TWO OPERATOR FIELDS TENSORIAL AND HADAMARD PRODUCTS INTEGRAL REVERSES OF YOUNG'S INEQUALITY IN HILBERT SPACES

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Abstract. Let H be a Hilbert space and Ω a locally compact Hausdorff space endowed with a Radon measure μ with $\int_{\Omega} 1d\mu(t) = 1$. In this paper we show among others that, if $(A_\tau)_{\tau\in\Omega}$ and $(B_\tau)_{\tau\in\Omega}$are continuous fields of positive operators in $B(H)$ such that $\mathrm{Sp}(A_\tau) \subset [m_1, M_1]$, $\mathrm{Sp}(B_\tau) \subset [m_2, M_2] \subset (0,\infty)$ for each $\tau \in \Omega$, then for all $\nu \in [0,1]$ we have

$$\begin{aligned}0 &\le (1-\nu)\int_{\Omega} A_\tau d\mu(\tau) \otimes 1 + \nu 1 \otimes \int_{\Omega} B_\tau d\mu(\tau) \\ &\quad - \int_{\Omega} A_\tau^{1-\nu} d\mu(\tau) \otimes \int_{\Omega} B_\tau^{\nu} d\mu(\tau) \\ &\le \Gamma_\nu(m_1, M_1, m_2, M_2)\int_{\Omega} A_\tau d\mu(\tau) \otimes 1,\end{aligned}$$

where

$$\Gamma_\nu(m_1, M_1, m_2, M_2) := \begin{cases} \ln\left(\frac{M_1}{m_2}\right)^{\nu(1-\nu)\left(\frac{M_1}{m_2}-1\right)} \text{ if } 1 \le \frac{m_1}{M_2} \\[2ex] \max\left\{\ln\left(\frac{m_1}{M_2}\right)^{\nu(1-\nu)\left(\frac{m_1}{M_2}-1\right)}, \ln\left(\frac{M_1}{m_2}\right)^{\nu(1-\nu)\left(\frac{M_1}{m_2}-1\right)}\right\} \\ \text{if } \frac{m_1}{M_2} < 1 < \frac{M_1}{m_2} \\[2ex] \ln\left(\frac{m_1}{M_2}\right)^{\nu(1-\nu)\left(\frac{m_1}{M_2}-1\right)} \text{ if } \frac{M_1}{m_2} \le 1. \end{cases}$$

The above inequality also holds for the Hadamard product " $\circ$ ".

1. Introduction

The famous *Young's inequality* for scalars says that if $a, b > 0$ and $\nu \in [0,1]$, then

$$a^{1-\nu} b^{\nu} \le (1-\nu)a + \nu b \tag{1.1}$$

with equality if and only if $a = b$. The inequality (1.1) is also called *ν-weighted arithmetic-geometric mean inequality.*

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We recall that *Specht's ratio* is defined by [13]

$$S(h) := \begin{cases} \dfrac{h^{\frac{1}{h-1}}}{e\ln\left(h^{\frac{1}{h-1}}\right)} \text{ if } h \in (0,1)\cup(1,\infty) \\ \\ 1 \text{ if } h = 1. \end{cases} \tag{1.2}$$

It is well known that $\lim_{h\to 1} S(h) = 1$, $S(h) = S\left(\frac{1}{h}\right) > 1$ for $h > 0$, $h \neq 1$. The function is decreasing on $(0,1)$ and increasing on $(1,\infty)$.

The following inequality provides a refinement and a multiplicative reverse for Young's inequality

$$S\left(\left(\frac{a}{b}\right)^r\right) a^{1-\nu}b^\nu \le (1-\nu)a + \nu b \le S\left(\frac{a}{b}\right) a^{1-\nu}b^\nu, \tag{1.3}$$

where $a, b > 0$, $\nu \in [0,1]$, $r = \min\{1-\nu, \nu\}$.

The second inequality in (1.3) is due to Tominaga [14] while the first one is due to Furuichi [5].

It is an open question for the author if in the right hand side of (1.3) we can replace $S\left(\frac{a}{b}\right)$ by $S\left(\left(\frac{a}{b}\right)^R\right)$ where $R = \max\{1-\nu, \nu\}$.

Kittaneh and Manasrah [9], [10] provided a refinement and an additive reverse for Young inequality as follows:

$$r\left(\sqrt{a} - \sqrt{b}\right)^2 \le (1-\nu)a + \nu b - a^{1-\nu}b^\nu \le R\left(\sqrt{a} - \sqrt{b}\right)^2 \tag{1.4}$$

where $a, b > 0$, $\nu \in [0,1]$, $r = \min\{1-\nu, \nu\}$ and $R = \max\{1-\nu, \nu\}$.

We also consider the *Kantorovich's constant* defined by

$$K(h) := \frac{(h+1)^2}{4h}, \ h > 0. \tag{1.5}$$

The function K is decreasing on $(0,1)$ and increasing on $[1,\infty)$, $K(h) \ge 1$ for any $h > 0$ and $K(h) = K\left(\frac{1}{h}\right)$ for any $h > 0$.

The following multiplicative refinement and reverse of Young inequality in terms of Kantorovich's constant holds

$$K^r\left(\frac{a}{b}\right) a^{1-\nu}b^\nu \le (1-\nu)a + \nu b \le K^R\left(\frac{a}{b}\right) a^{1-\nu}b^\nu \tag{1.6}$$

where $a, b > 0$, $\nu \in [0,1]$, $r = \min\{1-\nu, \nu\}$ and $R = \max\{1-\nu, \nu\}$.

The first inequality in (1.6) was obtained by Zou et al. in [16] while the second by Liao et al. [12].

In the recent paper [4] we obtained the following reverses of Young's inequality as well:

$$0 \le (1-\nu)a + \nu b - a^{1-\nu}b^\nu \le \nu(1-\nu)(a-b)(\ln a - \ln b) \tag{1.7}$$

and

$$1 \le \frac{(1-\nu)a + \nu b}{a^{1-\nu}b^\nu} \le \exp\left[4\nu(1-\nu)\left(K\left(\frac{a}{b}\right) - 1\right)\right], \tag{1.8}$$

where $a, b > 0$, $\nu \in [0,1]$.

It has been shown in [4] that there is no ordering for the upper bounds of the quantity $(1-\nu)a + \nu b - a^{1-\nu}b^\nu$ as provided by the inequalities (1.4) and (1.7). The same conclusion is true for the upper bounds of the quantity $\frac{(1-\nu)a+\nu b}{a^{1-\nu}b^\nu}$ incorporated in the inequalities (1.3), (1.6) and (1.8).

Let $I_1, ..., I_k$ be intervals from $\mathbb{R}$ and let $f : I_1 \times ... \times I_k \to \mathbb{R}$ be an essentially bounded real function defined on the product of the intervals. Let $A = (A_1, ..., A_n)$ be a k-tuple of bounded selfadjoint operators on Hilbert spaces $H_1, ..., H_k$ such that the spectrum of A_i is contained in I_i for $i = 1, ..., k$. We say that such a k-tuple is in the domain of f. If

$$A_i = \int_{I_i} \lambda_i dE_i(\lambda_i)$$

is the spectral resolution of A_i for $i = 1, ..., k$; by following [2], we define

$$f(A_1, ..., A_k) := \int_{I_1} ... \int_{I_k} f(\lambda_1, ..., \lambda_1) \, dE_1(\lambda_1) \otimes ... \otimes dE_k(\lambda_k) \tag{1.9}$$

as a bounded selfadjoint operator on the tensorial product $H_1 \otimes ... \otimes H_k$.

If the Hilbert spaces are of finite dimension, then the above integrals become finite sums, and we may consider the functional calculus for arbitrary real functions. This construction [2] extends the definition of Korányi [11] for functions of two variables and have the property that

$$f(A_1, ..., A_k) = f_1(A_1) \otimes ... \otimes f_k(A_k),$$

whenever f can be separated as a product $f(t_1, ..., t_k) = f_1(t_1)...f_k(t_k)$ of k functions each depending on only one variable.

It is know that, *if f is super-multiplicative (sub-multiplicative)* on $[0, \infty)$, namely

$$f(st) \geq (\leq) f(s) f(t) \text{ for all } s, t \in [0, \infty)$$

and if f is continuous on $[0, \infty)$, then [7, p. 173]

$$f(A \otimes B) \geq (\leq) f(A) \otimes f(B) \text{ for all } A, \ B \geq 0. \tag{1.10}$$

This follows by observing that, if

$$A = \int_{[0,\infty)} t dE(t) \text{ and } B = \int_{[0,\infty)} s dF(s)$$

are the spectral resolutions of A and B, then

$$f(A \otimes B) = \int_{[0,\infty)} \int_{[0,\infty)} f(st) \, dE(t) \otimes dF(s) \tag{1.11}$$

for the continuous function f on $[0, \infty)$.

Recall the *geometric operator mean* for the positive operators $A, \ B > 0$

$$A \#_t B := A^{1/2}(A^{-1/2}BA^{-1/2})^t A^{1/2},$$

where $t \in [0, 1]$ and

$$A \# B := A^{1/2}(A^{-1/2}BA^{-1/2})^{1/2} A^{1/2}.$$

By the definitions of $\#$ and $\otimes$ we have

$$A \# B = B \# A \text{ and } (A \# B) \otimes (B \# A) = (A \otimes B) \# (B \otimes A).$$

In 2007, S. Wada [15] obtained the following *Callebaut type inequalities* for tensorial product

$$\begin{aligned} (A \# B) \otimes (A \# B) &\leq \frac{1}{2} \left[(A \#_\alpha B) \otimes (A \#_{1-\alpha} B) + (A \#_{1-\alpha} B) \otimes (A \#_\alpha B) \right] \\ &\leq \frac{1}{2} (A \otimes B + B \otimes A) \end{aligned} \tag{1.12}$$

for $A, \ B > 0$ and $\alpha \in [0, 1]$.

Recall that the *Hadamard product* of A and B in $B(H)$ is defined to be the operator $A\circ B\in B(H)$ satisfying

$$\langle (A\circ B)\,e_j,e_j\rangle=\langle Ae_j,e_j\rangle\,\langle Be_j,e_j\rangle$$

for all $j\in\mathbb{N}$, where $\{e_j\}_{j\in\mathbb{N}}$ is an *orthonormal basis* for the separable Hilbert space H.

It is known that, see [6], we have the representation

$$A\circ B=\mathcal{U}^{*}\,(A\otimes B)\,\mathcal{U} \tag{1.13}$$

where $\mathcal{U}:H\to H\otimes H$ is the isometry defined by $\mathcal{U}e_j=e_j\otimes e_j$ for all $j\in\mathbb{N}$.

If f *is super-multiplicative (sub-multiplicative)* on $[0,\infty)$, then also [7, p. 173]

$$f\,(A\circ B)\geq(\leq)\,f\,(A)\circ f\,(B)\ \text{ for all } A,\ B\geq 0. \tag{1.14}$$

We recall the following elementary inequalities for the Hadamard product

$$A^{1/2}\circ B^{1/2}\leq\left(\frac{A+B}{2}\right)\circ 1\text{ for }A,\ B\geq 0$$

and *Fiedler inequality*

$$A\circ A^{-1}\geq 1\text{ for }A>0. \tag{1.15}$$

As extension of Kadison's Schwarz inequality on the Hadamard product, Ando [1] showed that

$$A\circ B\leq\left(A^{2}\circ 1\right)^{1/2}\left(B^{2}\circ 1\right)^{1/2}\ \text{ for }A,\ B\geq 0$$

and Aujla and Vasudeva [3] gave an alternative upper bound

$$A\circ B\leq\left(A^{2}\circ B^{2}\right)^{1/2}\ \text{ for }A,\ B\geq 0.$$

It has been shown in [8] that $\left(A^{2}\circ 1\right)^{1/2}\left(B^{2}\circ 1\right)^{1/2}$ and $\left(A^{2}\circ B^{2}\right)^{1/2}$ are incomparable for 2-square positive definite matrices A and B.

Let Ω be a locally compact Hausdorff space endowed with a Radon measure μ. A field $(A_t)_{t\in\Omega}$ of operators in $B\,(H)$ is called a continuous field of operators if the parametrization $t\longmapsto A_t$ is norm continuous on $B\,(H)$. If, in addition, the norm function $t\longmapsto\|A_t\|$ is Lebesgue integrable on Ω, we can form the Bochner integral $\int_\Omega A_t d\mu\,(t)$, which is the unique operator in $B\,(H)$ such that $\varphi\left(\int_\Omega A_t d\mu\,(t)\right)=\int_\Omega\varphi\,(A_t)\,d\mu\,(t)$ for every bounded linear functional φ on $B\,(H)$. Assume also that, $\int_\Omega 1d\mu\,(t)=1$.

Motivated by the above results, in this paper we show among others that, if $(A_\tau)_{\tau\in\Omega}$ and $(B_\tau)_{\tau\in\Omega}$are continuous fields of positive operators in $B\,(H)$ such that $\operatorname{Sp}(A_\tau)\subset[m_1,M_1]$, $\operatorname{Sp}(B_\tau)\subset[m_2,M_2]\subset(0,\infty)$ for each $\tau\in\Omega$, then for all $\nu\in[0,1]$ we have

$$\begin{aligned}0&\leq(1-\nu)\int_\Omega A_\tau d\mu\,(\tau)\otimes 1+\nu 1\otimes\int_\Omega B_\tau d\mu\,(\tau)\\&\quad-\int_\Omega A_\tau^{1-\nu}d\mu\,(\tau)\otimes\int_\Omega B_\tau^{\nu}d\mu\,(\tau)\\&\leq\Gamma_\nu\,(m_1,M_1,m_2,M_2)\int_\Omega A_\tau d\mu\,(\tau)\otimes 1,\end{aligned}$$

where $\Gamma_\nu\,(m_1,M_1,m_2,M_2)$ is defined by (2.2). The above inequality also holds for the Hadamard product " $\circ$ ".

2. Main Results

We have the following tensorial reverse of Young's inequality:

Lemma 1. *Assume that the selfadjoint operators A and B satisfy the condition $0 < m_1 \le A \le M_1$, $0 < m_2 \le B \le M_2$, then*

$$0 \le (1-\nu) A \otimes 1 + \nu 1 \otimes B - A^{1-\nu} \otimes B^{\nu} \le \Gamma_{\nu}(m_1, M_1, m_2, M_2) A \otimes 1, \tag{2.1}$$

where

$$\Gamma_{\nu}(m_1, M_1, m_2, M_2) \tag{2.2}$$
$$:= \begin{cases} \ln\left(\frac{M_1}{m_2}\right)^{\nu(1-\nu)\left(\frac{M_1}{m_2}-1\right)} & \text{if } 1 \le \frac{m_1}{M_2} \\ \max\left\{\ln\left(\frac{m_1}{M_2}\right)^{\nu(1-\nu)\left(\frac{m_1}{M_2}-1\right)}, \ln\left(\frac{M_1}{m_2}\right)^{\nu(1-\nu)\left(\frac{M_1}{m_2}-1\right)}\right\} \\ \text{if } \frac{m_1}{M_2} < 1 < \frac{M_1}{m_2} \\ \ln\left(\frac{m_1}{M_2}\right)^{\nu(1-\nu)\left(\frac{m_1}{M_2}-1\right)} & \text{if } \frac{M_1}{m_2} \le 1 \end{cases}$$

for all $\nu \in [0,1]$.

In particular,

$$0 \le \frac{A \otimes 1 + 1 \otimes B}{2} - A^{1/2} \otimes B^{1/2} \le \Gamma_{1/2}(m_1, M_1, m_2, M_2) A \otimes 1. \tag{2.3}$$

Proof. We consider the function $f(t) = (t-1)\ln t$, $t > 0$. We observe that

$$f'(t) = \ln t + 1 - \frac{1}{t}$$

and

$$f''(t) = \frac{1}{t} + \frac{1}{t^2} = \frac{t+1}{t^2}$$

for $t > 0$.

This shows that the function f is strictly convex on $(0,\infty)$, decreasing on $(0,1)$, increasing on $(1,\infty)$ with

$$\min_{t \in (0,\infty)} f(t) = f(1) = 0.$$

If $0 < m_1 \le t \le M_1$, $0 < m_2 \le s \le M_2$, then $u := \frac{t}{s} \in \left[\frac{m_1}{M_2}, \frac{M_1}{m_2}\right]$, which shows that

$$\max_{u \in \left[\frac{m_1}{M_2}, \frac{M_1}{m_2}\right]} f(u) = \begin{cases} f\left(\frac{M_1}{m_2}\right) \text{ if } 1 \le \frac{m_1}{M_2} \\ \max\left\{f\left(\frac{m_1}{M_2}\right), f\left(\frac{M_1}{m_2}\right)\right\} \text{ if } \frac{m_1}{M_2} < 1 < \frac{M_1}{m_2} \\ f\left(\frac{m_1}{M_2}\right) \text{ if } \frac{M_1}{m_2} \le 1 \end{cases}$$

$$= \begin{cases} \ln\left(\frac{M_1}{m_2}\right)^{\frac{M_1}{m_2}-1} \text{ if } 1 \le \frac{m_1}{M_2} \\ \\ \max\left\{\ln\left(\frac{m_1}{M_2}\right)^{\frac{m_1}{M_2}-1}, \ln\left(\frac{M_1}{m_2}\right)^{\frac{M_1}{m_2}-1}\right\} \\ \text{if } \frac{m_1}{M_2} < 1 < \frac{M_1}{m_2} \\ \\ \ln\left(\frac{m_1}{M_2}\right)^{\frac{m_1}{M_2}-1} \text{ if } \frac{M_1}{m_2} \le 1. \end{cases} .$$

By (1.7) we get

$$\begin{aligned} (2.4) \qquad 0 &\le (1-\nu)\,t + \nu s - t^{1-\nu}s^{\nu} \le \nu(1-\nu)\,t\left(\frac{s}{t}-1\right)\ln\frac{s}{t} \\ &\le \nu(1-\nu)\,t\Gamma(m_1, M_1, m_2, M_2) \end{aligned}$$

for $0 < m_1 \le t \le M_1$, $0 < m_2 \le s \le M_2$.

If

$$A = \int_{m_1}^{M_1} t dE(t) \text{ and } B = \int_{m_2}^{M_2} s dF(s)$$

are the spectral resolutions of A and B, then by taking the double integral $\int_{m_1}^{M_1}\int_{m_2}^{M_2}$ over $dE(t) \otimes dF(s)$ in (2.4) we get

$$\begin{aligned} (2.5) \qquad 0 &\le \int_{m_1}^{M_1}\int_{m_2}^{M_2}\left[(1-\nu)\,t + \nu s - t^{1-\nu}s^{\nu}\right] dE(t) \otimes dF(s) \\ &\le \nu(1-\nu)\,\Gamma(m_1, M_1, m_2, M_2)\int_{m_1}^{M_1}\int_{m_2}^{M_2} t dE(t) \otimes dF(s). \end{aligned}$$

Since

$$\begin{aligned} &\int_{m_1}^{M_1}\int_{m_2}^{M_2}\left[(1-\nu)\,t + \nu s - t^{1-\nu}s^{\nu}\right] dE(t) \otimes dF(s) \\ &= (1-\nu)\,A \otimes 1 + \nu 1 \otimes B - A^{1-\nu} \otimes B^{\nu} \end{aligned}$$

and

$$\int_{m_1}^{M_1}\int_{m_2}^{M_2} t dE(t) \otimes dF(s) = A \otimes 1,$$

hence by (2.5) we get (2.1). □

Corollary 1. *Assume that the selfadjoint operators A and B satisfy the condition $0 < m \le A$, $B \le M$, then*

$$\begin{aligned} (2.6) \qquad 0 &\le (1-\nu)\,A \otimes 1 + \nu 1 \otimes B - A^{1-\nu} \otimes B^{\nu} \\ &\le \nu(1-\nu)\left(\frac{M-m}{M}\right)\ln\left(\frac{M}{m}\right) A \otimes 1, \end{aligned}$$

for all $\nu \in [0,1]$.

In particular,

$$(2.7) \qquad 0 \le \frac{A \otimes 1 + 1 \otimes B}{2} - A^{1/2} \otimes B^{1/2} \le \frac{M-m}{4M}\ln\left(\frac{M}{m}\right) A \otimes 1.$$

The proof follows by lemma 1 for $m_1 = m_2 = m$ and $M_1 = M_2 = M$.

Corollary 2. *With the assumptions of Lemma 1, we have the following inequalities for the Hadamard product*

$$0 \leq [(1-\nu)A + \nu B] \circ 1 - A^{1-\nu} \circ B^{\nu} \leq \Gamma_{\nu}(m_1, M_1, m_2, M_2) A \circ 1 \tag{2.8}$$

for all $\nu \in [0,1]$.

In particular,

$$0 \leq \frac{A+B}{2} \circ 1 - A^{1/2} \circ B^{1/2} \leq \Gamma_{1/2}(m_1, M_1, m_2, M_2) A \circ 1. \tag{2.9}$$

If $0 < m \leq A,\ B \leq M,$ *then*

$$0 \leq [(1-\nu)A + \nu B] \circ 1 - A^{1-\nu} \circ B^{\nu} \leq \nu(1-\nu)\left(\frac{M-m}{M}\right)\ln\left(\frac{M}{m}\right) A \circ 1, \tag{2.10}$$

for all $\nu \in [0,1]$.

In particular,

$$0 \leq \frac{A+B}{2} \circ 1 - A^{1/2} \circ B^{1/2} \leq \frac{M-m}{4M}\ln\left(\frac{M}{m}\right) A \circ 1. \tag{2.11}$$

If we take $B = A$ in Corollary 2, then we get

$$0 \leq A \circ 1 - A^{1-\nu} \circ A^{\nu} \leq \nu(1-\nu)\left(\frac{M-m}{M}\right)\ln\left(\frac{M}{m}\right) A \circ 1, \tag{2.12}$$

for all $\nu \in [0,1]$.

In particular,

$$0 \leq A \circ 1 - A^{1/2} \circ A^{1/2} \leq \frac{M-m}{4M}\ln\left(\frac{M}{m}\right) A \circ 1. \tag{2.13}$$

Our first main result is as follows:

Theorem 1. *Let* $(A_\tau)_{\tau\in\Omega}$ *and* $(B_\tau)_{\tau\in\Omega}$ *be continuous fields of positive operators in* $B(H)$ *such that* $\mathrm{Sp}(A_\tau) \subset [m_1, M_1]$, $\mathrm{Sp}(B_\tau) \subset [m_2, M_2] \subset (0,\infty)$ *for each* $\tau \in \Omega$. *Then for all* $\nu \in [0,1]$ *we have*

$$\begin{aligned} 0 &\leq (1-\nu)\int_\Omega A_\tau d\mu(\tau) \otimes 1 + \nu 1 \otimes \int_\Omega B_\tau d\mu(\tau) \\ &\quad - \int_\Omega A_\tau^{1-\nu} d\mu(\tau) \otimes \int_\Omega B_\tau^{\nu} d\mu(\tau) \\ &\leq \Gamma_\nu(m_1, M_1, m_2, M_2) \int_\Omega A_\tau d\mu(\tau) \otimes 1 \end{aligned} \tag{2.14}$$

and, in particular

$$\begin{aligned} 0 &\leq \frac{1}{2}\left[\int_\Omega A_\tau d\mu(\tau) \otimes 1 + 1 \otimes \int_\Omega B_\tau d\mu(\tau)\right] \\ &\quad - \int_\Omega A_\tau^{1/2} d\mu(\tau) \otimes \int_\Omega B_\tau^{1/2} d\mu(\tau) \\ &\leq \Gamma_{1/2}(m_1, M_1, m_2, M_2) \int_\Omega A_\tau d\mu(\tau) \otimes 1, \end{aligned} \tag{2.15}$$

where $\Gamma_\nu(m_1, M_1, m_2, M_2)$ *is defined by (2.2).*

Proof. From (2.1) we have

$$0\le(1-\nu)A_{\tau}\otimes1+\nu1\otimes B_{\gamma}-A_{\tau}^{1-\nu}\otimes B_{\gamma}^{\nu}\le\Gamma_{\nu}(m_1,M_1,m_2,M_2)A_{\tau}\otimes1, \tag{2.16}$$

for all $\tau,\gamma\in\Omega$.

If we take the integral $\int_{\Omega}$ over $d\mu(\tau)$, then we get

$$\begin{aligned}0&\le\int_{\Omega}\left[(1-\nu)A_{\tau}\otimes1+\nu1\otimes B_{\gamma}-A_{\tau}^{1-\nu}\otimes B_{\gamma}^{\nu}\right]d\mu(\tau)\\&\le\Gamma_{\nu}(m_1,M_1,m_2,M_2)\int_{\Omega}A_{\tau}d\mu(\tau)\otimes1\end{aligned} \tag{2.17}$$

for all $\gamma\in\Omega$.

Using the properties of the Bochner's integral and the tensorial product we have

$$\begin{aligned}&\int_{\Omega}\left[(1-\nu)A_{\tau}\otimes1+\nu1\otimes B_{\gamma}-A_{\tau}^{1-\nu}\otimes B_{\gamma}^{\nu}\right]d\mu(\tau)\\&=(1-\nu)\int_{\Omega}A_{\tau}d\mu(\tau)\otimes1+\nu1\otimes B_{\gamma}-\int_{\Omega}A_{\tau}^{1-\nu}d\mu(\tau)\otimes B_{\gamma}^{\nu}\end{aligned}$$

for all $\gamma\in\Omega$.

By (2.17) we derive

$$\begin{aligned}0&\le(1-\nu)\int_{\Omega}A_{\tau}d\mu(\tau)\otimes1+\nu1\otimes B_{\gamma}-\int_{\Omega}A_{\tau}^{1-\nu}d\mu(\tau)\otimes B_{\gamma}^{\nu}\\&\le\Gamma_{\nu}(m_1,M_1,m_2,M_2)\int_{\Omega}A_{\tau}d\mu(\tau)\otimes1\end{aligned}$$

for all $\gamma\in\Omega$.

If we take the integral $\int_{\Omega}$ over $d\mu(\gamma)$ and utilizing the properties of the Bochner's integral and the tensorial product, we derive the desired result (2.14). □

Corollary 3. *With the assumptions of Theorem 1 we have the following inequalities for the Hadamard product*

$$\begin{aligned}0&\le\int_{\Omega}((1-\nu)A_{\tau}+\nu B_{\tau})\,d\mu(\tau)\circ1-\int_{\Omega}A_{\tau}^{1-\nu}d\mu(\tau)\circ\int_{\Omega}B_{\tau}^{\nu}d\mu(\tau)\\&\le\Gamma_{\nu}(m_1,M_1,m_2,M_2)\int_{\Omega}A_{\tau}d\mu(\tau)\circ1\end{aligned} \tag{2.18}$$

for all $\nu\in[0,1]$.

In particular,

$$\begin{aligned}0&\le\int_{\Omega}\frac{A_{\tau}+B_{\tau}}{2}d\mu(\tau)\circ1-\int_{\Omega}A_{\tau}^{1/2}d\mu(\tau)\circ\int_{\Omega}B_{\tau}^{1/2}d\mu(\tau)\\&\le\Gamma_{1/2}(m_1,M_1,m_2,M_2)\int_{\Omega}A_{\tau}d\mu(\tau)\circ1.\end{aligned} \tag{2.19}$$

The proof follows by taking $\mathcal{U}^*$ to the left and $\mathcal{U}$ to the right in the inequality (2.16) and using the properties of the integral.

Remark 1. *If we take $B_\tau = A_\tau$, $\tau \in \Omega$ in Corollary 3, then we obtain the following inequalities of interest*

$$0 \leq \int_\Omega A_\tau d\mu(\tau) \circ 1 - \int_\Omega A_\tau^{1-\nu} d\mu(\tau) \circ \int_\Omega A_\tau^\nu d\mu(\tau) \leq \Gamma_\nu(m_1, M_1, m_2, M_2) \int_\Omega A_\tau d\mu(\tau) \circ 1 \tag{2.20}$$

for all $\nu \in [0,1]$.

In particular,

$$0 \leq \int_\Omega A_\tau d\mu(\tau) \circ 1 - \int_\Omega A_\tau^{1/2} d\mu(\tau) \circ \int_\Omega B_\tau^{1/2} d\mu(\tau) \leq \Gamma_{1/2}(m_1, M_1, m_2, M_2) \int_\Omega A_\tau d\mu(\tau) \circ 1. \tag{2.21}$$

3. Related Results

Further, we can also state the following multiplicative reverse of Young's inequality:

Lemma 2. *Assume that the selfadjoint operators A and B satisfy the condition $0 < m_1 \leq A \leq M_1$, $0 < m_2 \leq B \leq M_2$, then*

$$A^{1-\nu} \otimes B^\nu \leq (1-\nu) A \otimes 1 + \nu 1 \otimes B \leq \exp[\Delta_\nu(m_1, M_1, m_2, M_2)] A^{1-\nu} \otimes B^\nu, \tag{3.1}$$

where

$$\Delta_\nu(m_1, M_1, m_2, M_2) := \nu(1-\nu) \times \begin{cases} \frac{(M_1-m_2)^2}{m_2 M_1} \text{ if } 1 \leq \frac{m_1}{M_2} \\ \\ \max\left\{\frac{(M_2-m_1)^2}{m_1 M_2}, \frac{(M_1-m_2)^2}{m_2 M_1}\right\} \\ \text{if } \frac{m_1}{M_2} < 1 < \frac{M_1}{m_2} \\ \\ \frac{(M_2-m_1)^2}{m_1 M_2} \text{ if } \frac{M_1}{m_2} \leq 1 \end{cases} \tag{3.2}$$

for all $\nu \in [0,1]$.

In particular,

$$A^{1/2} \otimes B^{1/2} \leq \frac{A \otimes 1 + 1 \otimes B}{2} \leq \exp\left[\Delta_{1/2}(m_1, M_1, m_2, M_2)\right] A^{1/2} \otimes B^{1/2}. \tag{3.3}$$

Proof. If $0 < m_1 \leq t \leq M_1$, $0 < m_2 \leq s \leq M_2$, then $u := \frac{t}{s} \in \left[\frac{m_1}{M_2}, \frac{M_1}{m_2}\right]$, which shows that

$$\max_{u \in \left[\frac{m_1}{M_2}, \frac{M_1}{m_2}\right]} K(u) = \begin{cases} K\left(\frac{M_1}{m_2}\right) \text{ if } 1 \leq \frac{m_1}{M_2} \\ \\ \max\left\{K\left(\frac{m_1}{M_2}\right), K\left(\frac{M_1}{m_2}\right)\right\} \text{ if } \frac{m_1}{M_2} < 1 < \frac{M_1}{m_2} \\ \\ K\left(\frac{m_1}{M_2}\right) \text{ if } \frac{M_1}{m_2} \leq 1, \end{cases}$$

which gives that

$$
\begin{aligned}
&\max_{u\in\left[\frac{m_1}{M_2},\frac{M_1}{m_2}\right]} K(u)-1\\
&=\begin{cases}
\frac{(M_1-m_2)^2}{4m_2M_1} \text{ if } 1\le \frac{m_1}{M_2}\\[2mm]
\max\left\{\frac{(M_2-m_1)^2}{4m_1M_2},\frac{(M_1-m_2)^2}{4m_2M_1}\right\} \text{ if } \frac{m_1}{M_2}<1<\frac{M_1}{m_2}\\[2mm]
\frac{(M_2-m_1)^2}{4m_1M_2} \text{ if } \frac{M_1}{m_2}\le 1,
\end{cases}\\
&=\frac{1}{4}\times\begin{cases}
\frac{(M_1-m_2)^2}{m_2M_1} \text{ if } 1\le \frac{m_1}{M_2}\\[2mm]
\max\left\{\frac{(M_2-m_1)^2}{m_1M_2},\frac{(M_1-m_2)^2}{m_2M_1}\right\} \text{ if } \frac{m_1}{M_2}<1<\frac{M_1}{m_2}\\[2mm]
\frac{(M_2-m_1)^2}{m_1M_2} \text{ if } \frac{M_1}{m_2}\le 1.
\end{cases}
\end{aligned}
$$

From (1.8) we derive

$$
\begin{aligned}
1&\le\frac{(1-\nu)t+\nu s}{t^{1-\nu}s^{\nu}}\le\exp\left[4\nu(1-\nu)\left(K\left(\frac{t}{s}\right)-1\right)\right]\\
&\le\exp\left[\Delta_\nu(m_1,M_1,m_2,M_2)\right],
\end{aligned}
$$

which gives

$$1\le(1-\nu)t+\nu s\le\exp\left[\Delta_\nu(m_1,M_1,m_2,M_2)\right]t^{1-\nu}s^{\nu}, \tag{3.4}$$

for $0<m_1\le t\le M_1$, $0<m_2\le s\le M_2$.

Now, by making a similar argument to the one in the proof of Lemma 1 and utilizing inequality (3.4), we deduce (3.1). □

Corollary 4. *Assume that the selfadjoint operators A and B satisfy the condition $0<m\le A$, $B\le M$, then*

$$
\begin{aligned}
A^{1-\nu}\otimes B^{\nu}&\le(1-\nu)A\otimes 1+\nu 1\otimes B\\
&\le\exp\left[\nu(1-\nu)\frac{(M-m)^2}{mM}\right]A^{1-\nu}\otimes B^{\nu}.
\end{aligned}\tag{3.5}
$$

In particular,

$$A^{1/2}\otimes B^{1/2}\le\frac{A\otimes 1+1\otimes B}{2}\le\exp\left[\frac{(M-m)^2}{4mM}\right]A^{1/2}\otimes B^{1/2}. \tag{3.6}$$

We also have the following inequalities for the Hadamard product:

Corollary 5. *With the assumptions of Lemma 2, we have the following inequalities*

$$A^{1-\nu}\circ B^{\nu}\le[(1-\nu)A+\nu B]\circ 1\le\exp\left[\Delta_\nu(m_1,M_1,m_2,M_2)\right]A^{1-\nu}\circ B^{\nu}, \tag{3.7}$$

for all $\nu\in[0,1]$.

In particular,

$$A^{1/2}\circ B^{1/2}\le\frac{A+B}{2}\circ 1\le\exp\left[\Delta_{1/2}(m_1,M_1,m_2,M_2)\right]A^{1/2}\circ B^{1/2}. \tag{3.8}$$

With the assumptions of Corollary 4,

$$A^{1-\nu}\circ B^{\nu}\leq\left[\left(1-\nu\right)A+\nu B\right]\circ 1 \leq \exp\left[\nu\left(1-\nu\right)\frac{\left(M-m\right)^{2}}{mM}\right]A^{1-\nu}\circ B^{\nu}. \tag{3.9}$$

In particular,

$$A^{1/2}\circ B^{1/2}\leq\frac{A+B}{2}\circ 1\leq\exp\left[\frac{\left(M-m\right)^{2}}{4mM}\right]A^{1/2}\circ B^{1/2}. \tag{3.10}$$

If we take $B=A$ in (3.9) and (3.10), then we get for $\nu\in[0,1]$ that

$$A^{1-\nu}\circ A^{\nu}\leq A\circ 1\leq\exp\left[\nu\left(1-\nu\right)\frac{\left(M-m\right)^{2}}{mM}\right]A^{1-\nu}\circ A^{\nu}. \tag{3.11}$$

In particular,

$$A^{1/2}\circ A^{1/2}\leq A\circ 1\leq\exp\left[\frac{\left(M-m\right)^{2}}{4mM}\right]A^{1/2}\circ A^{1/2}. \tag{3.12}$$

Our second main result is as follows:

Theorem 2. *Let $(A_\tau)_{\tau\in\Omega}$ and $(B_\tau)_{\tau\in\Omega}$ be continuous fields of positive operators in $B(H)$ such that $\operatorname{Sp}(A_\tau)\subset[m_1,M_1]$, $\operatorname{Sp}(B_\tau)\subset[m_2,M_2]\subset(0,\infty)$ for each $\tau\in\Omega$. Then for all $\nu\in[0,1]$ we have*

$$\begin{aligned}
&\int_\Omega A_\tau^{1-\nu}d\mu(\tau)\otimes\int_\Omega B_\tau^{\nu}d\mu(\tau)\\
&\leq(1-\nu)\int_\Omega A_\tau d\mu(\tau)\otimes 1+\nu 1\otimes\int_\Omega B_\tau d\mu(\tau)\\
&\leq\exp\left[\Delta_\nu\left(m_1,M_1,m_2,M_2\right)\right]\int_\Omega A_\tau^{1-\nu}d\mu(\tau)\otimes\int_\Omega B_\tau^{\nu}d\mu(\tau),
\end{aligned} \tag{3.13}$$

for all $\nu\in[0,1]$, where $\Delta_\nu(m_1,M_1,m_2,M_2)$ is defined in (3.2).

In particular, we have

$$\begin{aligned}
&\int_\Omega A_\tau^{1/2}d\mu(\tau)\otimes\int_\Omega B_\tau^{1/2}d\mu(\tau)\\
&\leq\frac{1}{2}\left[\int_\Omega A_\tau d\mu(\tau)\otimes 1+1\otimes\int_\Omega B_\tau d\mu(\tau)\right]\\
&\leq\exp\left[\Delta_{1/2}\left(m_1,M_1,m_2,M_2\right)\right]\int_\Omega A_\tau^{1/2}d\mu(\tau)\otimes\int_\Omega B_\tau^{1/2}d\mu(\tau),
\end{aligned} \tag{3.14}$$

The proof follows from Lemma 2 in a similar way to the one in the proof of Theorem 1 and the details are omitted.

Corollary 6. *With the assumptions of Theorem 2 we have the following inequalities for the Hadamard product*

$$
\begin{aligned}
(3.15)\qquad & \int_{\Omega} A_{\tau}^{1-\nu} d\mu(\tau) \circ \int_{\Omega} B_{\tau}^{\nu} d\mu(\tau) \\
& \leq \int_{\Omega} ((1-\nu) A_{\tau} + \nu B_{\tau}) \, d\mu(\tau) \circ 1 \\
& \leq \exp\left[\Delta_{\nu}(m_1, M_1, m_2, M_2)\right] \int_{\Omega} A_{\tau}^{1-\nu} d\mu(\tau) \circ \int_{\Omega} B_{\tau}^{\nu} d\mu(\tau),
\end{aligned}
$$

for all $\nu \in [0,1]$.

In particular,

$$
\begin{aligned}
(3.16)\qquad & \int_{\Omega} A_{\tau}^{1/2} d\mu(\tau) \circ \int_{\Omega} B_{\tau}^{1/2} d\mu(\tau) \\
& \leq \int_{\Omega} \left(\frac{A_{\tau} + B_{\tau}}{2}\right) d\mu(\tau) \circ 1 \\
& \leq \exp\left[\Delta_{1/2}(m_1, M_1, m_2, M_2)\right] \int_{\Omega} A_{\tau}^{1/2} d\mu(\tau) \circ \int_{\Omega} B_{\tau}^{1/2} d\mu(\tau).
\end{aligned}
$$

If we take $B_{\tau} = A_{\tau}$, $\tau \in \Omega$ in Corollary 6, then we get

$$
\begin{aligned}
(3.17)\qquad & \int_{\Omega} A_{\tau}^{1-\nu} d\mu(\tau) \circ \int_{\Omega} A_{\tau}^{\nu} d\mu(\tau) \\
& \leq \int_{\Omega} A_{\tau} d\mu(\tau) \circ 1 \\
& \leq \exp\left[\Delta_{\nu}(m_1, M_1, m_2, M_2)\right] \int_{\Omega} A_{\tau}^{1-\nu} d\mu(\tau) \circ \int_{\Omega} A_{\tau}^{\nu} d\mu(\tau),
\end{aligned}
$$

for all $\nu \in [0,1]$.

In particular,

$$
\begin{aligned}
(3.18)\qquad & \int_{\Omega} A_{\tau}^{1/2} d\mu(\tau) \circ \int_{\Omega} A_{\tau}^{1/2} d\mu(\tau) \\
& \leq \int_{\Omega} A_{\tau} d\mu(\tau) \circ 1 \\
& \leq \exp\left[\Delta_{1/2}(m_1, M_1, m_2, M_2)\right] \int_{\Omega} A_{\tau}^{1/2} d\mu(\tau) \circ \int_{\Omega} A_{\tau}^{1/2} d\mu(\tau).
\end{aligned}
$$

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