TENSORIAL AND HADAMARD PRODUCT INTEGRAL INEQUALITIES FOR CONVEX FUNCTIONS OF CONTINUOUS FIELDS OF OPERATORS IN HILBERT SPACES

SILVESTRU SEVER DRAGOMIR[1,2]

Abstract. Let H be a Hilbert space and Ω a locally compact Hausdorff space endowed with a Radon measure μ with $\int_{\Omega} 1d\mu(t)=1$. In this paper we show among others that, if f is continuous differentiable convex on the open interval I, $(A_{\tau})_{\tau\in\Omega}$ is a continuous field of positive operators in $B(H)$ such that $\operatorname{Sp}(A_{\tau})\subset I$ for each $\tau\in\Omega$ and B and operator such that $\operatorname{Sp}(B)\subset I$, then we have

$$\begin{aligned}&\int_{\Omega}\left(f'(A_{\tau})A_{\tau}\right)d\mu(\tau)\otimes 1-\int_{\Omega}f'(A_{\tau})d\mu(\tau)\otimes B\\ &\geq\int_{\Omega}f(A_{\tau})d\mu(\tau)\otimes 1-1\otimes f(B)\\ &\geq\left(\int_{\Omega}A_{\tau}d\mu(\tau)\otimes 1-(1\otimes B)\right)(1\otimes f'(B))\end{aligned}$$

and the Hadamard product inequality

$$\begin{aligned}&\int_{\Omega}\left(f'(A_{\tau})A_{\tau}\right)d\mu(\tau)\circ 1-\int_{\Omega}f'(A_{\tau})d\mu(\tau)\circ B\\ &\geq\int_{\Omega}f(A_{\tau})d\mu(\tau)\circ 1-1\circ f(B)\\ &\geq\int_{\Omega}A_{\tau}d\mu(\tau)\circ f'(B)-1\circ\left(f'(B)B\right).\end{aligned}$$

1. Introduction

Let $I_1,...,I_k$ be intervals from $\mathbb{R}$ and let $f:I_1\times...\times I_k\to\mathbb{R}$ be an essentially bounded real function defined on the product of the intervals. Let $A=(A_1,...,A_n)$ be a k-tuple of bounded selfadjoint operators on Hilbert spaces $H_1,...,H_k$ such that the spectrum of A_i is contained in I_i for $i=1,...,k$. We say that such a k-tuple is in the domain of f. If

$$A_i=\int_{I_i}\lambda_i dE_i(\lambda_i)$$

is the spectral resolution of A_i for $i=1,...,k$; by following [2], we define

$$f(A_1,...,A_k):=\int_{I_1}...\int_{I_k}f(\lambda_1,...,\lambda_1)\,dE_1(\lambda_1)\otimes...\otimes dE_k(\lambda_k) \tag{1.1}$$

as a bounded selfadjoint operator on the tensorial product $H_1\otimes...\otimes H_k$.

If the Hilbert spaces are of finite dimension, then the above integrals become finite sums, and we may consider the functional calculus for arbitrary real functions.

1991 *Mathematics Subject Classification.* 47A63; 47A99.

Key words and phrases. Tensorial product, Hadamard Product, Selfadjoint operators, Convex functions.

RGMIA Res. Rep. Coll. **25** (2022), Art. 128, 10 pp. Received 23/11/22

This construction [2] extends the definition of Korányi [4] for functions of two variables and have the property that

$$f(A_1, ..., A_k) = f_1(A_1) \otimes ... \otimes f_k(A_k),$$

whenever f can be separated as a product $f(t_1, ..., t_k) = f_1(t_1)...f_k(t_k)$ of k functions each depending on only one variable.

It is know that, *if f is super-multiplicative (sub-multiplicative)* on $[0, \infty)$, namely

$$f(st) \geq (\leq) f(s) f(t) \text{ for all } s, t \in [0, \infty)$$

and if f is continuous on $[0, \infty)$, then [6, p. 173]

$$f(A \otimes B) \geq (\leq) f(A) \otimes f(B) \text{ for all } A,\ B \geq 0. \tag{1.2}$$

This follows by observing that, if

$$A = \int_{[0,\infty)} t dE(t) \text{ and } B = \int_{[0,\infty)} s dF(s)$$

are the spectral resolutions of A and B, then

$$f(A \otimes B) = \int_{[0,\infty)} \int_{[0,\infty)} f(st) dE(t) \otimes dF(s) \tag{1.3}$$

for the continuous function f on $[0, \infty)$.

Recall the *geometric operator mean* for the positive operators A, $B > 0$

$$A \#_t B := A^{1/2}(A^{-1/2} B A^{-1/2})^t A^{1/2}$$

where $t \in [0, 1]$ and

$$A \# B := A^{1/2}(A^{-1/2} B A^{-1/2})^{1/2} A^{1/2}.$$

By the definitions of $\#$ and $\otimes$ we have

$$A \# B = B \# A \text{ and } (A \# B) \otimes (B \# A) = (A \otimes B) \# (B \otimes A).$$

In 2007, S. Wada [8] obtained the following *Callebaut type inequalities* for tensorial product

$$\begin{aligned} (A \# B) \otimes (A \# B) &\leq \frac{1}{2} \left[(A \#_\alpha B) \otimes (A \#_{1-\alpha} B) + (A \#_{1-\alpha} B) \otimes (A \#_\alpha B) \right] \\ &\leq \frac{1}{2} (A \otimes B + B \otimes A) \end{aligned} \tag{1.4}$$

for A, $B > 0$ and $\alpha \in [0, 1]$.

Recall that the *Hadamard product* of A and B in $B(H)$ is defined to be the operator $A \circ B \in B(H)$ satisfying

$$\langle (A \circ B) e_j, e_j \rangle = \langle A e_j, e_j \rangle \langle B e_j, e_j \rangle$$

for all $j \in \mathbb{N}$, where $\{e_j\}_{j \in \mathbb{N}}$ is an *orthonormal basis* for the separable Hilbert space H.

It is known that, see [5], we have the representation

$$A \circ B = \mathcal{U}^* (A \otimes B) \mathcal{U} \tag{1.5}$$

where $\mathcal{U} : H \to H \otimes H$ is the isometry defined by $\mathcal{U} e_j = e_j \otimes e_j$ for all $j \in \mathbb{N}$.

If f *is super-multiplicative (sub-multiplicative)* on $[0, \infty)$, then also [6, p. 173]

$$f(A \circ B) \geq (\leq) f(A) \circ f(B) \text{ for all } A,\ B \geq 0. \tag{1.6}$$

We recall the following elementary inequalities for the Hadamard product

$$A^{1/2} \circ B^{1/2} \leq \left(\frac{A+B}{2} \right) \circ 1 \text{ for } A,\ B \geq 0$$

and Fiedler inequality

$$A \circ A^{-1} \geq 1 \text{ for } A > 0.$$

As extension of Kadison's Schwarz inequality on the Hadamard product, Ando [1] showed that

$$A \circ B \leq \left(A^2 \circ 1 \right)^{1/2} \left(B^2 \circ 1 \right)^{1/2} \text{ for } A,\ B \geq 0$$

and Aujla and Vasudeva [3] gave an alternative upper bound

$$A \circ B \leq \left(A^2 \circ B^2 \right)^{1/2} \text{ for } A,\ B \geq 0.$$

It has been shown in [7] that $\left(A^2 \circ 1 \right)^{1/2} \left(B^2 \circ 1 \right)^{1/2}$ and $\left(A^2 \circ B^2 \right)^{1/2}$ are incomparable for 2-square positive definite matrices A and B.

Let Ω be a locally compact Hausdorff space endowed with a Radon measure μ. A field $(A_t)_{t \in \Omega}$ of operators in $B(H)$ is called a continuous field of operators if the parametrization $t \longmapsto A_t$ is norm continuous on $B(H)$. If, in addition, the norm function $t \longmapsto \|A_t\|$ is Lebesgue integrable on Ω, we can form the Bochner integral $\int_\Omega A_t d\mu(t)$, which is the unique operator in $B(H)$ such that $\varphi\left(\int_\Omega A_t d\mu(t)\right) = \int_\Omega \varphi(A_t) d\mu(t)$ for every bounded linear functional φ on $B(H)$. Assume also that, $\int_\Omega 1 d\mu(t) = 1$.

Motivated by the above results, in this paper we show among others that, if f is continuous differentiable convex on the open interval I, $(A_\tau)_{\tau \in \Omega}$ is a continuous field of positive operators in $B(H)$ such that $\operatorname{Sp}(A_\tau) \subset I$ for each $\tau \in \Omega$ and B and operator such that $\operatorname{Sp}(B) \subset I$, then we have

$$\begin{aligned}
&\int_\Omega \left(f'(A_\tau) A_\tau \right) d\mu(\tau) \otimes 1 - \int_\Omega f'(A_\tau) d\mu(\tau) \otimes B \\
&\geq \int_\Omega f(A_\tau) d\mu(\tau) \otimes 1 - 1 \otimes f(B) \\
&\geq \left(\int_\Omega A_\tau d\mu(\tau) \otimes 1 - (1 \otimes B) \right) (1 \otimes f'(B))
\end{aligned}$$

and the Hadamard product inequality

$$\begin{aligned}
&\int_\Omega \left(f'(A_\tau) A_\tau \right) d\mu(\tau) \circ 1 - \int_\Omega f'(A_\tau) d\mu(\tau) \circ B \\
&\geq \int_\Omega f(A_\tau) d\mu(\tau) \circ 1 - 1 \circ f(B) \\
&\geq \int_\Omega A_\tau d\mu(\tau) \circ f'(B) - 1 \circ \left(f'(B) B \right).
\end{aligned}$$

2. Main Results

We also have the following double inequality for tensorial product of operators:

Lemma 1. *Assume that f is continuous differentiable convex on the open interval I and A, B are selfadjoint operators in $B(H)$ with spectra $\operatorname{Sp}(A), \operatorname{Sp}(B) \subset I$, then*

$$\begin{aligned}
(f'(A) \otimes 1)(A \otimes 1 - 1 \otimes B) &\geq f(A) \otimes 1 - 1 \otimes f(B) \\
&\geq (A \otimes 1 - 1 \otimes B)(1 \otimes f'(B)).
\end{aligned} \tag{2.1}$$

Proof. Using the gradient inequality for the differentiable convex f on I we have

$$f'(t)(t-s) \geq f(t) - f(s) \geq f'(s)(t-s)$$

for all t, $s \in I$.

Assume that

$$A = \int_I t dE(t) \text{ and } B = \int_I s dF(s)$$

are the spectral resolutions of A and B.

These imply that

$$\int_I \int_I f'(t)(t-s)\, dE(t) \otimes dF(s) \geq \int_I \int_I (f(t) - f(s))\, dE(t) \otimes dF(s) \geq \int_I \int_I f'(s)(t-s)\, dE(t) \otimes dF(s). \tag{2.2}$$

Observe that

$$\begin{aligned} &\int_I \int_I f'(t)(t-s)\, dE(t) \otimes dF(s) \\ &= \int_I \int_I (f'(t)t - f'(t)s)\, dE(t) \otimes dF(s) \\ &= \int_I \int_I f'(t) t dE(t) \otimes dF(s) - \int_I \int_I f'(t) s dE(t) \otimes dF(s) \\ &= (f'(A)A) \otimes 1 - f'(A) \otimes B, \end{aligned} \tag{2.3}$$

$$\int_I \int_I (f(t) - f(s))\, dE(t) \otimes dF(s) = f(A) \otimes 1 - 1 \otimes f(B)$$

and

$$\begin{aligned} &\int_I \int_I f'(s)(t-s)\, dE(t) \otimes dF(s) \\ &= \int_I \int_I (tf'(s) - f'(s)s)\, dE(t) \otimes dF(s) \\ &= \int_I \int_I tf'(s)\, dE(t) \otimes dF(s) - \int_I \int_I f'(s) s dE(t) \otimes dF(s) \\ &= A \otimes f'(B) - 1 \otimes (f'(B)B) \end{aligned}$$

and by (2.3) we derive the inequality of interest:

$$\begin{aligned} (f'(A)A) \otimes 1 - f'(A) \otimes B &\geq f(A) \otimes 1 - 1 \otimes f(B) \\ &\geq A \otimes f'(B) - 1 \otimes (f'(B)B). \end{aligned} \tag{2.4}$$

Now, by utilizing the tensorial property

$$(XU) \otimes (YV) = (X \otimes Y)(U \otimes V),$$

for any X, U, Y, $V \in B(H)$, we have

$$\begin{aligned} (f'(A)A) \otimes 1 &= (f'(A) \otimes 1)(A \otimes 1), \\ f'(A) \otimes B &= (f'(A) \otimes 1)(1 \otimes B), \\ A \otimes f'(B) &= (A \otimes 1)(1 \otimes f'(B)) \end{aligned}$$

and

$$1 \otimes (f'(B)B) = 1 \otimes (Bf'(B)) = (1 \otimes B)(1 \otimes f'(B)).$$

Therefore

$$\begin{aligned}(f'(A)A)\otimes 1-f'(A)\otimes B &= (f'(A)\otimes 1)(A\otimes 1)-(f'(A)\otimes 1)(1\otimes B)\\ &= (f'(A)\otimes 1)(A\otimes 1-1\otimes B)\end{aligned}$$

and

$$\begin{aligned}A\otimes f'(B)-1\otimes (f'(B)B) &= (A\otimes 1)(1\otimes f'(B))-(1\otimes B)(1\otimes f'(B))\\ &= (A\otimes 1-1\otimes B)(1\otimes f'(B))\end{aligned}$$

and by (2.4) we derive (2.1). □

Corollary 1. *Assume that f is continuous differentiable convex on the open interval I and A, B are selfadjoint operators in $B(H)$ with spectra $\operatorname{Sp}(A), \operatorname{Sp}(B)\subset I$, then*

$$\begin{aligned}(f'(A)A)\circ 1-f'(A)\circ B &\geq (f(A)-f(B))\circ 1\\ &\geq A\circ f'(B)-(f'(B)B)\circ 1.\end{aligned}\tag{2.5}$$

Proof. If we multiply the inequality (2.4) to the left with $\mathcal{U}^*$ and at the right with $\mathcal{U}$, we get

$$\begin{aligned}&\mathcal{U}^*[(f'(A)A)\otimes 1-f'(A)\otimes B]\mathcal{U}\\ &\geq \mathcal{U}^*[f(A)\otimes 1-1\otimes f(B)]\mathcal{U}\\ &\geq \mathcal{U}^*[A\otimes f'(B)-1\otimes (f'(B)B)]\mathcal{U},\end{aligned}$$

namely

$$\begin{aligned}&\mathcal{U}^*((f'(A)A)\otimes 1)\mathcal{U}-\mathcal{U}^*(f'(A)\otimes B)\mathcal{U}\\ &\geq \mathcal{U}^*(f(A)\otimes 1)\mathcal{U}-\mathcal{U}^*(1\otimes f(B))\mathcal{U}\\ &\geq \mathcal{U}^*(A\otimes f'(B))\mathcal{U}-\mathcal{U}^*(1\otimes (f'(B)B))\mathcal{U}.\end{aligned}$$

Using representation (1.5) we get

$$\begin{aligned}(f'(A)A)\circ 1-f'(A)\circ B &\geq f(A)\circ 1-1\circ f(B)\\ &\geq A\circ f'(B)-1\circ (f'(B)B),\end{aligned}\tag{2.6}$$

which gives (2.5). □

In what follows, we assume that, $\int_\Omega 1d\mu(t)=1$.

Theorem 1. *Assume that f is continuous differentiable convex on the open interval I. Let $(A_\tau)_{\tau\in\Omega}$ and $(B_\tau)_{\tau\in\Omega}$ be continuous fields of positive operators in $B(H)$ such that $\operatorname{Sp}(A_\tau), \operatorname{Sp}(B_\tau)\subset I$ for each $\tau\in\Omega$. Then we have*

$$\begin{aligned}&\int_\Omega (f'(A_\tau)A_\tau)d\mu(\tau)\otimes 1-\int_\Omega f'(A_\tau)d\mu(\tau)\otimes\int_\Omega B_\tau d\mu(\tau)\\ &\geq \int_\Omega f(A_\tau)d\mu(\tau)\otimes 1-1\otimes\int_\Omega f(B_\tau)d\mu(\tau)\\ &\geq \int_\Omega A_\tau d\mu(\tau)\otimes\int_\Omega f'(B_\tau)d\mu(\tau)-1\otimes\int_\Omega f'(B_\tau)B_\tau d\mu(\tau)\end{aligned}\tag{2.7}$$

and the Hadamard product inequality

$$\begin{aligned}\int_{\Omega}\left(f'\left(A_{\tau}\right)A_{\tau}\right)d\mu\left(\tau\right)\circ 1-\int_{\Omega}f'\left(A_{\tau}\right)d\mu\left(\tau\right)\circ\int_{\Omega}B_{\tau}d\mu\left(\tau\right)\\ \geq\int_{\Omega}f\left(A_{\tau}\right)d\mu\left(\tau\right)\circ 1-1\circ\int_{\Omega}f\left(B_{\tau}\right)d\mu\left(\tau\right)\\ \geq\int_{\Omega}A_{\tau}d\mu\left(\tau\right)\circ\int_{\Omega}f'\left(B_{\tau}\right)d\mu\left(\tau\right)-1\circ\int_{\Omega}f'\left(B_{\tau}\right)B_{\tau}d\mu\left(\tau\right).\end{aligned}\tag{2.8}$$

Proof. From Lemma 1 we have

$$\begin{aligned}\left(f'\left(A_{\tau}\right)A_{\tau}\right)\otimes 1-f'\left(A_{\tau}\right)\otimes B_{\gamma}\geq f\left(A_{\tau}\right)\otimes 1-1\otimes f\left(B_{\gamma}\right)\\ \geq A_{\tau}\otimes f'\left(B_{\gamma}\right)-1\otimes\left(f'\left(B_{\gamma}\right)B_{\gamma}\right).\end{aligned}\tag{2.9}$$

for all τ, $\gamma\in\Omega$.

If we take the integral $\int_{\Omega}$ over $d\mu\left(\tau\right)$ in (2.9), then we get

$$\begin{aligned}\int_{\Omega}\left[\left(f'\left(A_{\tau}\right)A_{\tau}\right)\otimes 1-f'\left(A_{\tau}\right)\otimes B_{\gamma}\right]d\mu\left(\tau\right)\\ \geq\int_{\Omega}\left[f\left(A_{\tau}\right)\otimes 1-1\otimes f\left(B_{\gamma}\right)\right]d\mu\left(\tau\right)\\ \geq\int_{\Omega}\left[A_{\tau}\otimes f'\left(B_{\gamma}\right)-1\otimes\left(f'\left(B_{\gamma}\right)B_{\gamma}\right)\right]d\mu\left(\tau\right)\end{aligned}\tag{2.10}$$

for all $\gamma\in\Omega$.

By using the properties of integral and tensorial product, we derive that

$$\begin{aligned}\int_{\Omega}\left[\left(f'\left(A_{\tau}\right)A_{\tau}\right)\otimes 1-f'\left(A_{\tau}\right)\otimes B_{\gamma}\right]d\mu\left(\tau\right)\\ =\int_{\Omega}\left(f'\left(A_{\tau}\right)A_{\tau}\right)d\mu\left(\tau\right)\otimes 1-\int_{\Omega}f'\left(A_{\tau}\right)d\mu\left(\tau\right)\otimes B_{\gamma},\end{aligned}$$

$$\begin{aligned}\int_{\Omega}\left[f\left(A_{\tau}\right)\otimes 1-1\otimes f\left(B_{\gamma}\right)\right]d\mu\left(\tau\right)\\ =\int_{\Omega}f\left(A_{\tau}\right)d\mu\left(\tau\right)\otimes 1-1\otimes f\left(B_{\gamma}\right)\end{aligned}$$

and

$$\begin{aligned}\int_{\Omega}\left[A_{\tau}\otimes f'\left(B_{\gamma}\right)-1\otimes\left(f'\left(B_{\gamma}\right)B_{\gamma}\right)\right]d\mu\left(\tau\right)\\ =\int_{\Omega}A_{\tau}d\mu\left(\tau\right)\otimes f'\left(B_{\gamma}\right)-1\otimes\left(f'\left(B_{\gamma}\right)B_{\gamma}\right).\end{aligned}$$

By utilizing (2.10) we derive

$$\begin{aligned}\int_{\Omega}\left(f'\left(A_{\tau}\right)A_{\tau}\right)d\mu\left(\tau\right)\otimes 1-\int_{\Omega}f'\left(A_{\tau}\right)d\mu\left(\tau\right)\otimes B_{\gamma}\\ \geq\int_{\Omega}f\left(A_{\tau}\right)d\mu\left(\tau\right)\otimes 1-1\otimes f\left(B_{\gamma}\right)\\ \geq\int_{\Omega}A_{\tau}d\mu\left(\tau\right)\otimes f'\left(B_{\gamma}\right)-1\otimes\left(f'\left(B_{\gamma}\right)B_{\gamma}\right)\end{aligned}\tag{2.11}$$

for all $\gamma\in\Omega$.

If we take the integral $\int_\Omega$ over $d\mu(\gamma)$ in (2.11) and use the properties of the integral and tensorial product, we derive (2.7).

If we multiply the inequality (2.7) to the left with $\mathcal{U}^*$ and at the right with $\mathcal{U}$, use the properties of the integral, the we also get the inequality (2.8). □

Corollary 2. *Assume that f is continuous differentiable convex on the open interval I. Let $(A_\tau)_{\tau\in\Omega}$ be a continuous field of positive operators in $B(H)$ such that $\operatorname{Sp}(A_\tau) \subset I$ for each $\tau \in \Omega$ and B and operator such that $\operatorname{Sp}(B) \subset I$. Then we have*

$$
\begin{aligned}
(2.12)\qquad & \int_\Omega (f'(A_\tau) A_\tau) d\mu(\tau) \otimes 1 - \int_\Omega f'(A_\tau) d\mu(\tau) \otimes B \\
& \geq \int_\Omega f(A_\tau) d\mu(\tau) \otimes 1 - 1 \otimes f(B) \\
& \geq \int_\Omega A_\tau d\mu(\tau) \otimes f'(B) - 1 \otimes (f'(B) B)
\end{aligned}
$$

and the Hadamard product inequality

$$
\begin{aligned}
(2.13)\qquad & \int_\Omega (f'(A_\tau) A_\tau) d\mu(\tau) \circ 1 - \int_\Omega f'(A_\tau) d\mu(\tau) \circ B \\
& \geq \int_\Omega f(A_\tau) d\mu(\tau) \circ 1 - 1 \circ f(B) \\
& \geq \int_\Omega A_\tau d\mu(\tau) \circ f'(B) - 1 \circ (f'(B) B).
\end{aligned}
$$

The proof follows by Theorem 1 for $B_\tau = B$ for $\tau \in \Omega$.

We observe that

$$\int_\Omega A_\tau d\mu(\tau) \otimes f'(B) = \left(\int_\Omega A_\tau d\mu(\tau) \otimes 1\right)(1 \otimes f'(B))$$

and

$$1 \otimes (f'(B) B) = 1 \otimes (B f'(B)) = (1 \otimes B)(1 \otimes f'(B)),$$

therefore

$$
\begin{aligned}
& \int_\Omega A_\tau d\mu(\tau) \otimes f'(B) - 1 \otimes (f'(B) B) \\
& = \left(\int_\Omega A_\tau d\mu(\tau) \otimes 1\right)(1 \otimes f'(B)) - (1 \otimes B)(1 \otimes f'(B)) \\
& = \left(\int_\Omega A_\tau d\mu(\tau) \otimes 1 - (1 \otimes B)\right)(1 \otimes f'(B))
\end{aligned}
$$

and from (2.12) we get

$$
\begin{aligned}
(2.14)\qquad & \int_\Omega f(A_\tau) d\mu(\tau) \otimes 1 - 1 \otimes f(B) \\
& \geq \left(\int_\Omega A_\tau d\mu(\tau) \otimes 1 - (1 \otimes B)\right)(1 \otimes f'(B)).
\end{aligned}
$$

Remark 1. *With the assumptions of Corollary 2 and if we take $B = \int_\Omega A_\tau d\mu(\tau)$, for which have that $\operatorname{Sp}(B) \subset I$, then we have the following Jensen's type tensorial* *inequalities*

$$
\begin{aligned}
(2.15)\qquad & \int_{\Omega}\left(f'\left(A_{\tau}\right)A_{\tau}\right)d\mu\left(\tau\right)\otimes 1-\int_{\Omega}f'\left(A_{\tau}\right)d\mu\left(\tau\right)\otimes\int_{\Omega}A_{\tau}d\mu\left(\tau\right)\\
&\geq\int_{\Omega}f\left(A_{\tau}\right)d\mu\left(\tau\right)\otimes 1-1\otimes f\left(\int_{\Omega}A_{\tau}d\mu\left(\tau\right)\right)\\
&\geq\left(\int_{\Omega}A_{\tau}d\mu\left(\tau\right)\otimes 1-\left(1\otimes\int_{\Omega}A_{\tau}d\mu\left(\tau\right)\right)\right)\left(1\otimes f'\left(\int_{\Omega}A_{\tau}d\mu\left(\tau\right)\right)\right)
\end{aligned}
$$

and the Hadamard product inequalities

$$
\begin{aligned}
(2.16)\qquad & \int_{\Omega}\left(f'\left(A_{\tau}\right)A_{\tau}\right)d\mu\left(\tau\right)\circ 1-\int_{\Omega}f'\left(A_{\tau}\right)d\mu\left(\tau\right)\circ\int_{\Omega}A_{\tau}d\mu\left(\tau\right)\\
&\geq\int_{\Omega}f\left(A_{\tau}\right)d\mu\left(\tau\right)\circ 1-1\circ f\left(\int_{\Omega}A_{\tau}d\mu\left(\tau\right)\right)\\
&\geq\int_{\Omega}A_{\tau}d\mu\left(\tau\right)\circ f'\left(\int_{\Omega}A_{\tau}d\mu\left(\tau\right)\right)\\
&-1\circ\left(f'\left(\int_{\Omega}A_{\tau}d\mu\left(\tau\right)\right)\int_{\Omega}A_{\tau}d\mu\left(\tau\right)\right).
\end{aligned}
$$

3. Some Examples

Assume that A, B are such that $\operatorname{Sp}(A), \operatorname{Sp}(B)\subset I$, then $\operatorname{Sp}((1-t)A+tB)\subset I$ for all $t\in[0,1]$. By taking $A_{\tau}=(1-t)A+tB$ in (2.15) and (2.16) we get

$$
\begin{aligned}
(3.1)\qquad & \int_0^1 f'\left(\left(1-t\right)A+tB\right)\left(\left(1-t\right)A+tB\right)dt\otimes 1\\
&-\int_0^1 f'\left(\left(1-t\right)A+tB\right)dt\otimes\frac{A+B}{2}\\
&\geq\int_0^1 f\left(\left(1-t\right)A+tB\right)dt\otimes 1-1\otimes f\left(\frac{A+B}{2}\right)\\
&\geq\left(\frac{A+B}{2}\otimes 1-1\otimes\frac{A+B}{2}\right)\left(1\otimes f'\left(\frac{A+B}{2}\right)\right)
\end{aligned}
$$

and the Hadamard product inequalities

$$
\begin{aligned}
(3.2)\qquad & \int_0^1 f'\left(\left(1-t\right)A+tB\right)\left(\left(1-t\right)A+tB\right)dt\circ 1\\
&-\int_0^1 f'\left(\left(1-t\right)A+tB\right)dt\circ\frac{A+B}{2}\\
&\geq\int_0^1 f\left(\left(1-t\right)A+tB\right)dt\circ 1-1\circ f\left(\frac{A+B}{2}\right)\\
&\geq\frac{A+B}{2}\circ f'\left(\frac{A+B}{2}\right)-1\circ\left(f'\left(\frac{A+B}{2}\right)\frac{A+B}{2}\right).
\end{aligned}
$$

For $f(x) = \exp x$, $x \in \mathbb{R}$ and from (3.1) and (3.2) we derive the exponential inequalities

$$
\begin{aligned}
(3.3) \qquad & \int_0^1 \exp((1-t)A + tB)((1-t)A + tB)\,dt \otimes 1 \\
& - \int_0^1 \exp((1-t)A + tB)\,dt \otimes \frac{A+B}{2} \\
& \geq \int_0^1 \exp((1-t)A + tB)\,dt \otimes 1 - 1 \otimes f\left(\frac{A+B}{2}\right) \\
& \geq \left(\frac{A+B}{2} \otimes 1 - 1 \otimes \frac{A+B}{2}\right)\left(1 \otimes \exp\left(\frac{A+B}{2}\right)\right)
\end{aligned}
$$

and the Hadamard product inequalities

$$
\begin{aligned}
(3.4) \qquad & \int_0^1 \exp((1-t)A + tB)((1-t)A + tB)\,dt \circ 1 \\
& - \int_0^1 \exp((1-t)A + tB)\,dt \circ \frac{A+B}{2} \\
& \geq \int_0^1 \exp((1-t)A + tB)\,dt \circ 1 - 1 \circ f\left(\frac{A+B}{2}\right) \\
& \geq \frac{A+B}{2} \circ \exp\left(\frac{A+B}{2}\right) - 1 \circ \left(\exp\left(\frac{A+B}{2}\right)\frac{A+B}{2}\right).
\end{aligned}
$$

It is known that if A and B are commuting, i.e. $AB = BA$, then the exponential function satisfies the property

$$\exp(A)\exp(B) = \exp(B)\exp(A) = \exp(A+B).$$

Also, if A is invertible and $a, b \in \mathbb{R}$ with $a < b$ then

$$\int_a^b \exp(tA)\,dt = A^{-1}\left[\exp(bA) - \exp(aA)\right].$$

Moreover, if A and B are commuting and $B - A$ is invertible, then

$$
\begin{aligned}
\int_0^1 \exp((1-s)A + sB)\,ds &= \int_0^1 \exp(s(B-A))\exp(A)\,ds \\
&= \left(\int_0^1 \exp(s(B-A))\,ds\right)\exp(A) \\
&= (B-A)^{-1}\left[\exp(B-A) - I\right]\exp(A) \\
&= (B-A)^{-1}\left[\exp(B) - \exp(A)\right].
\end{aligned}
$$

So, if A and B are commuting and $B-A$ is invertible, then by (3.3) and (3.4) we get

$$\begin{aligned}(3.5)\qquad &\int_0^1 \exp\left(\left(1-t\right)A+tB\right)\left(\left(1-t\right)A+tB\right)dt\otimes 1\\ &-\left(B-A\right)^{-1}\left[\exp\left(B\right)-\exp\left(A\right)\right]\otimes\frac{A+B}{2}\\ &\geq\left(B-A\right)^{-1}\left[\exp\left(B\right)-\exp\left(A\right)\right]\otimes 1-1\otimes\exp\left(\frac{A+B}{2}\right)\\ &\geq\left(\frac{A+B}{2}\otimes 1-1\otimes\frac{A+B}{2}\right)\left(1\otimes\exp\left(\frac{A+B}{2}\right)\right)\end{aligned}$$

and the Hadamard product inequalities

$$\begin{aligned}(3.6)\qquad &\int_0^1 \exp\left(\left(1-t\right)A+tB\right)\left(\left(1-t\right)A+tB\right)dt\circ 1\\ &-\left(B-A\right)^{-1}\left[\exp\left(B\right)-\exp\left(A\right)\right]\circ\frac{A+B}{2}\\ &\geq\left(B-A\right)^{-1}\left[\exp\left(B\right)-\exp\left(A\right)\right]\circ 1-1\circ\exp\left(\frac{A+B}{2}\right)\\ &\geq\frac{A+B}{2}\circ\exp\left(\frac{A+B}{2}\right)-1\circ\left(\exp\left(\frac{A+B}{2}\right)\frac{A+B}{2}\right).\end{aligned}$$

References

[1] T. Ando, Concavity of certain maps on positive definite matrices and applications to Hadamard products, *Lin. Alg. & Appl.* **26** (1979), 203-241.

[2] H. Araki and F. Hansen, Jensen's operator inequality for functions of several variables, *Proc. Amer. Math. Soc.* **128** (2000), No. 7, 2075-2084.

[3] J. S. Aujila and H. L. Vasudeva, Inequalities involving Hadamard product and operator means, *Math. Japon.* **42** (1995), 265-272.

[4] A. Korányi. On some classes of analytic functions of several variables. *Trans. Amer. Math. Soc.,* **101** (1961), 520–554.

[5] J. I. Fujii, The Marcus-Khan theorem for Hilbert space operators. *Math. Jpn.* **41** (1995), 531-535

[6] T. Furuta, J. Mićić Hot, J. Pečarić and Y. Seo, *Mond-Pečarić Method in Operator Inequalities. Inequalities for Bounded Selfadjoint Operators on a Hilbert Space*, Element, Zagreb, 2005.

[7] K. Kitamura and Y. Seo, Operator inequalities on Hadamard product associated with Kadison's Schwarz inequalities, *Scient. Math.* **1** (1998), No. 2, 237-241.

[8] S. Wada, On some refinement of the Cauchy-Schwarz Inequality, *Lin. Alg. & Appl.* **420** (2007), 433-440.

[1]Mathematics, College of Engineering & Science, Victoria University, PO Box 14428, Melbourne City, MC 8001, Australia.

E-mail address: `sever.dragomir@vu.edu.au`

URL: `http://rgmia.org/dragomir`

[2]DST-NRF Centre of Excellence in the Mathematical, and Statistical Sciences, School of Computer Science, & Applied Mathematics, University of the Witwatersrand,, Private Bag 3, Johannesburg 2050, South Africa