TENSORIAL AND HADAMARD PRODUCT INTEGRAL INEQUALITIES FOR SYNCHRONOUS FUNCTIONS OF CONTINUOUS FIELDS OF OPERATORS IN HILBERT SPACES

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Abstract. Let H be a Hilbert space and Ω a locally compact Hausdorff space endowed with a Radon measure μ with $\int_\Omega 1d\mu(t)=1$. In this paper we show among others that, if f, g are synchronous and continuous on I and h is nonnegative and continuous on the same interval while $(A_\tau)_{\tau\in\Omega}$ and $(B_\tau)_{\tau\in\Omega}$ are continuous fields of selfadjoint operators in $B(H)$ such that $\operatorname{Sp}(A_\tau), \operatorname{Sp}(B_\tau)\subset I$ for each $\tau\in\Omega$, then

$$\begin{aligned}&\int_\Omega h(A_\tau) f(A_\tau) g(A_\tau) d\mu(\tau)\otimes\int_\Omega h(B_\tau) d\mu(\tau)\\&+\int_\Omega h(A_\tau) d\mu(\tau)\otimes\int_\Omega h(B_\tau) f(B_\tau) g(B_\tau) d\mu(\tau)\\&\geq\int_\Omega h(A_\tau) f(A_\tau) d\mu(\tau)\otimes\int_\Omega h(B_\tau) g(B_\tau) d\mu(\tau)\\&+\int_\Omega h(A_\tau) g(A_\tau) d\mu(\tau)\otimes\int_\Omega h(B_\tau) f(B_\tau) d\mu(\tau).\end{aligned}$$

We also have the similar inequalities for the Hadamard product "$\circ$"..

1. Introduction

Let $I_1, ..., I_k$ be intervals from $\mathbb{R}$ and let $f: I_1\times ...\times I_k\to\mathbb{R}$ be an essentially bounded real function defined on the product of the intervals. Let $A=(A_1, ..., A_n)$ be a k-tuple of bounded selfadjoint operators on Hilbert spaces $H_1, ..., H_k$ such that the spectrum of A_i is contained in I_i for $i=1, ..., k$. We say that such a k-tuple is in the domain of f. If

$$A_i=\int_{I_i}\lambda_i dE_i(\lambda_i)$$

is the spectral resolution of A_i for $i=1, ..., k$; by following [2], we define

$$f(A_1, ..., A_k):=\int_{I_1}...\int_{I_k} f(\lambda_1, ..., \lambda_1) dE_1(\lambda_1)\otimes ...\otimes dE_k(\lambda_k) \tag{1.1}$$

as a bounded selfadjoint operator on the tensorial product $H_1\otimes ...\otimes H_k$.

If the Hilbert spaces are of finite dimension, then the above integrals become finite sums, and we may consider the functional calculus for arbitrary real functions. This construction [2] extends the definition of Korányi [4] for functions of two variables and have the property that

$$f(A_1, ..., A_k)=f_1(A_1)\otimes ...\otimes f_k(A_k),$$

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whenever f can be separated as a product $f(t_1, ..., t_k) = f_1(t_1)...f_k(t_k)$ of k functions each depending on only one variable.

It is know that, *if f is super-multiplicative (sub-multiplicative)* on $[0, \infty)$, namely

$$f(st) \geq (\leq) f(s) f(t) \text{ for all } s, t \in [0, \infty)$$

and if f is continuous on $[0, \infty)$, then [6, p. 173]

$$f(A \otimes B) \geq (\leq) f(A) \otimes f(B) \text{ for all } A,\ B \geq 0. \tag{1.2}$$

This follows by observing that, if

$$A = \int_{[0,\infty)} t dE(t) \text{ and } B = \int_{[0,\infty)} s dF(s)$$

are the spectral resolutions of A and B, then

$$f(A \otimes B) = \int_{[0,\infty)} \int_{[0,\infty)} f(st)\, dE(t) \otimes dF(s) \tag{1.3}$$

for the continuous function f on $[0, \infty)$.

Recall the *geometric operator mean* for the positive operators $A,\ B > 0$

$$A \#_t B := A^{1/2}(A^{-1/2} B A^{-1/2})^t A^{1/2}$$

where $t \in [0, 1]$ and

$$A \# B := A^{1/2}(A^{-1/2} B A^{-1/2})^{1/2} A^{1/2}.$$

By the definitions of $\#$ and $\otimes$ we have

$$A \# B = B \# A \text{ and } (A \# B) \otimes (B \# A) = (A \otimes B) \# (B \otimes A).$$

In 2007, S. Wada [8] obtained the following *Callebaut type inequalities* for tensorial product

$$\begin{aligned} (A \# B) \otimes (A \# B) &\leq \frac{1}{2} \left[(A \#_\alpha B) \otimes (A \#_{1-\alpha} B) + (A \#_{1-\alpha} B) \otimes (A \#_\alpha B) \right] \\ &\leq \frac{1}{2} (A \otimes B + B \otimes A) \end{aligned} \tag{1.4}$$

for $A,\ B > 0$ and $\alpha \in [0, 1]$.

Recall that the *Hadamard product* of A and B in $B(H)$ is defined to be the operator $A \circ B \in B(H)$ satisfying

$$\langle (A \circ B) e_j, e_j \rangle = \langle A e_j, e_j \rangle \langle B e_j, e_j \rangle$$

for all $j \in \mathbb{N}$, where $\{e_j\}_{j \in \mathbb{N}}$ is an *orthonormal basis* for the separable Hilbert space H.

It is known that, see [5], we have the representation

$$A \circ B = \mathcal{U}^* (A \otimes B) \mathcal{U} \tag{1.5}$$

where $\mathcal{U} : H \to H \otimes H$ is the isometry defined by $\mathcal{U} e_j = e_j \otimes e_j$ for all $j \in \mathbb{N}$.

If f *is super-multiplicative (sub-multiplicative)* on $[0, \infty)$, then also [6, p. 173]

$$f(A \circ B) \geq (\leq) f(A) \circ f(B) \text{ for all } A,\ B \geq 0. \tag{1.6}$$

We recall the following elementary inequalities for the Hadamard product

$$A^{1/2} \circ B^{1/2} \leq \left(\frac{A + B}{2} \right) \circ 1 \text{ for } A,\ B \geq 0$$

and Fiedler inequality

$$A \circ A^{-1} \geq 1 \text{ for } A > 0.$$

As extension of Kadison's Schwarz inequality on the Hadamard product, Ando [1] showed that

$$A \circ B \leq \left(A^2 \circ 1\right)^{1/2} \left(B^2 \circ 1\right)^{1/2} \text{ for } A,\ B \geq 0$$

and Aujla and Vasudeva [3] gave an alternative upper bound

$$A \circ B \leq \left(A^2 \circ B^2\right)^{1/2} \text{ for } A,\ B \geq 0.$$

It has been shown in [7] that $\left(A^2 \circ 1\right)^{1/2} \left(B^2 \circ 1\right)^{1/2}$ and $\left(A^2 \circ B^2\right)^{1/2}$ are incomparable for 2-square positive definite matrices A and B.

Let Ω be a locally compact Hausdorff space endowed with a Radon measure μ. A field $(A_t)_{t\in\Omega}$ of operators in $B(H)$ is called a continuous field of operators if the parametrization $t \longmapsto A_t$ is norm continuous on $B(H)$. If, in addition, the norm function $t \longmapsto \|A_t\|$ is Lebesgue integrable on Ω, we can form the Bochner integral $\int_\Omega A_t d\mu(t)$, which is the unique operator in $B(H)$ such that $\varphi\left(\int_\Omega A_t d\mu(t)\right) = \int_\Omega \varphi(A_t)\, d\mu(t)$ for every bounded linear functional φ on $B(H)$. Assume also that, $\int_\Omega 1 d\mu(t) = 1$.

Motivated by the above results, in this paper we show among others that, if f, g are synchronous and continuous on I and h is nonnegative and continuous on the same interval while $(A_\tau)_{\tau\in\Omega}$ and $(B_\tau)_{\tau\in\Omega}$ are continuous fields of selfadjoint operators in $B(H)$ such that $\operatorname{Sp}(A_\tau)$, $\operatorname{Sp}(B_\tau) \subset I$ for each $\tau \in \Omega$, then

$$\begin{aligned}
&\int_\Omega h(A_\tau) f(A_\tau) g(A_\tau)\, d\mu(\tau) \otimes \int_\Omega h(B_\tau)\, d\mu(\tau) \\
&+ \int_\Omega h(A_\tau)\, d\mu(\tau) \otimes \int_\Omega h(B_\tau) f(B_\tau) g(B_\tau)\, d\mu(\tau) \\
&\geq \int_\Omega h(A_\tau) f(A_\tau)\, d\mu(\tau) \otimes \int_\Omega h(B_\tau) g(B_\tau)\, d\mu(\tau) \\
&+ \int_\Omega h(A_\tau) g(A_\tau)\, d\mu(\tau) \otimes \int_\Omega h(B_\tau) f(B_\tau)\, d\mu(\tau).
\end{aligned}$$

We also have the similar inequalities for the Hadamard product "$\circ$".

2. Main Results

We recall that the functions f, g are *synchronous (asynchronous)* on the interval I if

$$(f(t) - f(s))(g(t) - g(s)) \geq (\leq)\, 0$$

for all t, $s \in I$. If f and g have the same monotonicity on I, then they are synchronous.

We start to the following result:

Lemma 1. *Assume that f, g are synchronous and continuous on I and h, k nonnegative and continuous on the same interval. If A, B are selfadjoint with spectra $\operatorname{Sp}(A)$, $\operatorname{Sp}(B) \subset I$, then*

$$\begin{aligned}
&[h(A) f(A) g(A)] \otimes k(B) + h(A) \otimes [k(B) f(B) g(B)] \\
&\geq [h(A) f(A)] \otimes [k(B) g(B)] + [h(A) g(A)] \otimes [k(B) f(B)]
\end{aligned} \tag{2.1}$$

or, equivalently

$$\begin{aligned}(2.2)\qquad &(h(A)\otimes k(B))[(f(A)g(A))\otimes 1+1\otimes(f(B)g(B))]\\ &\geq (h(A)\otimes k(B))[f(A)\otimes g(B)+g(A)\otimes f(B)].\end{aligned}$$

If f, g are asynchronous on I, then the inequality reverses in (2.1) and (2.2).

Proof. Assume that f and g are synchronous on I, then

$$f(t)g(t)+f(s)g(s)\geq f(t)g(s)+f(s)g(t)$$

for all $t,s\in I$.

We multiply this inequality by $h(t)k(s)\geq 0$ to get

$$\begin{aligned}&f(t)g(t)h(t)k(s)+h(t)f(s)g(s)k(s)\\ &\geq f(t)h(t)g(s)k(s)+f(s)k(s)g(t)h(t)\end{aligned}$$

for all $t,s\in I$.

If we take the double integral, then we get

$$\begin{aligned}(2.3)\qquad &\int_I\int_I[f(t)g(t)h(t)k(s)+h(t)f(s)g(s)k(s)]\,dE(t)\otimes dF(s)\\ &\geq \int_I\int_I[f(t)h(t)g(s)k(s)+f(s)k(s)g(t)h(t)]\,dE(t)\otimes dF(s).\end{aligned}$$

Observe that

$$\begin{aligned}&\int_I\int_I[f(t)g(t)h(t)k(s)+h(t)f(s)g(s)k(s)]\,dE(t)\otimes dF(s)\\ &=\int_I\int_I f(t)g(t)h(t)k(s)\,dE(t)\otimes dF(s)\\ &+\int_I\int_I h(t)f(s)g(s)k(s)\,dE(t)\otimes dF(s)\\ &=[h(A)f(A)g(A)]\otimes k(B)+h(A)\otimes[k(B)f(B)g(B)]\end{aligned}$$

and

$$\begin{aligned}&\int_I\int_I[f(t)h(t)g(s)k(s)+f(s)k(s)g(t)h(t)]\,dE(t)\otimes dF(s)\\ &=\int_I\int_I f(t)h(t)g(s)k(s)\,dE(t)\otimes dF(s)\\ &+\int_I\int_I g(t)h(t)f(s)k(s)\,dE(t)\otimes dF(s)\\ &=[h(A)f(A)]\otimes[k(B)g(B)]+[h(A)g(A)]\otimes[k(B)f(B)].\end{aligned}$$

By utilizing (2.3) we derive (2.2).

Now, by making use of the tensorial property

$$(XU)\otimes(YV)=(X\otimes Y)(U\otimes V),$$

for any X, U, Y, $V\in B(H)$, we obtain

$$\begin{aligned}&[h(A)f(A)g(A)]\otimes k(B)+h(A)\otimes[k(B)f(B)g(B)]\\ &=(h(A)\otimes k(B))[(f(A)g(A))\otimes 1]+(h(A)\otimes k(B))[1\otimes(f(B)g(B))]\\ &=(h(A)\otimes k(B))[(f(A)g(A))\otimes 1+1\otimes(f(B)g(B))]\end{aligned}$$

and

$$\begin{aligned}
&[h(A) f(A)] \otimes [k(B) g(B)] + [h(A) g(A)] \otimes [k(B) f(B)] \\
&= (h(A) \otimes k(B)) (f(A) \otimes g(B)) + (h(A) \otimes k(B)) (g(A) \otimes f(B)) \\
&= (h(A) \otimes k(B)) [f(A) \otimes g(B) + g(A) \otimes f(B)],
\end{aligned}$$

which proves (2.2). □

Remark 1. *With the assumptions of Lemma 1 and if we take $k = h$, then we get*

$$\begin{aligned}
&[h(A) f(A) g(A)] \otimes h(B) + h(A) \otimes [h(B) f(B) g(B)] \\
&\geq [h(A) f(A)] \otimes [h(B) g(B)] + [h(A) g(A)] \otimes [h(B) f(B)],
\end{aligned} \tag{2.4}$$

where f, g are synchronous and continuous on I and h is nonnegative and continuous on the same interval.

Moreover, if we take $h \equiv 1$ in (2.4), then we get

$$(f(A) g(A)) \otimes 1 + 1 \otimes (f(B) g(B)) \geq f(A) \otimes g(B) + g(A) \otimes f(B), \tag{2.5}$$

where f, g are synchronous and continuous on I

We have the following result for Hadamard product as well:

Corollary 1. *Assume that f, g are synchronous and continuous on I and h, k nonnegative and continuous on the same interval. If A, B are selfadjoint with spectra $\operatorname{Sp}(A), \operatorname{Sp}(B) \subset I$, then*

$$\begin{aligned}
&k(B) \circ [h(A) f(A) g(A)] + h(A) \circ [k(B) f(B) g(B)] \\
&\geq [h(A) f(A)] \circ [k(B) g(B)] + [k(B) f(B)] \circ [h(A) g(A)].
\end{aligned} \tag{2.6}$$

If f, g are asynchronous on I, then the inequality reverses in (2.6).

In particular, we have

$$\begin{aligned}
&h(B) \circ [h(A) f(A) g(A)] + h(A) \circ [h(B) f(B) g(B)] \\
&\geq [h(A) f(A)] \circ [h(B) g(B)] + [h(B) f(B)] \circ [h(A) g(A)]
\end{aligned} \tag{2.7}$$

and

$$(f(A) g(A) + (f(B) g(B))) \circ 1 \geq f(A) \circ g(B) + f(B) \circ g(A). \tag{2.8}$$

Proof. If we take $\mathcal{U}^*$ to the left and $\mathcal{U}$ to the right in the inequality (2.1), we get

$$\begin{aligned}
&\mathcal{U}^* ([h(A) f(A) g(A)] \otimes k(B)) \mathcal{U} \\
&+ \mathcal{U}^* (h(A) \otimes [k(B) f(B) g(B)]) \mathcal{U} \\
&\geq \mathcal{U}^* ([h(A) f(A)] \otimes [k(B) g(B)]) \mathcal{U} \\
&+ \mathcal{U}^* ([h(A) g(A)] \otimes [k(B) f(B)]) \mathcal{U},
\end{aligned}$$

namely

$$\begin{aligned}
&[h(A) f(A) g(A)] \circ k(B) + h(A) \circ [k(B) f(B) g(B)] \\
&\geq [h(A) f(A)] \circ [k(B) g(B)] + [h(A) g(A)] \circ [k(B) f(B)],
\end{aligned}$$

which is equivalent to (2.6). □

Theorem 1. *Assume that f, g are synchronous and continuous on I and h, k nonnegative and continuous on the same interval. If $(A_\tau)_{\tau\in\Omega}$ and $(B_\tau)_{\tau\in\Omega}$ are continuous fields of selfadjoint operators in $B(H)$ such that $\mathrm{Sp}(A_\tau), \mathrm{Sp}(B_\tau) \subset I$ for each $\tau \in \Omega$, then*

$$\begin{aligned} &\int_\Omega h(A_\tau) f(A_\tau) g(A_\tau) d\mu(\tau) \otimes \int_\Omega k(B_\tau) d\mu(\tau) \\ &+ \int_\Omega h(A_\tau) d\mu(\tau) \otimes \int_\Omega k(B_\tau) f(B_\tau) g(B_\tau) d\mu(\tau) \\ &\geq \int_\Omega h(A_\tau) f(A_\tau) d\mu(\tau) \otimes \int_\Omega k(B_\tau) g(B_\tau) d\mu(\tau) \\ &+ \int_\Omega h(A_\tau) g(A_\tau) d\mu(\tau) \otimes \int_\Omega k(B_\tau) f(B_\tau) d\mu(\tau). \end{aligned} \tag{2.9}$$

In particular, for $k = h$, we have

$$\begin{aligned} &\int_\Omega h(A_\tau) f(A_\tau) g(A_\tau) d\mu(\tau) \otimes \int_\Omega h(B_\tau) d\mu(\tau) \\ &+ \int_\Omega h(A_\tau) d\mu(\tau) \otimes \int_\Omega h(B_\tau) f(B_\tau) g(B_\tau) d\mu(\tau) \\ &\geq \int_\Omega h(A_\tau) f(A_\tau) d\mu(\tau) \otimes \int_\Omega h(B_\tau) g(B_\tau) d\mu(\tau) \\ &+ \int_\Omega h(A_\tau) g(A_\tau) d\mu(\tau) \otimes \int_\Omega h(B_\tau) f(B_\tau) d\mu(\tau). \end{aligned} \tag{2.10}$$

Proof. We have from (2.1) that

$$\begin{aligned} &[h(A_\tau) f(A_\tau) g(A_\tau)] \otimes k(B_\gamma) + h(A_\tau) \otimes [k(B_\gamma) f(B_\gamma) g(B_\gamma)] \\ &\geq [h(A_\tau) f(A_\tau)] \otimes [k(B_\gamma) g(B_\gamma)] + [h(A_\tau) g(A_\tau)] \otimes [k(B_\gamma) f(B_\gamma)] \end{aligned} \tag{2.11}$$

for all τ, $\gamma \in \Omega$.

If we take the integral $\int_\Omega$ over $d\mu(\tau)$ in (2.11), then we get

$$\begin{aligned} &\int_\Omega \{[h(A_\tau) f(A_\tau) g(A_\tau)] \otimes k(B_\gamma) \\ &+ h(A_\tau) \otimes [k(B_\gamma) f(B_\gamma) g(B_\gamma)]\} d\mu(\tau) \\ &\geq \int_\Omega \{[h(A_\tau) f(A_\tau)] \otimes [k(B_\gamma) g(B_\gamma)] \\ &+ [h(A_\tau) g(A_\tau)] \otimes [k(B_\gamma) f(B_\gamma)]\} d\mu(\tau). \end{aligned} \tag{2.12}$$

Using the properties of integral and tensorial products, we have

$$\begin{aligned} &\int_\Omega \{[h(A_\tau) f(A_\tau) g(A_\tau)] \otimes k(B_\gamma) \\ &+ h(A_\tau) \otimes [k(B_\gamma) f(B_\gamma) g(B_\gamma)]\} d\mu(\tau) \\ &= \int_\Omega h(A_\tau) f(A_\tau) g(A_\tau) d\mu(\tau) \otimes k(B_\gamma) \\ &+ \int_\Omega h(A_\tau) d\mu(\tau) \otimes [k(B_\gamma) f(B_\gamma) g(B_\gamma)], \end{aligned}$$

$$\begin{aligned}&\int_{\Omega}\{[h(A_\tau)f(A_\tau)]\otimes[k(B_\gamma)g(B_\gamma)]\\&+[h(A_\tau)g(A_\tau)]\otimes[k(B_\gamma)f(B_\gamma)]\}\,d\mu(\tau)\\&=\int_{\Omega}h(A_\tau)f(A_\tau)\,d\mu(\tau)\otimes[k(B_\gamma)g(B_\gamma)]\\&+\int_{\Omega}h(A_\tau)g(A_\tau)\,d\mu(\tau)\otimes[k(B_\gamma)f(B_\gamma)]\end{aligned}$$

and by (2.12) we get

$$\begin{aligned}&\int_{\Omega}h(A_\tau)f(A_\tau)g(A_\tau)\,d\mu(\tau)\otimes k(B_\gamma)\\&+\int_{\Omega}h(A_\tau)\,d\mu(\tau)\otimes[k(B_\gamma)f(B_\gamma)g(B_\gamma)]\\&\geq\int_{\Omega}h(A_\tau)f(A_\tau)\,d\mu(\tau)\otimes[k(B_\gamma)g(B_\gamma)]\\&+\int_{\Omega}h(A_\tau)g(A_\tau)\,d\mu(\tau)\otimes[k(B_\gamma)f(B_\gamma)]\end{aligned}\tag{2.13}$$

for all $\gamma\in\Omega$.

If we take the integral $\int_{\Omega}$ over $d\mu(\gamma)$ in (2.13), then we get the desired result (2.9). □

Remark 2. *Moreover, if we take $h\equiv 1$ in (2.10), then we get*

$$\begin{aligned}&\int_{\Omega}f(A_\tau)g(A_\tau)\,d\mu(\tau)\otimes 1+1\otimes\int_{\Omega}f(B_\tau)g(B_\tau)\,d\mu(\tau)\\&\geq\int_{\Omega}f(A_\tau)\,d\mu(\tau)\otimes\int_{\Omega}g(B_\tau)\,d\mu(\tau)\\&+\int_{\Omega}g(A_\tau)\,d\mu(\tau)\otimes\int_{\Omega}f(B_\tau)\,d\mu(\tau).\end{aligned}\tag{2.14}$$

If we take $B_\tau=A_\tau$, $\tau\in\Omega$ in (2.14) then we obtain

$$\begin{aligned}&\int_{\Omega}f(A_\tau)g(A_\tau)\,d\mu(\tau)\otimes 1+1\otimes\int_{\Omega}f(A_\tau)g(A_\tau)\,d\mu(\tau)\\&\geq\int_{\Omega}f(A_\tau)\,d\mu(\tau)\otimes\int_{\Omega}g(A_\tau)\,d\mu(\tau)\\&+\int_{\Omega}g(A_\tau)\,d\mu(\tau)\otimes\int_{\Omega}f(A_\tau)\,d\mu(\tau).\end{aligned}\tag{2.15}$$

Corollary 2. *With the assumptions of Theorem 1,*

$$\begin{aligned}&\int_{\Omega}h(A_\tau)f(A_\tau)g(A_\tau)\,d\mu(\tau)\circ\int_{\Omega}k(B_\tau)\,d\mu(\tau)\\&+\int_{\Omega}h(A_\tau)\,d\mu(\tau)\circ\int_{\Omega}k(B_\tau)f(B_\tau)g(B_\tau)\,d\mu(\tau)\\&\geq\int_{\Omega}h(A_\tau)f(A_\tau)\,d\mu(\tau)\circ\int_{\Omega}k(B_\tau)g(B_\tau)\,d\mu(\tau)\\&+\int_{\Omega}h(A_\tau)g(A_\tau)\,d\mu(\tau)\circ\int_{\Omega}k(B_\tau)f(B_\tau)\,d\mu(\tau).\end{aligned}\tag{2.16}$$

In particular, for $k = h$, we have

$$\begin{aligned}(2.17)\qquad & \int_{\Omega} h(A_\tau) f(A_\tau) g(A_\tau) d\mu(\tau) \circ \int_{\Omega} h(B_\tau) d\mu(\tau) \\ & + \int_{\Omega} h(A_\tau) d\mu(\tau) \circ \int_{\Omega} h(B_\tau) f(B_\tau) g(B_\tau) d\mu(\tau) \\ & \geq \int_{\Omega} h(A_\tau) f(A_\tau) d\mu(\tau) \circ \int_{\Omega} h(B_\tau) g(B_\tau) d\mu(\tau) \\ & + \int_{\Omega} h(A_\tau) g(A_\tau) d\mu(\tau) \circ \int_{\Omega} h(B_\tau) f(B_\tau) d\mu(\tau).\end{aligned}$$

Remark 3. *By taking $B_\tau = A_\tau$, $\tau \in \Omega$ in (2.17) and using the commutativity of the Hadamard product, we get*

$$\begin{aligned}(2.18)\qquad & \int_{\Omega} h(A_\tau) d\mu(\tau) \circ \int_{\Omega} h(A_\tau) f(A_\tau) g(A_\tau) d\mu(\tau) \\ & \geq \int_{\Omega} h(A_\tau) f(A_\tau) d\mu(\tau) \circ \int_{\Omega} h(A_\tau) g(A_\tau) d\mu(\tau).\end{aligned}$$

In particular, if we take $h \equiv 1$ in (2.18), then we get

$$(2.19)\qquad \int_{\Omega} f(A_\tau) g(A_\tau) d\mu(\tau) \circ 1 \geq \int_{\Omega} f(A_\tau) d\mu(\tau) \circ \int_{\Omega} g(A_\tau) d\mu(\tau).$$

Assume that A, B are such that $\operatorname{Sp}(A), \operatorname{Sp}(B) \subset I$, then $\operatorname{Sp}((1-t)A + tB) \subset I$ for all $t \in [0,1]$. By taking $A_\tau = (1-t)A + tB$ in (2.19), we get

$$\begin{aligned}(2.20)\qquad & \int_0^1 f((1-t)A + tB) g((1-t)A + tB) dt \circ 1 \\ & \geq \int_0^1 f((1-t)A + tB) dt \circ \int_0^1 g((1-t)A + tB) dt\end{aligned}$$

for all continuous and synchronous functions on I.

For $f(x) = \exp(\alpha x)$, $g(x) = \exp(\beta x)$ with $\alpha\beta > 0$, we get from (2.20) that

$$\begin{aligned}(2.21)\qquad & \int_0^1 \exp[(\alpha+\beta)((1-t)A + tB)] dt \circ 1 \\ & \geq \int_0^1 \exp[\alpha((1-t)A + tB)] dt \circ \int_0^1 \exp\beta((1-t)A + tB) dt\end{aligned}$$

It is known that if U and V are commuting, i.e. $UV = VU$, then the exponential function satisfies the property

$$\exp(U)\exp(V) = \exp(V)\exp(U) = \exp(U+V).$$

Also, if U is invertible and $a, b \in \mathbb{R}$ with $a < b$ then

$$\int_a^b \exp(tU) dt = U^{-1}[\exp(bU) - \exp(aU)].$$

Moreover, if U and V are commuting and $V-U$ is invertible, then

$$\begin{aligned}\int_0^1 \exp\left(\left(1-s\right)U+sV\right)ds &= \int_0^1 \exp\left(s\left(V-U\right)\right)\exp\left(U\right)ds\\ &= \left(\int_0^1 \exp\left(s\left(V-U\right)\right)ds\right)\exp\left(U\right)\\ &= \left(V-U\right)^{-1}\left[\exp\left(V-U\right)-I\right]\exp\left(U\right)\\ &= \left(V-U\right)^{-1}\left[\exp\left(V\right)-\exp\left(U\right)\right].\end{aligned}$$

Therefore

$$\int_0^1 \exp\left[k\left(\left(1-s\right)U+sV\right)\right]ds = k^{-1}\left(V-U\right)^{-1}\left[\exp\left(kV\right)-\exp\left(kU\right)\right]$$

for $k\neq 0$.

Now, if A and B are commutative with $B-A$ is invertible, then

$$\begin{aligned}&\int_0^1 \exp\left[\left(\alpha+\beta\right)\left(\left(1-t\right)A+tB\right)\right]dt\circ 1\\ &= \left(\alpha+\beta\right)^{-1}\left(B-A\right)^{-1}\left[\exp\left(\left(\alpha+\beta\right)B\right)-\exp\left(\left(\alpha+\beta\right)A\right)\right],\end{aligned}$$

$$\int_0^1 \exp\left[\alpha\left(\left(1-t\right)A+tB\right)\right]dt = \alpha^{-1}\left(B-A\right)^{-1}\left[\exp\left(\alpha B\right)-\exp\left(\alpha A\right)\right]$$

and

$$\int_0^1 \exp\left[\beta\left(\left(1-t\right)A+tB\right)\right]dt = \beta^{-1}\left(B-A\right)^{-1}\left[\exp\left(\beta B\right)-\exp\left(\beta A\right)\right].$$

From (2.21) we then get

$$\begin{aligned}&\left(\alpha+\beta\right)^{-1}\left\{\left(B-A\right)^{-1}\left[\exp\left(\left(\alpha+\beta\right)B\right)-\exp\left(\left(\alpha+\beta\right)A\right)\right]\right\}\circ 1\\ &\geq \alpha^{-1}\beta^{-1}\left\{\left(B-A\right)^{-1}\left[\exp\left(\alpha B\right)-\exp\left(\alpha A\right)\right]\right\}\\ &\circ\left\{\left(B-A\right)^{-1}\left[\exp\left(\beta B\right)-\exp\left(\beta A\right)\right]\right\},\end{aligned} \tag{2.22}$$

where A and B are commutative with $B-A$ is invertible.

3. Related Results

We also have:

Lemma 2. *Let $f,\ g:[m,M]\subset\mathbb{R}\to\mathbb{R}$ be continuous on $[m,M]$ and differentiable on (m,M) with $g'(t)\neq 0$ for $t\in(m,M)$. Assume that*

$$-\infty<\gamma=\inf_{t\in(m,M)}\frac{f'(t)}{g'(t)},\quad \sup_{t\in(m,M)}\frac{f'(t)}{g'(t)}=\Gamma<\infty,$$

and A, B *selfadjoint operators with spectra* $\operatorname{Sp}(A), \operatorname{Sp}(B) \subseteq [m, M]$, *then for any continuous and nonnegative function* h *defined on* $[m, M]$,

$$
\begin{aligned}
(3.1)\qquad & 2\gamma \left[\frac{\left(h(A) g^2(A)\right) \otimes h(B) + h(A) \otimes \left(h(B) g^2(B)\right)}{2} \right. \\
& \left. - (g(A) h(A)) \otimes (h(B) g(B)) \right] \\
& \leq [h(A) f(A) g(A)] \otimes h(B) + h(A) \otimes [h(B) f(B) g(B)] \\
& - [h(A) f(A)] \otimes [h(B) g(B)] - [h(A) g(A)] \otimes [h(B) f(B)] \\
& \leq 2\Gamma \left[\frac{\left(h(A) g^2(A)\right) \otimes h(B) + h(A) \otimes \left(h(B) g^2(B)\right)}{2} \right. \\
& \left. - (g(A) h(A)) \otimes (h(B) g(B)) \right].
\end{aligned}
$$

In particular,

$$
\begin{aligned}
(3.2)\qquad & 2\gamma \left[\frac{g^2(A) \otimes 1 + 1 \otimes g^2(B)}{2} - g(A) \otimes g(B) \right] \\
& \leq [f(A) g(A)] \otimes 1 + 1 \otimes [f(B) g(B)] - f(A) \otimes g(B) - g(A) \otimes f(B) \\
& \leq 2\Gamma \left[\frac{g^2(A) \otimes 1 + 1 \otimes g^2(B)}{2} - g(A) \otimes g(B) \right].
\end{aligned}
$$

Proof. Using the Cauchy mean value theorem, for all t, $s \in [m, M]$ with $t \neq s$ there exists ξ between t and s such that

$$
\frac{f(t) - f(s)}{g(t) - g(s)} = \frac{f'(\xi)}{g'(\xi)} \in [\gamma, \Gamma].
$$

Therefore

$$
\gamma [g(t) - g(s)]^2 \leq [f(t) - f(s)] [g(t) - g(s)] \leq \Gamma [g(t) - g(s)]^2
$$

for all t, $s \in [m, M]$, which is equivalent to

$$
\begin{aligned}
& \gamma \left[g^2(t) - 2g(t) g(s) + g^2(s)\right] \\
& \leq f(t) g(t) + f(s) g(s) - f(t) g(s) - f(s) g(t) \\
& \leq \Gamma \left[g^2(t) - 2g(t) g(s) + g^2(s)\right]
\end{aligned}
$$

for all t, $s \in [m, M]$.

If we multiply by $h(t) h(s) \geq 0$, then we get

$$
\begin{aligned}
& \gamma \left[h(t) g^2(t) h(s) - 2g(t) h(t) h(s) g(s) + h(t) h(s) g^2(s)\right] \\
& \leq h(t) f(t) g(t) h(s) + h(t) h(s) f(s) g(s) \\
& - h(t) f(t) h(s) g(s) - h(t) g(t) h(s) f(s) \\
& \leq \Gamma \left[h(t) g^2(t) h(s) - 2g(t) h(t) h(s) g(s) + h(t) h(s) g^2(s)\right]
\end{aligned}
$$

for all t, $s \in [m, M]$.

This implies that

$$\begin{aligned}
&\gamma \int_m^M \int_m^M \left[h(t)\, g^2(t)\, h(s) - 2g(t)\, h(t)\, h(s)\, g(s) + h(t)\, h(s)\, g^2(s)\right] \\
&\times dE(t) \otimes dF(s) \\
&\leq \int_m^M \int_m^M [h(t)\, f(t)\, g(t)\, h(s) + h(t)\, h(s)\, f(s)\, g(s) \\
&-h(t)\, f(t)\, h(s)\, g(s) - h(t)\, g(t)\, h(s)\, f(s)]\, dE(t) \otimes dF(s) \\
&\leq \Gamma \int_m^M \int_m^M \left[h(t)\, g^2(t)\, h(s) - 2g(t)\, h(t)\, h(s)\, g(s) + h(t)\, h(s)\, g^2(s)\right] \\
&\times dE(t) \otimes dF(s)
\end{aligned}$$

and by performing the calculations as in the proof of Lemma 1, we derive (3.1). $\square$

Corollary 3. *With the assumptions of Lemma 1 we have*

$$\begin{aligned}
&2\gamma \left[\frac{h(B) \circ \left(h(A)\, g^2(A)\right) + h(A) \circ \left(h(B)\, g^2(B)\right)}{2}\right. \\
&\left. - (g(A)\, h(A)) \circ (h(B)\, g(B))\right] \\
&\leq h(B) \circ [h(A)\, f(A)\, g(A)] + h(A) \circ [h(B)\, f(B)\, g(B)] \\
&- [h(A)\, f(A)] \circ [h(B)\, g(B)] - [h(A)\, g(A)] \circ [h(B)\, f(B)] \\
&\leq 2\Gamma \left[\frac{h(B) \circ \left(h(A)\, g^2(A)\right) + h(A) \circ \left(h(B)\, g^2(B)\right)}{2}\right. \\
&\left. - (g(A)\, h(A)) \circ (h(B)\, g(B))\right].
\end{aligned} \tag{3.3}$$

In particular,

$$\begin{aligned}
&2\gamma \left[\frac{g^2(A) + g^2(B)}{2} \circ 1 - g(A) \circ g(B)\right] \\
&\leq [f(A)\, g(A) + f(B)\, g(B)] \circ 1 - f(A) \circ g(B) - g(A) \circ f(B) \\
&\leq 2\Gamma \left[\frac{g^2(A) + g^2(B)}{2} \circ 1 - g(A) \circ g(B)\right].
\end{aligned} \tag{3.4}$$

Theorem 2. *Let f and g be as in Lemma 2. If $(A_\tau)_{\tau\in\Omega}$ and $(B_\tau)_{\tau\in\Omega}$ are continuous fields of selfadjoint operators in $B(H)$ such that* $\operatorname{Sp}(A_\tau), \operatorname{Sp}(B_\tau) \subset [m, M]$

for each $\tau \in \Omega$, then we have

$$
\begin{aligned}
(3.5)\qquad & 2\gamma \left[\frac{1}{2}\left(\int_{\Omega} h(A_\tau) g^2(A_\tau) d\mu(\tau) \otimes \int_{\Omega} h(B_\tau) d\mu(\tau)\right.\right.\\
&+ \left.\int_{\Omega} h(A_\tau) d\mu(\tau) \otimes \int_{\Omega} h(B_\tau) g^2(B_\tau) d\mu(\tau)\right)\\
&- \left.\int_{\Omega} g(A_\tau) h(A_\tau) d\mu(\tau) \otimes \int_{\Omega} h(B_\tau) g(B_\tau) d\mu(\tau)\right]\\
&\le \int_{\Omega} h(A_\tau) f(A_\tau) g(A_\tau) d\mu(\tau) \otimes \int_{\Omega} h(B_\tau) d\mu(\tau)\\
&+ \int_{\Omega} h(A_\tau) d\mu(\tau) \otimes \int_{\Omega} h(B_\tau) f(B_\tau) g(B_\tau) d\mu(\tau)\\
&- \int_{\Omega} h(A_\tau) f(A_\tau) d\mu(\tau) \otimes \int_{\Omega} h(B_\tau) g(B_\tau) d\mu(\tau)\\
&- \int_{\Omega} h(A_\tau) g(A_\tau) d\mu(\tau) \otimes \int_{\Omega} h(B_\tau) f(B_\tau) d\mu(\tau)\\
&\le 2\Gamma \left[\frac{1}{2}\left(\int_{\Omega} h(A_\tau) g^2(A_\tau) d\mu(\tau) \otimes \int_{\Omega} h(B_\tau) d\mu(\tau)\right.\right.\\
&+ \left.\int_{\Omega} h(A_\tau) d\mu(\tau) \otimes \int_{\Omega} h(B_\tau) g^2(B_\tau) d\mu(\tau)\right)\\
&- \left.\int_{\Omega} g(A_\tau) h(A) d\mu(\tau) \otimes \int_{\Omega} h(B_\tau) g(B_\tau) d\mu(\tau)\right].
\end{aligned}
$$

The proof follows from Lemma 2 by using a similar argument to the one in the proof of Theorem 1 and we omit the details.

If we take $h \equiv 1$ in (3.5), then we get

$$
\begin{aligned}
(3.6)\quad & 2\gamma \left[\frac{1}{2}\left(\int_{\Omega} g^2(A_\tau) d\mu(\tau) \otimes 1 + 1 \otimes \int_{\Omega} g^2(B_\tau) d\mu(\tau)\right)\right.\\
&- \left.\int_{\Omega} g(A_\tau) d\mu(\tau) \otimes \int_{\Omega} g(B_\tau) d\mu(\tau)\right]\\
&\le \int_{\Omega} f(A_\tau) g(A_\tau) d\mu(\tau) \otimes 1 + 1 \otimes \int_{\Omega} f(B_\tau) g(B_\tau) d\mu(\tau)\\
&- \int_{\Omega} f(A_\tau) d\mu(\tau) \otimes \int_{\Omega} g(B_\tau) d\mu(\tau) - \int_{\Omega} g(A_\tau) d\mu(\tau) \otimes \int_{\Omega} f(B_\tau) d\mu(\tau)\\
&\le 2\Gamma \left[\frac{1}{2}\left(\int_{\Omega} g^2(A_\tau) d\mu(\tau) \otimes 1 + 1 \otimes \int_{\Omega} g^2(B_\tau) d\mu(\tau)\right)\right.\\
&- \left.\int_{\Omega} g(A_\tau) d\mu(\tau) \otimes \int_{\Omega} g(B_\tau) d\mu(\tau)\right].
\end{aligned}
$$

From (3.6) we derive the following result for the Hadamard product

$$
\begin{aligned}
(3.7)\quad & 2\gamma \\
& \times\left[\int_{\Omega}\frac{g^{2}(A_{\tau})+g^{2}(B_{\tau})}{2}d\mu(\tau)\circ 1-\int_{\Omega}g(A_{\tau})\,d\mu(\tau)\circ\int_{\Omega}g(B_{\tau})\,d\mu(\tau)\right] \\
& \leq\int_{\Omega}\left[f(A_{\tau})\,g(A_{\tau})+f(B_{\tau})\,g(B_{\tau})\right]d\mu(\tau)\circ 1 \\
& -\int_{\Omega}f(A_{\tau})\,d\mu(\tau)\circ\int_{\Omega}g(B_{\tau})\,d\mu(\tau)-\int_{\Omega}g(A_{\tau})\,d\mu(\tau)\circ\int_{\Omega}f(B_{\tau})\,d\mu(\tau) \\
& \leq 2\Gamma \\
& \times\left[\int_{\Omega}\frac{g^{2}(A_{\tau})+g^{2}(B_{\tau})}{2}d\mu(\tau)\circ 1-\int_{\Omega}g(A_{\tau})\,d\mu(\tau)\circ\int_{\Omega}g(B_{\tau})\,d\mu(\tau)\right].
\end{aligned}
$$

If in this inequality we take $B_{\tau}=A_{\tau}$, $\tau\in\Omega$, then we get

$$
\begin{aligned}
(3.8)\qquad & \gamma\left[\int_{\Omega}g^{2}(A_{\tau})\,d\mu(\tau)\circ 1-\int_{\Omega}g(A_{\tau})\,d\mu(\tau)\circ\int_{\Omega}g(A_{\tau})\,d\mu(\tau)\right] \\
& \leq\int_{\Omega}f(A_{\tau})\,g(A_{\tau})\,d\mu(\tau)\circ 1-\int_{\Omega}f(A_{\tau})\,d\mu(\tau)\circ\int_{\Omega}g(A_{\tau})\,d\mu(\tau) \\
& \leq\Gamma\left[\int_{\Omega}g^{2}(A_{\tau})\,d\mu(\tau)\circ 1-\int_{\Omega}g(A_{\tau})\,d\mu(\tau)\circ\int_{\Omega}g(A_{\tau})\,d\mu(\tau)\right].
\end{aligned}
$$

Consider the functions $f(t)=t^{p}$, $g(t)=t^{q}$ defined on $(0,\infty)$. Then $f'(t)=pt^{p-1}$, $g'(t)=qt^{q-1}$ for $t>0$ and

$$\frac{f'(t)}{g'(t)}=\frac{p}{q}t^{p-q},\ t>0.$$

Assume that either $p,q\in(0,\infty)$ or $p,q\in(-\infty,0)$. Then $\frac{p}{q}>0$ and $\frac{f'(t)}{g'(t)}$ is increasing for $p>q$ and decreasing for $p<q$ and constant 1 for $p=q$.

Observe that for $[m,M]\subset(0,\infty)$,

$$\inf_{t\in[m,M]}\frac{f'(t)}{g'(t)}=\frac{p}{q}m^{p-q}\text{ and }\sup_{t\in[m,M]}\frac{f'(t)}{g'(t)}=\frac{p}{q}M^{p-q}\text{ for }p>q$$

and

$$\inf_{t\in[m,M]}\frac{f'(t)}{g'(t)}=\frac{p}{q}M^{p-q}\text{ and }\sup_{t\in[m,M]}\frac{f'(t)}{g'(t)}=\frac{p}{q}m^{p-q}\text{ for }p<q.$$

Assume that either $p,q\in(0,\infty)$ or $p,q\in(-\infty,0)$ and $(A_{\tau})_{\tau\in\Omega}$ and $(B_{\tau})_{\tau\in\Omega}$ are continuous fields of selfadjoint operators in $B(H)$ such that $\mathrm{Sp}(A_{\tau})$, $\mathrm{Sp}(B_{\tau})$ $[m,M]\subset$

$(0,\infty)$ for each $\tau\in\Omega$. From (3.5) we get for $p>q$ that

$$\begin{aligned}(3.9)\quad & 0\le 2\frac{p}{q}m^{p-q}\\ &\times\left(\frac{\int_\Omega A_\tau^{2q}d\mu(\tau)\otimes 1+1\otimes\int_\Omega B_\tau^{2q}d\mu(\tau)}{2}-\int_\Omega A_\tau^q d\mu(\tau)\otimes\int_\Omega B_\tau^q d\mu(\tau)\right)\\ &\le\int_\Omega A_\tau^{p+q}d\mu(\tau)\otimes 1+1\otimes\int_\Omega B_\tau^{p+q}d\mu(\tau)\\ &-\int_\Omega A_\tau^p d\mu(\tau)\otimes\int_\Omega B_\tau^q d\mu(\tau)-\int_\Omega A_\tau^q d\mu(\tau)\otimes\int_\Omega B_\tau^p d\mu(\tau)\\ &\le 2\frac{p}{q}M^{p-q}\\ &\times\left(\frac{\int_\Omega A_\tau^{2q}d\mu(\tau)\otimes 1+1\otimes\int_\Omega B_\tau^{2q}d\mu(\tau)}{2}-\int_\Omega A_\tau^q d\mu(\tau)\otimes\int_\Omega B_\tau^q d\mu(\tau)\right)\end{aligned}$$

and for $p<q$

$$\begin{aligned}(3.10)\quad & 0\le 2\frac{p}{q}M^{p-q}\\ &\times\left(\frac{\int_\Omega A_\tau^{2q}d\mu(\tau)\otimes 1+1\otimes\int_\Omega B_\tau^{2q}d\mu(\tau)}{2}-\int_\Omega A_\tau^q d\mu(\tau)\otimes\int_\Omega B_\tau^q d\mu(\tau)\right)\\ &\le\int_\Omega A_\tau^{p+q}\otimes 1+1\otimes\int_\Omega B_\tau^{p+q}-\int_\Omega A_\tau^p\otimes\int_\Omega B_\tau^q-\int_\Omega A_\tau^q\otimes\int_\Omega B_\tau^p\\ &\le 2\frac{p}{q}m^{p-q}\\ &\times\left(\frac{\int_\Omega A_\tau^{2q}d\mu(\tau)\otimes 1+1\otimes\int_\Omega B_\tau^{2q}d\mu(\tau)}{2}-\int_\Omega A_\tau^q d\mu(\tau)\otimes\int_\Omega B_\tau^q d\mu(\tau)\right).\end{aligned}$$

From (3.7) we also have the inequalities for the Hadamard product for $p>q$ that

$$\begin{aligned}(3.11)\quad & 0\le 2\frac{p}{q}m^{p-q}\left(\int_\Omega\left(\frac{A_\tau^{2q}+B_\tau^{2q}}{2}\right)d\mu(\tau)\circ 1-\int_\Omega A_\tau^q d\mu(\tau)\circ\int_\Omega B_\tau^q d\mu(\tau)\right)\\ &\le\int_\Omega\left(A_\tau^{p+q}+B_\tau^{p+q}\right)d\mu(\tau)\circ 1\\ &-\int_\Omega A_\tau^p d\mu(\tau)\circ\int_\Omega B_\tau^q d\mu(\tau)-\int_\Omega A_\tau^q d\mu(\tau)\circ\int_\Omega B_\tau^p d\mu(\tau)\\ &\le 2\frac{p}{q}M^{p-q}\left(\int_\Omega\left(\frac{A_\tau^{2q}+B_\tau^{2q}}{2}\right)d\mu(\tau)\circ 1-\int_\Omega A_\tau^q d\mu(\tau)\circ\int_\Omega B_\tau^q d\mu(\tau)\right)\end{aligned}$$

and for $p<q$

$$\begin{aligned}(3.12)\quad & 0\le 2\frac{p}{q}M^{p-q}\left(\int_\Omega\left(\frac{A_\tau^{2q}+B_\tau^{2q}}{2}\right)d\mu(\tau)\circ 1-\int_\Omega A_\tau^q d\mu(\tau)\circ\int_\Omega B_\tau^q d\mu(\tau)\right)\\ &\le\int_\Omega\left(A_\tau^{p+q}+B_\tau^{p+q}\right)d\mu(\tau)\circ 1\\ &-\int_\Omega A_\tau^p d\mu(\tau)\circ\int_\Omega B_\tau^q d\mu(\tau)-\int_\Omega A_\tau^q d\mu(\tau)\circ\int_\Omega B_\tau^p d\mu(\tau)\\ &\le 2\frac{p}{q}m^{p-q}\left(\int_\Omega\left(\frac{A_\tau^{2q}+B_\tau^{2q}}{2}\right)d\mu(\tau)\circ 1-\int_\Omega A_\tau^q d\mu(\tau)\circ\int_\Omega B_\tau^q d\mu(\tau)\right).\end{aligned}$$

Finally, for $B_\tau = A_\tau$ in (3.11) and (3.12), we get for $p > q$ that

$$0 \leq \frac{p}{q} m^{p-q} \left(\int_\Omega A_\tau^{2q} d\mu(\tau) \circ 1 - \int_\Omega A_\tau^{q} d\mu(\tau) \circ \int_\Omega A_\tau^{q} d\mu(\tau) \right) \tag{3.13}$$
$$\leq \int_\Omega A_\tau^{p+q} d\mu(\tau) \circ 1 - \int_\Omega A_\tau^{p} d\mu(\tau) \circ \int_\Omega A_\tau^{q} d\mu(\tau)$$
$$\leq \frac{p}{q} M^{p-q} \left(\int_\Omega A_\tau^{2q} d\mu(\tau) \circ 1 - \int_\Omega A_\tau^{q} d\mu(\tau) \circ \int_\Omega A_\tau^{q} d\mu(\tau) \right)$$

and for $p < q$

$$0 \leq \frac{p}{q} M^{p-q} \left(\int_\Omega A_\tau^{2q} d\mu(\tau) \circ 1 - \int_\Omega A_\tau^{q} d\mu(\tau) \circ \int_\Omega A_\tau^{q} d\mu(\tau) \right) \tag{3.14}$$
$$\leq \int_\Omega A_\tau^{p+q} d\mu(\tau) \circ 1 - \int_\Omega A_\tau^{p} d\mu(\tau) \circ \int_\Omega A_\tau^{q} d\mu(\tau)$$
$$\leq \frac{p}{q} m^{p-q} \left(\int_\Omega A_\tau^{2q} d\mu(\tau) \circ 1 - \int_\Omega A_\tau^{q} d\mu(\tau) \circ \int_\Omega A_\tau^{q} d\mu(\tau) \right).$$

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