TENSORIAL AND HADAMARD PRODUCTS INTEGRAL REVERSES OF YOUNG'S INEQUALITY FOR CONTINUOUS FIELDS OF OPERATORS IN HILBERT SPACES

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Abstract. Let H be a Hilbert space and Ω a locally compact Hausdorff space endowed with a Radon measure μ with $\int_{\Omega} 1 d\mu(t) = 1$. In this paper we show among others that, if $(A_\tau)_{\tau\in\Omega}$ and $(B_\tau)_{\tau\in\Omega}$ are continuous fields of positive operators in $B(H)$, then for all $\nu \in [0,1]$ we have the tensorial inequality

$$\begin{aligned}
0 &\leq (1-\nu) \int_{\Omega} A_\tau d\mu(\tau) \otimes 1 + \nu 1 \otimes \int_{\Omega} B_\tau d\mu(\tau) \\
&- \int_{\Omega} A_\tau^{1-\nu} d\mu(\tau) \otimes \int_{\Omega} B_\tau^{\nu} d\mu(\tau) \\
&\leq \nu(1-\nu) \\
&\times \left[\int_{\Omega} (A_\tau \ln A_\tau) d\mu(\tau) \otimes 1 + 1 \otimes \left(\int_{\Omega} B_\tau \ln B_\tau d\mu(\tau)\right)\right. \\
&\left. - \int_{\Omega} A_\tau d\mu(\tau) \otimes \int_{\Omega} \ln B_\tau d\mu(\tau) - \int_{\Omega} \ln A_\tau d\mu(\tau) \otimes \int_{\Omega} B_\tau d\mu(\tau)\right].
\end{aligned}$$

We also have the following inequalities for the Hadamard product

$$\begin{aligned}
0 &\leq \int_{\Omega} ((1-\nu) A_\tau + \nu B_\tau) d\mu(\tau) \circ 1 - \int_{\Omega} A_\tau^{1-\nu} d\mu(\tau) \circ \int_{\Omega} B_\tau^{\nu} d\mu(\tau) \\
&\leq \nu(1-\nu) \left[\int_{\Omega} (A_\tau \ln A_\tau + B_\tau \ln B_\tau) d\mu(\tau) \circ 1\right. \\
&\left. - \int_{\Omega} A_\tau d\mu(\tau) \circ \int_{\Omega} \ln B_\tau d\mu(\tau) - \int_{\Omega} \ln A_\tau d\mu(\tau) \circ \int_{\Omega} B_\tau d\mu(\tau)\right]
\end{aligned}$$

for all $\nu \in [0,1]$.

1. Introduction

The famous *Young's inequality* for scalars says that if $a, b > 0$ and $\nu \in [0,1]$, then

$$a^{1-\nu} b^{\nu} \leq (1-\nu) a + \nu b \tag{1.1}$$

with equality if and only if $a = b$. The inequality (1.1) is also called *ν-weighted arithmetic-geometric mean inequality.*

We recall that *Specht's ratio* is defined by [13]

$$S(h) := \begin{cases} \dfrac{h^{\frac{1}{h-1}}}{e \ln\left(h^{\frac{1}{h-1}}\right)} & \text{if } h \in (0,1) \cup (1,\infty) \\ 1 & \text{if } h = 1. \end{cases} \tag{1.2}$$

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It is well known that $\lim_{h\to 1} S(h)=1$, $S(h)=S\left(\frac{1}{h}\right)>1$ for $h>0$, $h\neq 1$. The function is decreasing on $(0,1)$ and increasing on $(1,\infty)$.

The following inequality provides a refinement and a multiplicative reverse for Young's inequality

$$S\left(\left(\frac{a}{b}\right)^{r}\right)a^{1-\nu}b^{\nu}\leq(1-\nu)a+\nu b\leq S\left(\frac{a}{b}\right)a^{1-\nu}b^{\nu}, \tag{1.3}$$

where $a,b>0$, $\nu\in[0,1]$, $r=\min\{1-\nu,\nu\}$.

The second inequality in (1.3) is due to Tominaga [14] while the first one is due to Furuichi [5].

It is an open question for the author if in the right hand side of (1.3) we can replace $S\left(\frac{a}{b}\right)$ by $S\left(\left(\frac{a}{b}\right)^{R}\right)$ where $R=\max\{1-\nu,\nu\}$.

Kittaneh and Manasrah [9], [10] provided a refinement and an additive reverse for Young inequality as follows:

$$r\left(\sqrt{a}-\sqrt{b}\right)^{2}\leq(1-\nu)a+\nu b-a^{1-\nu}b^{\nu}\leq R\left(\sqrt{a}-\sqrt{b}\right)^{2} \tag{1.4}$$

where $a,b>0$, $\nu\in[0,1]$, $r=\min\{1-\nu,\nu\}$ and $R=\max\{1-\nu,\nu\}$.

We also consider the *Kantorovich's constant* defined by

$$K(h):=\frac{(h+1)^{2}}{4h},\ \ h>0. \tag{1.5}$$

The function K is decreasing on $(0,1)$ and increasing on $[1,\infty)$, $K(h)\geq 1$ for any $h>0$ and $K(h)=K\left(\frac{1}{h}\right)$ for any $h>0$.

The following multiplicative refinement and reverse of Young inequality in terms of Kantorovich's constant holds

$$K^{r}\left(\frac{a}{b}\right)a^{1-\nu}b^{\nu}\leq(1-\nu)a+\nu b\leq K^{R}\left(\frac{a}{b}\right)a^{1-\nu}b^{\nu} \tag{1.6}$$

where $a,b>0$, $\nu\in[0,1]$, $r=\min\{1-\nu,\nu\}$ and $R=\max\{1-\nu,\nu\}$.

The first inequality in (1.6) was obtained by Zou et al. in [16] while the second by Liao et al. [12].

In the recent paper [4] we obtained the following reverses of Young's inequality as well:

$$0\leq(1-\nu)a+\nu b-a^{1-\nu}b^{\nu}\leq\nu(1-\nu)(a-b)(\ln a-\ln b) \tag{1.7}$$

and

$$1\leq\frac{(1-\nu)a+\nu b}{a^{1-\nu}b^{\nu}}\leq\exp\left[4\nu(1-\nu)\left(K\left(\frac{a}{b}\right)-1\right)\right], \tag{1.8}$$

where $a,b>0$, $\nu\in[0,1]$.

It has been shown in [4] that there is no ordering for the upper bounds of the quantity $(1-\nu)a+\nu b-a^{1-\nu}b^{\nu}$ as provided by the inequalities (1.4) and (1.7). The same conclusion is true for the upper bounds of the quantity $\frac{(1-\nu)a+\nu b}{a^{1-\nu}b^{\nu}}$ incorporated in the inequalities (1.3), (1.6) and (1.8).

Let $I_1,\ldots,I_k$ be intervals from $\mathbb{R}$ and let $f:I_1\times\ldots\times I_k\to\mathbb{R}$ be an essentially bounded real function defined on the product of the intervals. Let $A=(A_1,\ldots,A_n)$ be a k-tuple of bounded selfadjoint operators on Hilbert spaces $H_1,\ldots,H_k$ such that the spectrum of A_i is contained in I_i for $i = 1, ..., k$. We say that such a k-tuple is in the domain of f. If

$$A_i = \int_{I_i} \lambda_i dE_i (\lambda_i)$$

is the spectral resolution of A_i for $i = 1, ..., k$; by following [2], we define

$$f (A_1, ..., A_k) := \int_{I_1} ... \int_{I_k} f (\lambda_1, ..., \lambda_1) dE_1 (\lambda_1) \otimes ... \otimes dE_k (\lambda_k) \tag{1.9}$$

as a bounded selfadjoint operator on the tensorial product $H_1 \otimes ... \otimes H_k$.

If the Hilbert spaces are of finite dimension, then the above integrals become finite sums, and we may consider the functional calculus for arbitrary real functions. This construction [2] extends the definition of Korányi [11] for functions of two variables and have the property that

$$f (A_1, ..., A_k) = f_1(A_1) \otimes ... \otimes f_k(A_k),$$

whenever f can be separated as a product $f(t_1, ..., t_k) = f_1(t_1)...f_k(t_k)$ of k functions each depending on only one variable.

It is know that, *if f is super-multiplicative (sub-multiplicative)* on $[0, \infty)$, namely

$$f (st) \geq (\leq) f (s) f (t) \text{ for all } s, t \in [0, \infty)$$

and if f is continuous on $[0, \infty)$, then [7, p. 173]

$$f (A \otimes B) \geq (\leq) f (A) \otimes f (B) \text{ for all } A,\ B \geq 0. \tag{1.10}$$

This follows by observing that, if

$$A = \int_{[0,\infty)} tdE (t) \text{ and } B = \int_{[0,\infty)} sdF (s)$$

are the spectral resolutions of A and B, then

$$f (A \otimes B) = \int_{[0,\infty)} \int_{[0,\infty)} f (st) dE (t) \otimes dF (s) \tag{1.11}$$

for the continuous function f on $[0, \infty)$.

Recall the *geometric operator mean* for the positive operators $A,\ B > 0$

$$A\#_t B := A^{1/2}(A^{-1/2}BA^{-1/2})^t A^{1/2},$$

where $t \in [0, 1]$ and

$$A\#B := A^{1/2}(A^{-1/2}BA^{-1/2})^{1/2} A^{1/2}.$$

By the definitions of $\#$ and $\otimes$ we have

$$A\#B = B\#A \text{ and } (A\#B) \otimes (B\#A) = (A \otimes B) \# (B \otimes A).$$

In 2007, S. Wada [15] obtained the following *Callebaut type inequalities* for tensorial product

$$\begin{aligned} (A\#B) \otimes (A\#B) &\leq \frac{1}{2} [(A\#_\alpha B) \otimes (A\#_{1-\alpha} B) + (A\#_{1-\alpha} B) \otimes (A\#_\alpha B)] \\ &\leq \frac{1}{2} (A \otimes B + B \otimes A) \end{aligned} \tag{1.12}$$

for $A,\ B > 0$ and $\alpha \in [0, 1]$.

Recall that the *Hadamard product* of A and B in $B(H)$ is defined to be the operator $A \circ B \in B(H)$ satisfying

$$\langle (A \circ B) e_j, e_j \rangle = \langle A e_j, e_j \rangle \langle B e_j, e_j \rangle$$

for all $j \in \mathbb{N}$, where $\{e_j\}_{j \in \mathbb{N}}$ is an *orthonormal basis* for the separable Hilbert space H.

It is known that, see [6], we have the representation

$$A \circ B = \mathcal{U}^* (A \otimes B) \mathcal{U} \tag{1.13}$$

where $\mathcal{U} : H \to H \otimes H$ is the isometry defined by $\mathcal{U} e_j = e_j \otimes e_j$ for all $j \in \mathbb{N}$.

If f *is super-multiplicative (sub-multiplicative)* on $[0, \infty)$, then also [7, p. 173]

$$f (A \circ B) \geq (\leq) f (A) \circ f (B) \text{ for all } A,\ B \geq 0. \tag{1.14}$$

We recall the following elementary inequalities for the Hadamard product

$$A^{1/2} \circ B^{1/2} \leq \left(\frac{A + B}{2} \right) \circ 1 \text{ for } A,\ B \geq 0$$

and *Fiedler inequality*

$$A \circ A^{-1} \geq 1 \text{ for } A > 0. \tag{1.15}$$

As extension of Kadison's Schwarz inequality on the Hadamard product, Ando [1] showed that

$$A \circ B \leq \left(A^2 \circ 1 \right)^{1/2} \left(B^2 \circ 1 \right)^{1/2} \text{ for } A,\ B \geq 0$$

and Aujla and Vasudeva [3] gave an alternative upper bound

$$A \circ B \leq \left(A^2 \circ B^2 \right)^{1/2} \text{ for } A,\ B \geq 0.$$

It has been shown in [8] that $\left(A^2 \circ 1 \right)^{1/2} \left(B^2 \circ 1 \right)^{1/2}$ and $\left(A^2 \circ B^2 \right)^{1/2}$ are incomparable for 2-square positive definite matrices A and B.

Let Ω be a locally compact Hausdorff space endowed with a Radon measure μ. A field $(A_t)_{t \in \Omega}$ of operators in $B(H)$ is called a continuous field of operators if the parametrization $t \longmapsto A_t$ is norm continuous on $B(H)$. If, in addition, the norm function $t \longmapsto \|A_t\|$ is Lebesgue integrable on Ω, we can form the Bochner integral $\int_\Omega A_t d\mu (t)$, which is the unique operator in $B(H)$ such that $\varphi \left(\int_\Omega A_t d\mu (t) \right) = \int_\Omega \varphi (A_t) d\mu (t)$ for every bounded linear functional φ on $B(H)$. Assume also that, $\int_\Omega 1 d\mu (t) = 1$.

Motivated by the above results, in this paper we show among others that, if $(A_\tau)_{\tau \in \Omega}$ and $(B_\tau)_{\tau \in \Omega}$ are continuous fields of positive operators in $B(H)$, then for all $\nu \in [0, 1]$ we have the tensorial inequality

$$\begin{aligned}
0 &\leq (1 - \nu) \int_\Omega A_\tau d\mu (\tau) \otimes 1 + \nu 1 \otimes \int_\Omega B_\tau d\mu (\tau) \\
&- \int_\Omega A_\tau^{1-\nu} d\mu (\tau) \otimes \int_\Omega B_\tau^\nu d\mu (\tau) \\
&\leq \nu (1 - \nu) \\
&\times \left[\int_\Omega (A_\tau \ln A_\tau) d\mu (\tau) \otimes 1 + 1 \otimes \left(\int_\Omega B_\tau \ln B_\tau d\mu (\tau) \right) \right. \\
&\left. - \int_\Omega A_\tau d\mu (\tau) \otimes \int_\Omega \ln B_\tau d\mu (\tau) - \int_\Omega \ln A_\tau d\mu (\tau) \otimes \int_\Omega B_\tau d\mu (\tau) \right].
\end{aligned}$$

We also have the following inequalities for the Hadamard product

$$\begin{aligned}
0 &\leq \int_{\Omega} ((1-\nu) A_{\tau} + \nu B_{\tau}) \, d\mu(\tau) \circ 1 - \int_{\Omega} A_{\tau}^{1-\nu} d\mu(\tau) \circ \int_{\Omega} B_{\tau}^{\nu} d\mu(\tau) \\
&\leq \nu (1-\nu) \left[\int_{\Omega} (A_{\tau} \ln A_{\tau} + B_{\tau} \ln B_{\tau}) \, d\mu(\tau) \circ 1 \right. \\
&\left. - \int_{\Omega} A_{\tau} d\mu(\tau) \circ \int_{\Omega} \ln B_{\tau} d\mu(\tau) - \int_{\Omega} \ln A_{\tau} d\mu(\tau) \circ \int_{\Omega} B_{\tau} d\mu(\tau) \right]
\end{aligned}$$

for all $\nu \in [0,1]$.

2. Main Results

We start to the following tensorial inequality:

Lemma 1. *Assume that A, $B > 0$ and $\nu \in [0,1]$, then*

$$\begin{aligned}
0 &\leq (1-\nu) A \otimes 1 + \nu 1 \otimes B - A^{1-\nu} \otimes B^{\nu} \\
&\leq \nu (1-\nu) [(A \ln A) \otimes 1 + 1 \otimes (B \ln B) - A \otimes \ln B - (\ln A) \otimes B].
\end{aligned} \tag{2.1}$$

In particular,

$$\begin{aligned}
0 &\leq \frac{1}{2} (A \otimes 1 + 1 \otimes B) - A^{1/2} \otimes B^{1/2} \\
&\leq \frac{1}{4} [(A \ln A) \otimes 1 + 1 \otimes (B \ln B) - A \otimes \ln B - (\ln A) \otimes B].
\end{aligned} \tag{2.2}$$

Proof. From (1.7) we have

$$0 \leq (1-\nu) t + \nu s - t^{1-\nu} s^{\nu} \leq \nu (1-\nu) (t-s) (\ln t - \ln s) \tag{2.3}$$

for all t, $s > 0$ and $\nu \in [0,1]$.

If

$$A = \int_{[0,\infty)} t dE(t) \text{ and } B = \int_{[0,\infty)} s dF(s)$$

are the spectral resolutions of A and B, then by taking the double integral $\int_{[0,\infty)} \int_{[0,\infty)}$ over $dE(t) \otimes dF(s)$ in (2.3) we get

$$\begin{aligned}
0 &\leq \int_{[0,\infty)} \int_{[0,\infty)} \left[(1-\nu) t + \nu s - t^{1-\nu} s^{\nu} \right] dE(t) \otimes dF(s) \\
&\leq \nu (1-\nu) \int_{[0,\infty)} \int_{[0,\infty)} (t-s) (\ln t - \ln s) \, dE(t) \otimes dF(s).
\end{aligned} \tag{2.4}$$

Observe that, by (1.9)

$$\begin{aligned}
&\int_{[0,\infty)} \int_{[0,\infty)} \left[(1-\nu) t + \nu s - t^{1-\nu} s^{\nu} \right] dE(t) \otimes dF(s) \\
&= (1-\nu) \int_{[0,\infty)} \int_{[0,\infty)} t dE(t) \otimes dF(s) + \nu \int_{[0,\infty)} \int_{[0,\infty)} s dE(t) \otimes dF(s) \\
&- \int_{[0,\infty)} \int_{[0,\infty)} t^{1-\nu} s^{\nu} dE(t) \otimes dF(s) \\
&= (1-\nu) A \otimes 1 + \nu 1 \otimes B - A^{1-\nu} \otimes B^{\nu}
\end{aligned}$$

and

$$\begin{aligned}
&\int_{[0,\infty)}\int_{[0,\infty)}(t-s)(\ln t-\ln s)\,dE(t)\otimes dF(s)\\
&=\int_{[0,\infty)}\int_{[0,\infty)}(t\ln t+s\ln s-t\ln s-s\ln t)\,dE(t)\otimes dF(s)\\
&=\int_{[0,\infty)}\int_{[0,\infty)}t\ln t dE(t)\otimes dF(s)+\int_{[0,\infty)}\int_{[0,\infty)}s\ln s dE(t)\otimes dF(s)\\
&-\int_{[0,\infty)}\int_{[0,\infty)}t\ln s dE(t)\otimes dF(s)-\int_{[0,\infty)}\int_{[0,\infty)}s\ln t dE(t)\otimes dF(s)\\
&=(A\ln A)\otimes 1+1\otimes(B\ln B)-A\otimes\ln B-(\ln A)\otimes B
\end{aligned}$$

and by (2.4) we get (2.1). □

Corollary 1. *With the assumptions of Theorem 1 we have the following inequalities for the Hadamard product*

$$\begin{aligned}
0 &\le [(1-\nu)A+\nu B]\circ 1-A^{1-\nu}\circ B^{\nu}\\
&\le \nu(1-\nu)[(A\ln A+B\ln B)\circ 1-A\circ\ln B-(\ln A)\circ B]
\end{aligned}\tag{2.5}$$

for $\nu\in[0,1]$.

In particular,

$$\begin{aligned}
0 &\le \frac{A+B}{2}\circ 1-A^{1/2}\circ B^{1/2}\\
&\le \frac{1}{4}[(A\ln A+B\ln B)\circ 1-A\circ\ln B-(\ln A)\circ B].
\end{aligned}\tag{2.6}$$

Proof. For the operators X and Y we have the representation

$$X\circ Y=\mathcal{U}^*(X\otimes Y)\mathcal{U},$$

where $\mathcal{U}:H\to H\otimes H$ is the isometry defined by $\mathcal{U}e_j=e_j\otimes e_j$ for all $j\in\mathbb{N}$.

If we take $\mathcal{U}^*$ to the left and $\mathcal{U}$ to the right in the inequality (2.1), we get

$$\begin{aligned}
0 &\le \mathcal{U}^*\left[(1-\nu)A\otimes 1+\nu 1\otimes B-A^{1-\nu}\otimes B^{\nu}\right]\mathcal{U}\\
&\le \nu(1-\nu)\mathcal{U}^*[(A\ln A)\otimes 1+1\otimes(B\ln B)-A\otimes\ln B-(\ln A)\otimes B]\mathcal{U}.
\end{aligned}\tag{2.7}$$

Observe that

$$\begin{aligned}
&\mathcal{U}^*\left[(1-\nu)A\otimes 1+\nu 1\otimes B-A^{1-\nu}\otimes B^{\nu}\right]\mathcal{U}\\
&=(1-\nu)\mathcal{U}^*(A\otimes 1)\mathcal{U}+\nu\mathcal{U}^*(1\otimes B)\mathcal{U}-\mathcal{U}^*\left(A^{1-\nu}\otimes B^{\nu}\right)\mathcal{U}\\
&=(1-\nu)(A\circ 1)+\nu(1\circ B)-\left(A^{1-\nu}\circ B^{\nu}\right)
\end{aligned}$$

and

$$\begin{aligned}
&\mathcal{U}^*[(A\ln A)\otimes 1+1\otimes(B\ln B)-A\otimes\ln B-(\ln A)\otimes B]\mathcal{U}\\
&=\mathcal{U}^*((A\ln A)\otimes 1)\mathcal{U}+\mathcal{U}^*(1\otimes(B\ln B))\mathcal{U}\\
&-\mathcal{U}^*(A\otimes\ln B)\mathcal{U}-\mathcal{U}^*((\ln A)\otimes B)\mathcal{U}\\
&=(A\ln A)\circ 1+1\circ(B\ln B)-A\circ\ln B-(\ln A)\circ B
\end{aligned}$$

and by (2.7) we derive (2.5). □

Remark 1. *If we take $B = A$ in Corollary 1, then we get*

$$0 \leq A \circ 1 - A^{1-\nu} \circ A^{\nu} \leq 2\nu (1-\nu) [(A \ln A) \circ 1 - A \circ \ln A] \tag{2.8}$$

for all $\nu \in [0,1]$.

In particular,

$$0 \leq A \circ 1 - A^{1/2} \circ A^{1/2} \leq \frac{1}{2} [(A \ln A) \circ 1 - A \circ \ln A]. \tag{2.9}$$

Our first main result is as follows:

Theorem 1. *Let $(A_\tau)_{\tau\in\Omega}$ and $(B_\tau)_{\tau\in\Omega}$ be continuous fields of positive operators in $B(H)$ such that $\operatorname{Sp}(A_\tau), \operatorname{Sp}(B_\tau) \subset (0,\infty)$ for each $\tau \in \Omega$. Then for all $\nu \in [0,1]$ we have*

$$\begin{aligned} 0 &\leq (1-\nu) \int_\Omega A_\tau d\mu(\tau) \otimes 1 + \nu 1 \otimes \int_\Omega B_\tau d\mu(\tau) \\ &- \int_\Omega A_\tau^{1-\nu} d\mu(\tau) \otimes \int_\Omega B_\tau^{\nu} d\mu(\tau) \\ &\leq \nu(1-\nu) \\ &\times \left[\int_\Omega (A_\tau \ln A_\tau) d\mu(\tau) \otimes 1 + 1 \otimes \left(\int_\Omega B_\tau \ln B_\tau d\mu(\tau) \right) \right. \\ &\left. - \int_\Omega A_\tau d\mu(\tau) \otimes \int_\Omega \ln B_\tau d\mu(\tau) - \int_\Omega \ln A_\tau d\mu(\tau) \otimes \int_\Omega B_\tau d\mu(\tau) \right] \end{aligned} \tag{2.10}$$

In particular,

$$\begin{aligned} 0 &\leq \frac{1}{2} \left[\int_\Omega A_\tau d\mu(\tau) \otimes 1 + 1 \otimes \int_\Omega B_\tau d\mu(\tau) \right] \\ &- \int_\Omega A_\tau^{1/2} d\mu(\tau) \otimes \int_\Omega B_\tau^{1/2} d\mu(\tau) \\ &\leq \frac{1}{4} \left[\int_\Omega (A_\tau \ln A_\tau) d\mu(\tau) \otimes 1 + 1 \otimes \left(\int_\Omega B_\tau \ln B_\tau d\mu(\tau) \right) \right. \\ &\left. - \int_\Omega A_\tau d\mu(\tau) \otimes \int_\Omega \ln B_\tau d\mu(\tau) - \int_\Omega \ln A_\tau d\mu(\tau) \otimes \int_\Omega B_\tau d\mu(\tau) \right]. \end{aligned} \tag{2.11}$$

Proof. From (2.1) we get

$$\begin{aligned} 0 &\leq (1-\nu) A_\tau \otimes 1 + \nu 1 \otimes B_\gamma - A_\tau^{1-\nu} \otimes B_\gamma^{\nu} \\ &\leq \nu(1-\nu) [(A_\tau \ln A_\tau) \otimes 1 + 1 \otimes (B_\gamma \ln B_\gamma) - A_\tau \otimes \ln B_\gamma - (\ln A_\tau) \otimes B_\gamma] \end{aligned} \tag{2.12}$$

for all $\tau, \gamma \in \Omega$.

If we take the integral $\int_\Omega$ over $d\mu(\tau)$, then we get

$$\begin{aligned} 0 &\leq \int_\Omega \left[(1-\nu) A_\tau \otimes 1 + \nu 1 \otimes B_\gamma - A_\tau^{1-\nu} \otimes B_\gamma^{\nu} \right] d\mu(\tau) \\ &\leq \nu(1-\nu) \int_\Omega [(A_\tau \ln A_\tau) \otimes 1 + 1 \otimes (B_\gamma \ln B_\gamma) \\ &- A_\tau \otimes \ln B_\gamma - (\ln A_\tau) \otimes B_\gamma] d\mu(\tau) \end{aligned} \tag{2.13}$$

for all $\gamma \in \Omega$.

Using the properties of the Bochner's integral and the tensorial product we have

$$\int_{\Omega}\left[(1-\nu)A_{\tau}\otimes 1+\nu 1\otimes B_{\gamma}-A_{\tau}^{1-\nu}\otimes B_{\gamma}^{\nu}\right]d\mu(\tau)$$
$$=(1-\nu)\int_{\Omega}A_{\tau}d\mu(\tau)\otimes 1+\nu 1\otimes B_{\gamma}-\int_{\Omega}A_{\tau}^{1-\nu}d\mu(\tau)\otimes B_{\gamma}^{\nu}$$

and

$$\int_{\Omega}\left[(A_{\tau}\ln A_{\tau})\otimes 1+1\otimes(B_{\gamma}\ln B_{\gamma})-A_{\tau}\otimes\ln B_{\gamma}-(\ln A_{\tau})\otimes B_{\gamma}\right]d\mu(\tau)$$
$$=\int_{\Omega}(A_{\tau}\ln A_{\tau})\,d\mu(\tau)\otimes 1+1\otimes(B_{\gamma}\ln B_{\gamma})$$
$$-\int_{\Omega}A_{\tau}d\mu(\tau)\otimes\ln B_{\gamma}-\int_{\Omega}\ln A_{\tau}d\mu(\tau)\otimes B_{\gamma}$$

for all $\gamma\in\Omega$.

From (2.13) we then get

$$\begin{aligned}0&\leq(1-\nu)\int_{\Omega}A_{\tau}d\mu(\tau)\otimes 1+\nu 1\otimes B_{\gamma}-\int_{\Omega}A_{\tau}^{1-\nu}d\mu(\tau)\otimes B_{\gamma}^{\nu}\\&\leq\nu(1-\nu)\\&\times\left[\int_{\Omega}(A_{\tau}\ln A_{\tau})\,d\mu(\tau)\otimes 1+1\otimes(B_{\gamma}\ln B_{\gamma})\right.\\&\left.-\int_{\Omega}A_{\tau}d\mu(\tau)\otimes\ln B_{\gamma}-\int_{\Omega}\ln A_{\tau}d\mu(\tau)\otimes B_{\gamma}\right]\end{aligned}\tag{2.14}$$

for all $\gamma\in\Omega$.

If we take the integral $\int_{\Omega}$ over $d\mu(\gamma)$ and utilizing the properties of the Bochner's integral and the tensorial product, we derive the desired result (2.10). □

Corollary 2. *With the assumptions of Theorem 1 we have the following inequalities for the Hadamard product*

$$\begin{aligned}0&\leq\int_{\Omega}((1-\nu)A_{\tau}+\nu B_{\tau})\,d\mu(\tau)\circ 1-\int_{\Omega}A_{\tau}^{1-\nu}d\mu(\tau)\circ\int_{\Omega}B_{\tau}^{\nu}d\mu(\tau)\\&\leq\nu(1-\nu)\left[\int_{\Omega}(A_{\tau}\ln A_{\tau}+B_{\tau}\ln B_{\tau})\,d\mu(\tau)\circ 1\right.\\&\left.-\int_{\Omega}A_{\tau}d\mu(\tau)\circ\int_{\Omega}\ln B_{\tau}d\mu(\tau)-\int_{\Omega}\ln A_{\tau}d\mu(\tau)\circ\int_{\Omega}B_{\tau}d\mu(\tau)\right]\end{aligned}\tag{2.15}$$

for all $\nu\in[0,1]$.

In particular,

$$\begin{aligned}0&\leq\int_{\Omega}\frac{A_{\tau}+B_{\tau}}{2}d\mu(\tau)\circ 1-\int_{\Omega}A_{\tau}^{1/2}d\mu(\tau)\circ\int_{\Omega}B_{\tau}^{1/2}d\mu(\tau)\\&\leq\frac{1}{4}\left[\int_{\Omega}(A_{\tau}\ln A_{\tau}+B_{\tau}\ln B_{\tau})\,d\mu(\tau)\circ 1\right.\\&\left.-\int_{\Omega}A_{\tau}d\mu(\tau)\circ\int_{\Omega}\ln B_{\tau}d\mu(\tau)-\int_{\Omega}\ln A_{\tau}d\mu(\tau)\circ\int_{\Omega}B_{\tau}d\mu(\tau)\right]\end{aligned}\tag{2.16}$$

The proof follows by taking $\mathcal{U}^{*}$ to the left and $\mathcal{U}$ to the right in the inequality (2.10) and using the properties of the integral.

Remark 2. *If we take $B_\tau = A_\tau$, $\tau \in \Omega$ in Corollary 2, then we obtain the following inequalities of interest*

$$\begin{aligned} 0 &\le \int_\Omega A_\tau d\mu(\tau) \circ 1 - \int_\Omega A_\tau^{1-\nu} d\mu(\tau) \circ \int_\Omega A_\tau^{\nu} d\mu(\tau) \\ &\le 2\nu(1-\nu)\left[\int_\Omega A_\tau \ln A_\tau d\mu(\tau) \circ 1 - \int_\Omega A_\tau d\mu(\tau) \circ \int_\Omega \ln A_\tau d\mu(\tau)\right] \end{aligned} \tag{2.17}$$

for all $\nu \in [0,1]$.

In particular,

$$\begin{aligned} 0 &\le \int_\Omega A_\tau d\mu(\tau) \circ 1 - \int_\Omega A_\tau^{1/2} d\mu(\tau) \circ \int_\Omega B_\tau^{1/2} d\mu(\tau) \\ &\le \frac{1}{2}\left[\int_\Omega A_\tau \ln A_\tau d\mu(\tau) \circ 1 - \int_\Omega A_\tau d\mu(\tau) \circ \int_\Omega \ln A_\tau d\mu(\tau)\right] \end{aligned} \tag{2.18}$$

3. Related Results

In the case when the operators are bounded below and above we also have:

Lemma 2. *Assume that the selfadjoint operators A and B satisfy the condition $0 < m \le A$, $B \le M$, then*

$$0 \le (1-\nu) A \otimes 1 + \nu 1 \otimes B - A^{1-\nu} \otimes B^{\nu} \le \nu(1-\nu)(M-m)\ln\left(\frac{M}{m}\right) \tag{3.1}$$

for all $\nu \in [0,1]$.

In particular,

$$0 \le \frac{A \otimes 1 + 1 \otimes B}{2} - A^{1/2} \otimes B^{1/2} \le \frac{1}{4}(M-m)\ln\left(\frac{M}{m}\right). \tag{3.2}$$

Proof. If a, $b \in [m, M] \subset (0, \infty)$, then

$$\begin{aligned} 0 &\le (a-b)(\ln a - \ln b) = |(a-b)(\ln a - \ln b)| \\ &= |a-b|\,|\ln a - \ln b| \le (M-m)(\ln M - \ln m). \end{aligned}$$

By (2.3) we then get

$$0 \le (1-\nu) a + \nu b - a^{1-\nu} b^{\nu} \le \nu(1-\nu)(M-m)(\ln M - \ln m) \tag{3.3}$$

for all a, $b \in [m, M]$.

If

$$A = \int_m^M t dE(t) \text{ and } B = \int_m^M s dF(s)$$

are the spectral resolutions of A and B, then by taking the double integral $\int_m^M \int_m^M$ over $dE(t) \otimes dF(s)$ in (3.3) we get

$$\begin{aligned} 0 &\le \int_m^M \int_m^M \left[(1-\nu) t + \nu s - t^{1-\nu} s^{\nu}\right] dE(t) \otimes dF(s) \\ &\le \nu(1-\nu)(M-m)(\ln M - \ln m) \int_m^M \int_m^M dE(t) \otimes dF(s) \\ &= \nu(1-\nu)(M-m)(\ln M - \ln m). \end{aligned} \tag{3.4}$$

Observe that, by (1.9),

$$\begin{aligned}
&\int_{[0,\infty)}\int_{[0,\infty)}\left[(1-\nu)t+\nu s-t^{1-\nu}s^{\nu}\right]dE(t)\otimes dF(s)\\
&=(1-\nu)\int_{[0,\infty)}\int_{[0,\infty)}tdE(t)\otimes dF(s)+\nu\int_{[0,\infty)}\int_{[0,\infty)}sdE(t)\otimes dF(s)\\
&-\int_{[0,\infty)}\int_{[0,\infty)}t^{1-\nu}s^{\nu}dE(t)\otimes dF(s)\\
&=(1-\nu)A\otimes 1+\nu 1\otimes B-A^{1-\nu}\otimes B^{\nu},
\end{aligned}$$

which gives, by (3.4), the desired result (3.1). □

Corollary 3. *With the assumptions of Lemma 2, we have the following inequalities for the Hadamard product*

$$0\le[(1-\nu)A+\nu B]\circ 1-A^{1-\nu}\circ B^{\nu}\le\nu(1-\nu)(M-m)\ln\left(\frac{M}{m}\right)\tag{3.5}$$

for all $\nu\in[0,1]$.

In particular,

$$0\le\frac{A+B}{2}\circ 1-A^{1/2}\circ B^{1/2}\le\frac{1}{4}(M-m)\ln\left(\frac{M}{m}\right).\tag{3.6}$$

Remark 3. *If we take $B=A$ in Corollary 3, then we get*

$$0\le A\circ 1-A^{1-\nu}\circ A^{\nu}\le\nu(1-\nu)(M-m)\ln\left(\frac{M}{m}\right)\tag{3.7}$$

for all $\nu\in[0,1]$.

In particular,

$$0\le A\circ 1-A^{1/2}\circ A^{1/2}\le\frac{1}{4}(M-m)\ln\left(\frac{M}{m}\right).\tag{3.8}$$

We also have:

Theorem 2. *Let $(A_\tau)_{\tau\in\Omega}$ and $(B_\tau)_{\tau\in\Omega}$ be continuous fields of positive operators in $B(H)$ such that $\operatorname{Sp}(A_\tau),\operatorname{Sp}(B_\tau)\subseteq[m,M]\subset(0,\infty)$ for each $\tau\in\Omega$. Then for all $\nu\in[0,1]$ we have*

$$\begin{aligned}
0&\le(1-\nu)\int_\Omega A_\tau d\mu(\tau)\otimes 1+\nu 1\otimes\int_\Omega B_\tau d\mu(\tau)\\
&-\int_\Omega A_\tau^{1-\nu}d\mu(\tau)\otimes\int_\Omega B_\tau^{\nu}d\mu(\tau)\\
&\le\nu(1-\nu)(M-m)\ln\left(\frac{M}{m}\right).
\end{aligned}\tag{3.9}$$

In particular,

$$\begin{aligned}
0&\le\frac{1}{2}\left[\int_\Omega A_\tau d\mu(\tau)\otimes 1+1\otimes\int_\Omega B_\tau d\mu(\tau)\right]\\
&-\int_\Omega A_\tau^{1/2}d\mu(\tau)\otimes\int_\Omega B_\tau^{1/2}d\mu(\tau)\\
&\le\frac{1}{4}(M-m)\ln\left(\frac{M}{m}\right).
\end{aligned}\tag{3.10}$$

The proof follows from Lemma 2 in a similar way to the one in the proof of Theorem 1 and the details are omitted.

Corollary 4. *With the assumptions of Theorem 2 we have the following inequalities for the Hadamard product*

$$0 \le \int_\Omega ((1-\nu) A_\tau + \nu B_\tau) \, d\mu(\tau) \circ 1 - \int_\Omega A_\tau^{1-\nu} d\mu(\tau) \circ \int_\Omega B_\tau^{\nu} d\mu(\tau) \le \nu(1-\nu)(M-m)\ln\left(\frac{M}{m}\right). \tag{3.11}$$

In particular,

$$0 \le \int_\Omega \frac{A_\tau + B_\tau}{2} d\mu(\tau) \circ 1 - \int_\Omega A_\tau^{1/2} d\mu(\tau) \circ \int_\Omega B_\tau^{1/2} d\mu(\tau) \le \frac{1}{4}(M-m)\ln\left(\frac{M}{m}\right). \tag{3.12}$$

If we take $B_\tau = A_\tau$, $\tau \in \Omega$ in Corollary 4, then we get

$$0 \le \int_\Omega A_\tau d\mu(\tau) \circ 1 - \int_\Omega A_\tau^{1-\nu} d\mu(\tau) \circ \int_\Omega A_\tau^{\nu} d\mu(\tau) \le \nu(1-\nu)(M-m)\ln\left(\frac{M}{m}\right) \tag{3.13}$$

and

$$0 \le \int_\Omega A_\tau d\mu(\tau) \circ 1 - \int_\Omega A_\tau^{1/2} d\mu(\tau) \circ \int_\Omega A_\tau^{1/2} d\mu(\tau) \le \frac{1}{4}(M-m)\ln\left(\frac{M}{m}\right). \tag{3.14}$$

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