_x\left(B\right)\right), \tag{3.16}$$

where $x \in H$, $\|x\| = 1$.

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