q-Deformed hyperbolic tangent based Banach space valued ordinary and fractional neural network approximations

George A. Anastassiou
Department of Mathematical Sciences, University of Memphis,
Memphis, TN 38152, U.S.A.
ganastss@memphis.edu

Abstract

Here we research the univariate quantitative approximation, ordinary and fractional, of Banach space valued continuous functions on a compact interval or all the real line by quasi-interpolation Banach space valued neural network operators. These approximations are derived by establishing Jackson type inequalities involving the modulus of continuity of the engaged function or its Banach space valued high order derivative of fractional derivatives. Our operators are defined by using a density function generated by a q-deformed hyperbolic tangent function, which is a sigmoid function. The approximations are pointwise and of the uniform norm. The related Banach space valued feed-forward neural networks are with one hidden layer.

2020 Mathematics Subject Classification: 26A33, 41A17, 41A25, 41A30, 46B25.

Keywords and phrases: q-deformed hyperbolic tangent function, Banach space valued neural network approximation, Banach space valued quasi-interpolation operator, modulus of continuity, Banach space valued Caputo fractional derivative, Banach space valued fractional approximation.

1 Introduction

The author in [1] and [2], see Chapters 2-5, was the first to establish neural network approximation to continuous functions with rates by very specifically defined neural network operators of Cardaliagnet-Euvrard and "Squashing" types, by employing the modulus of continuity of the engaged function or its high order derivative, and producing very tight Jackson type inequalities. He treats there

RGMIA Res. Rep. Coll. **25** (2022), Art. 129, 24 pp. Received 24/11/22

both the univariate and multivariate cases. The defining these operators "bell-shaped" and "squashing" functions are assumed to be of compact suport. Also in [2] he gives the Nth order asymptotic expansion for the error of weak approximation of these two operators to a special natural class of smooth functions, see Chapters 4-5 there.

Again the author inspired by [16], continued his studies on neural networks approximation by introducing and using the proper quasi-interpolation operators of sigmoidal and hyperbolic tangent type which resulted into [3]-[7], by treating both the univariate and multivariate cases. He did also the corresponding fractional cases [8], [9], [13].

The author here performs q-deformed hyperbolic tangent function activated neural network approximations to continuous functions over compact intervals of the real line or over the whole $\mathbb{R}$ with valued to an arbitrary Banach space $(X, \|\cdot\|)$. Finally he treats completely the related X-valued fractional approximation. All convergences here are with rates expressed via the modulus of continuity of the involved function or its X-valued high order derivative, or X-valued fractional derivatives and given by very tight Jackson type inequalities.

Our compact intervals are not necessarily symmetric to the origin. Some of our upper bounds to error quantity are very flexible and general. In preparation to prove our results we establish important properties of the basic density function defining our operators which is induced by a q-deformed hyperbolic tangent function, which is a sigmoid function.

Feed-forward X-valued neural networks (FNNs) with one hidden layer, the only type of networks we deal with in this article, are mathematically expressed as

$$N_n(x) = \sum_{=0}^{n} c_j \sigma\left(\langle a_j \cdot x \rangle + b_j\right), \quad x \in \mathbb{R}^s, \; s \in \mathbb{N},$$

where for $0 \leq j \leq n$, $b_j \in \mathbb{R}$ are the thresholds, $a_j \in \mathbb{R}^s$ are the connection weights, $c_j \in X$ are the coefficients, $\langle a_j \cdot x \rangle$ is the inner product of a_j and x, and σ is the activation function of the network. About neural networks in general read [18], [20], [22].

2 About q-deformed $\tanh_q x$

We found $\tanh_q x$ in [17].

We will study $\tanh_q x$ and prove that it is a sigmoid function and we will give several of its properties related to the approximation by neural network operators. So the deformed $\tanh_q x$ is defined as follows:

$$h_q(x) := \tanh_q x := \frac{e^x - qe^{-x}}{e^x + qe^{-x}}, \tag{1}$$

$x \in \mathbb{R}$, where $q \in (0, +\infty) - \{1\}$.

We have that

$$h_q(0) = \frac{1-q}{1+q} \neq 0, \quad q \neq 1. \tag{2}$$

We notice that

$$h_q(-x) = \frac{e^{-x} - qe^{x}}{e^{-x} + qe^{x}} = \frac{\frac{1}{q}e^{-x} - e^{x}}{\frac{1}{q}e^{-x} + e^{x}} = -\left(\frac{e^{x} - \frac{1}{q}e^{-x}}{e^{x} + \frac{1}{q}e^{-x}}\right) = -h_{\frac{1}{q}}(x). \tag{3}$$

That is

$$h_q(-x) = -h_{\frac{1}{q}}(x), \quad \forall\, x \in \mathbb{R}, \tag{4}$$

and $h_{\frac{1}{q}}(x) = -h_q(-x)$, hence

$$h'_{\frac{1}{q}}(x) = h'_q(-x). \tag{5}$$

It is

$$h_q(x) = \frac{e^{2x} - q}{e^{2x} + q} = \frac{1 - \frac{q}{e^{2x}}}{1 + \frac{q}{e^{2x}}} \underset{(x \to +\infty)}{\rightarrow} 1,$$

i.e.

$$h_q(+\infty) = 1. \tag{6}$$

Furthermore

$$h_q(x) = \frac{e^{2x} - q}{e^{2x} + q} \underset{(x \to -\infty)}{\rightarrow} -\frac{q}{q} = -1,$$

i.e.

$$h_q(-\infty) = -1. \tag{7}$$

We find that

$$h'_q(x) = \frac{4qe^{2x}}{\left(e^{2x} + q\right)^2} > 0, \tag{8}$$

therefore h_q is striclty increasing.

Next we obtain ($x \in \mathbb{R}$)

$$h''_q(x) = 8qe^{2x}\left(\frac{q - e^{2x}}{\left(e^{2x} + q\right)^3}\right) \in C(\mathbb{R}). \tag{9}$$

We observe that

$$q - e^{2x} \gtreqless 0 \Leftrightarrow q \gtreqless e^{2x} \Leftrightarrow \ln q \gtreqless 2x \Leftrightarrow x \lesseqgtr \frac{\ln q}{2}.$$

So, in case of $x < \frac{\ln q}{2}$, we have that h_q is strictly concave up, with $h''_q\left(\frac{\ln q}{2}\right) = 0$.

And in case of $x > \frac{\ln q}{2}$, we have that h_q is strictly concave down.

So h_q is a shifted sigmoidal with $h_q(0) = \frac{1-q}{1+q} \neq 0$, and $h_q(-x) = -h_{q^{-1}}(x)$ (a semi-odd function), see also [14].

By $1 > -1$, $x+1 > x-1$, we consider the activation function

$$\psi_q(x) := \frac{1}{4}\left(h_q(x+1) - h_q(x-1)\right) > 0, \tag{10}$$

$\forall x \in \mathbb{R}$, $q > 0$, $q \neq 1$. Notice that $\psi_q(\pm\infty) = 0$, so the x-axis is horizontal asymptote.

We have that

$$\psi_q(-x) = \frac{1}{4}\left[h_q(-x+1) - h_q(-x-1)\right] =$$

$$\frac{1}{4}\left[\left(\frac{e^{-x+1} - qe^{x-1}}{e^{-x+1} + qe^{x-1}}\right) - \left(\frac{e^{-x-1} - qe^{x+1}}{e^{-x-1} + qe^{x+1}}\right)\right] =$$

$$\frac{1}{4}\left[-\left(\frac{qe^{x-1} - e^{-x+1}}{qe^{x-1} + e^{-x+1}}\right) + \left(\frac{qe^{x+1} - e^{-x-1}}{qe^{x+1} + e^{-x-1}}\right)\right] =$$

$$\frac{1}{4}\left[\left(\frac{qe^{x+1} - e^{-x-1}}{qe^{x+1} + e^{-x-1}}\right) - \left(\frac{qe^{x-1} - e^{-x+1}}{qe^{x-1} + e^{-x+1}}\right)\right] =$$

$$\frac{1}{4}\left[\left(\frac{e^{x+1} - \frac{1}{q}e^{-x-1}}{e^{x+1} + \frac{1}{q}e^{-x-1}}\right) - \left(\frac{e^{x-1} - \frac{1}{q}e^{-x+1}}{q^{x-1} + \frac{1}{q}e^{-x+1}}\right)\right] =$$

$$\frac{1}{4}\left[h_{q^{-1}}(x+1) - h_{q^{-1}}(x-1)\right] = \psi_{\frac{1}{q}}(x), \quad \forall\, x \in \mathbb{R}. \tag{11}$$

Thus

$$\psi_q(-x) = \psi_{q^{-1}}(x), \;\; \forall\, x \in \mathbb{R}, \; \forall\, q > 0, \, q \neq 1, \; \text{a deformed symmetry.} \tag{12}$$

Next we have that

$$\psi_q'(x) = \frac{1}{4}\left[h_q'(x+1) - h_q'(x-1)\right], \quad \forall\, x \in \mathbb{R}. \tag{13}$$

Let $x < \frac{\ln q}{2} - 1$, then $x - 1 < x + 1 < \frac{\ln q}{2}$ and $h_q'(x+1) > h_q'(x-1)$ (by h_q being strictly concave up for $x < \frac{\ln q}{2}$), that is $\psi_q'(x) > 0$. Hence ψ_q is striclty increasing over $\left(-\infty, \frac{\ln q}{2} - 1\right)$.

Let now $x - 1 > \frac{\ln q}{2}$, then $x + 1 > x - 1 > \frac{\ln q}{2}$, and $h_q'(x+1) < h_q'(x-1)$, that is $\psi_q'(x) < 0$.

Therefore ψ_q is striclty decreasing over $\left(\frac{\ln q}{2} + 1, +\infty\right)$.

Next, let $\frac{\ln q}{2} - 1 \leq x \leq \frac{\ln q}{2} + 1$. We have that

$$\psi_q''(x) = \frac{1}{4}\left[h_q''(x+1) - h_q''(x-1)\right] =$$

$$2q\left[e^{2(x+1)}\left(\frac{q-e^{2(x+1)}}{\left(e^{2(x+1)}+q\right)^{3}}\right)-e^{2(x-1)}\left(\frac{q-e^{2(x-1)}}{\left(e^{2(x-1)}+q\right)^{3}}\right)\right]. \tag{14}$$

By $\frac{\ln q}{2} \leq x \Leftrightarrow \frac{\ln q}{2} \leq x+1 \Leftrightarrow \ln q \leq 2(x+1) \Leftrightarrow q \leq e^{2(x+1)} \Leftrightarrow q-e^{2(x+1)} \leq 0$.

By $x \leq \frac{\ln q}{2}+1 \Leftrightarrow x-1 \leq \frac{\ln q}{2} \Leftrightarrow 2(x-1) \leq \ln q \Leftrightarrow e^{2(x-1)} \leq q \Leftrightarrow q-e^{2(x-1)} \geq 0$.

Clearly by (14) we get that $\psi_q''(x) \leq 0$, for $x \in \left[\frac{\ln q}{2}-1, \frac{\ln q}{2}+1\right]$.

More precisely ψ_q is concave down over $\left[\frac{\ln q}{2}-1, \frac{\ln q}{2}+1\right]$, and strictly concave down over $\left(\frac{\ln q}{2}-1, \frac{\ln q}{2}+1\right)$.

Consequently ψ_q has a bell-type shape over $\mathbb{R}$.

Of course it holds $\psi_q''\left(\frac{\ln q}{2}\right) < 0$.

At $x = \frac{\ln q}{2}$, we have

$$\psi_q'(x) = \frac{1}{4}\left[h_q'(x+1) - h_q'(x-1)\right] =$$

$$q\left[\frac{e^{2(x+1)}}{\left(e^{2(x+1)}+q\right)^{2}} - \frac{e^{2(x-1)}}{\left(e^{2(x-1)}+q\right)^{2}}\right]. \tag{15}$$

Thus

$$\psi_q'\left(\frac{\ln q}{2}\right) = q\left[\frac{e^{2\left(\frac{\ln q}{2}+1\right)}}{\left(e^{2\left(\frac{\ln q}{2}+1\right)}+q\right)^{2}} - \frac{e^{2\left(\frac{\ln q}{2}-1\right)}}{\left(e^{2\left(\frac{\ln q}{2}-1\right)}+q\right)^{2}}\right] =$$

$$q^2\left[\frac{e^2}{\left(qe^2+q\right)^2} - \frac{e^{-2}}{\left(qe^{-2}+q\right)^2}\right] = \left[\frac{e^2}{\left(e^2+1\right)^2} - \frac{e^{-2}}{\left(e^{-2}+1\right)^2}\right] \tag{16}$$

$$= \frac{e^2\left(e^{-2}+1\right)^2 - e^{-2}\left(e^2+1\right)^2}{\left(e^2+1\right)^2\left(e^{-2}+1\right)^2} = 0.$$

Therefore at $x = \frac{\ln q}{2}$, ψ_q achieves a maximum, which is

$$\psi(x) = \frac{1}{4}\left[h_q(x+1) - h_q(x-1)\right] =$$

$$\frac{1}{4}\left[\left(\frac{e^{x+1}-qe^{-x-1}}{e^{x+1}+qe^{-x-1}}\right) - \left(\frac{e^{x-1}-qe^{-x+1}}{q^{x-1}+qe^{-x+1}}\right)\right] =$$

$$\frac{1}{4}\left[\left(\frac{e^{\frac{\ln q}{2}+1}-qe^{-\frac{\ln q}{2}-1}}{e^{\frac{\ln q}{2}+1}+qe^{-\frac{\ln q}{2}-1}}\right) - \left(\frac{e^{\frac{\ln q}{2}-1}-qe^{-\frac{\ln q}{2}+1}}{q^{\frac{\ln q}{2}-1}+qe^{-\frac{\ln q}{2}+1}}\right)\right] = \tag{17}$$

$$\frac{1}{4}\left[\left(\frac{\sqrt{q}e-qq^{-\frac{1}{2}}e^{-1}}{\sqrt{q}e+qq^{-\frac{1}{2}}e^{-1}}\right) - \left(\frac{\sqrt{q}e^{-1}-qq^{-\frac{1}{2}}e}{\sqrt{q}e^{-1}+qq^{-\frac{1}{2}}e}\right)\right] =$$

$$\frac{1}{4}\left[\left(\frac{e-e^{-1}}{e+e^{-1}}\right)-\left(\frac{e^{-1}-e}{e^{-1}+e}\right)\right]=\frac{\left(e-e^{-1}\right)}{2\left(e+e^{-1}\right)}.$$

Conclusion: the maximum value of ψ_q is

$$\psi_q\left(\frac{\ln q}{2}\right)=\frac{\left(e-e^{-1}\right)}{2\left(e+e^{-1}\right)}=\frac{\tanh 1}{2}. \tag{18}$$

We give

Theorem 1 *We have that*

$$\sum_{i=-\infty}^{\infty}\psi_q\left(x-i\right)=1,\ \ \forall\ x\in\mathbb{R},\ \forall\ q>0,\ q\neq 1. \tag{19}$$

Proof. We notice that

$$\sum_{i=-\infty}^{\infty}\left(h_q\left(x-i\right)-h_q\left(x-1-i\right)\right)=$$

$$\sum_{i=0}^{\infty}\left(h_q\left(x-i\right)-h_q\left(x-1-i\right)\right)+\sum_{i=-\infty}^{-1}\left(h_q\left(x-i\right)-h_q\left(x-1-i\right)\right).$$

Furthermore ($\rho\in\mathbb{Z}^+$)

$$\sum_{i=0}^{\infty}\left(h_q\left(x-i\right)-h_q\left(x-1-i\right)\right)= \tag{20}$$

$$\lim_{\rho\to\infty}\sum_{i=0}^{\rho}\left(h_q\left(x-i\right)-h_q\left(x-1-i\right)\right)\quad\text{(telescoping sum)}$$
$$=\lim_{\rho\to\infty}\left(h_q\left(x\right)-h_q\left(x-\left(\rho+1\right)\right)\right)=1+h_q\left(x\right).$$

Similarly,

$$\sum_{i=-\infty}^{-1}\left(h_q\left(x-i\right)-h_q\left(x-1-i\right)\right)=\lim_{\rho\to\infty}\sum_{i=-\rho}^{-1}\left(h_q\left(x-i\right)-h_q\left(x-1-i\right)\right)=$$

$$\lim_{\rho\to\infty}\left(h_q\left(x+\rho\right)-h_q\left(x\right)\right)=1-h_q\left(x\right). \tag{21}$$

By adding the last two limits we derive

$$\sum_{i=-\infty}^{\infty}\left(h_q\left(x-i\right)-h_q\left(x-1-i\right)\right)=2,\ \ \forall\ x\in\mathbb{R}. \tag{22}$$

Consequently we get

$$\sum_{i=-\infty}^{\infty}\left(h_q\left(x+1-i\right)-h_q\left(x-i\right)\right)=2,\ \ \forall\ x\in\mathbb{R}.$$

Therefore it holds

$$\sum_{i=-\infty}^{\infty} \left(h_q\left(x+1-i\right) - h_q\left(x-1-i\right)\right) = 4, \ \forall \ x \in \mathbb{R}, \tag{23}$$

proving the claim. ■

Thus

$$\sum_{i=-\infty}^{\infty} \psi_q\left(nx-i\right) = 1, \forall \ n \in \mathbb{N}, \ \forall \ x \in \mathbb{R}. \tag{24}$$

Similarly, it holds

$$\sum_{i=-\infty}^{\infty} \psi_{\frac{1}{q}}\left(x-i\right) = 1, \forall \ x \in \mathbb{R}. \tag{25}$$

But $\psi_{\frac{1}{q}}\left(x-i\right) \overset{(12)}{=} \psi_q\left(i-x\right), \forall \ x \in \mathbb{R}$.

Hence

$$\sum_{i=-\infty}^{\infty} \psi_q\left(i-x\right) = 1, \forall \ x \in \mathbb{R}, \tag{26}$$

and

$$\sum_{i=-\infty}^{\infty} \psi_q\left(i+x\right) = 1, \forall \ x \in \mathbb{R}. \tag{27}$$

It follows

Theorem 2 *It holds*

$$\int_{-\infty}^{\infty} \psi_q\left(x\right) dx = 1, \quad q > 0, \ q \neq 1. \tag{28}$$

Proof. We observe that

$$\int_{-\infty}^{\infty} \psi_q\left(x\right) dx = \sum_{j=-\infty}^{\infty} \int_{j}^{j+1} \psi_q\left(x\right) dx = \sum_{j=-\infty}^{\infty} \int_{0}^{1} \psi_q\left(x+j\right) dx = \tag{29}$$

$$\int_{0}^{1} \left(\sum_{j=-\infty}^{\infty} \psi_q\left(x+j\right) dx \right) = \int_{0}^{1} 1 dx = 1.$$

So that ψ_q is a density function on $\mathbb{R}$; $q > 0$, $q \neq 1$. ■

We need the following result

Theorem 3 *Let $0 < \alpha < 1$, and $n \in \mathbb{N}$ with $n^{1-\alpha} > 2$; $q > 0$, $q \neq 1$. Then*

$$\sum_{\begin{cases} k = -\infty \\ : \left|nx-k\right| \geq n^{1-\alpha} \end{cases}}^{\infty} \psi_q\left(nx-k\right) < \max\left\{q, \frac{1}{q}\right\} e^{4} e^{-2n^{(1-\alpha)}} = Q e^{-2n^{(1-\alpha)}}; \tag{30}$$

$$Q := \max\left\{q, \frac{1}{q}\right\} e^4.$$

Proof. Let $x \geq 1$. That is $0 \leq x-1 < x+1$. Applying the mean value theorem we obtain

$$\psi_q(x) = \frac{1}{4}\left[h_q(x+1) - h_q(x-1)\right] = \frac{1}{4} \cdot 2 \cdot \frac{4qe^{2\xi}}{\left(e^{2\xi}+q\right)^2},$$

that is

$$\psi_q(x) = \frac{2qe^{2\xi}}{\left(e^{2\xi}+q\right)^2}, \tag{31}$$

for some $0 \leq x-1 < \xi < x+1$; $q > 0$, $q \neq 1$.

But $e^{2\xi} < e^{2\xi} + q$, and

$$\psi_q(x) < \frac{2q\left(e^{2\xi}+q\right)}{\left(e^{2\xi}+q\right)^2} = \frac{2q}{\left(e^{2\xi}+q\right)} < \frac{2q}{e^{2(x-1)}+q} < \frac{2q}{e^{2(x-1)}}, \quad x \geq 1. \tag{32}$$

That is

$$\psi_q(x) < \frac{2q}{e^{2(x-1)}}, \quad \forall\ x \geq 1,$$

or, better

$$\psi_q(x) < 2qe^2 e^{-2x}, \quad \forall\ x \geq 1. \tag{33}$$

Thus, we observe that

$$\sum_{\substack{k=-\infty \\ : |nx-k| \geq n^{1-\alpha}}}^{\infty} \psi_q(|nx-k|) <$$

$$2qe^2 \sum_{\substack{k=-\infty \\ : |nx-k| \geq n^{1-\alpha}}}^{\infty} e^{-2|nx-k|} \leq 2qe^2 \int_{n^{1-\alpha}-1}^{\infty} e^{-2x} dx =$$

$$qe^2 \int_{n^{1-\alpha}-1}^{\infty} e^{-2x} d(2x) \overset{(y=2x)}{=} qe^2 \int_{n^{1-\alpha}-1}^{\infty} e^{-y} dy = qe^2 \left\{-e^{-y}\big|_{n^{1-\alpha}-1}^{\infty}\right\} =$$

$$qe^2 \left\{-e^{-2x}\big|_{n^{1-\alpha}-1}^{\infty}\right\} = qe^2 \left\{e^{-2x}\big|_{\infty}^{n^{1-\alpha}-1}\right\} = qe^2 \left\{e^{-2\left(n^{1-\alpha}-1\right)} - e^{-2\cdot\infty}\right\} \tag{34}$$

$$= qe^2\left(e^{-2\left(n^{1-\alpha}-1\right)}\right) = qe^2 e^{-2n^{(1-\alpha)}} e^2 = qe^4 e^{-2n^{(1-\alpha)}}.$$

Therefore it holds

$$\sum_{\substack{k=-\infty \\ : |nx-k| \geq n^{1-\alpha}}}^{\infty} \psi_q(|nx-k|) < qe^4 e^{-2n^{(1-\alpha)}}, \quad \forall\ q > 0,\ q \neq 1. \tag{35}$$

If $(nx-k)>0$, then

$$\sum_{\begin{cases} k=-\infty \\ : |nx-k| \geq n^{1-\alpha} \end{cases}}^{\infty} \psi_q(nx-k) < qe^4 e^{-2n^{(1-\alpha)}}. \tag{36}$$

Similarly, it is valid (by (35))

$$\sum_{\begin{cases} k=-\infty \\ : |nx-k| \geq n^{1-\alpha} \end{cases}}^{\infty} \psi_{\frac{1}{q}}(|nx-k|) < \frac{1}{q} e^4 e^{-2n^{(1-\alpha)}}, \quad \forall\, q>0,\, q\neq 1. \tag{37}$$

Assume now that $nx-k\leq 0$, then

$$\sum_{\begin{cases} k=-\infty \\ : |nx-k| \geq n^{1-\alpha} \end{cases}}^{\infty} \psi_q(nx-k) \overset{(12)}{=} \sum_{\begin{cases} k=-\infty \\ : |nx-k| \geq n^{1-\alpha} \end{cases}}^{\infty} \psi_{\frac{1}{q}}(-(nx-k))$$

$$< \frac{1}{q} e^4 e^{-2n^{(1-\alpha)}}, \quad \forall\, q>0,\, q\neq 1. \tag{38}$$

Therefore, it holds (by (36), (38))

$$\sum_{\begin{cases} k=-\infty \\ : |nx-k| \geq n^{1-\alpha} \end{cases}}^{\infty} \psi_q(nx-k) < \max\left\{q, \frac{1}{q}\right\} e^4 e^{-2n^{(1-\alpha)}}, \quad \forall\, q>0,\, q\neq 1. \tag{39}$$

The claim is proved. ■

Let $\lceil\cdot\rceil$ the ceiling of the number, and $\lfloor\cdot\rfloor$ the integral part of the number.

Theorem 4 *Let $x\in[a,b]\subset\mathbb{R}$ and $n\in\mathbb{N}$ so that $\lceil na\rceil\leq\lfloor nb\rfloor$. For $q>0$, $q\neq 1$, we consider the number $\lambda_q>z_0>0$ with $\psi(z_0)=\psi_q(0)$ and $\lambda_q>1$. Then*

$$\frac{1}{\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}\psi_q(nx-k)} < max\left\{\frac{1}{\psi_q(\lambda_q)}, \frac{1}{\psi_{\frac{1}{q}}\left(\lambda_{\frac{1}{q}}\right)}\right\} =: \Phi(q). \tag{40}$$

Proof. By Theorem 1 we have

$$\sum_{i=-\infty}^{\infty}\psi_q(x-i)=1,\ \forall\, x\in\mathbb{R},\ \forall\, q>0,\ q\neq 1,$$

and by (26), we have that

$$\sum_{i=-\infty}^{\infty} \psi_q (i - x) = 1, \ \forall \ x \in \mathbb{R}, \ \forall \ q > 0, \ q \neq 1. \tag{41}$$

Therefore we get

$$\sum_{i=-\infty}^{\infty} \psi_q (|x - i|) = 1, \ \forall \ x \in \mathbb{R}, \ \forall \ q > 0, \ q \neq 1. \tag{42}$$

Hence

$$1 = \sum_{k=-\infty}^{\infty} \psi_q (|nx - k|) > \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \psi_q (|nx - k|) > \psi_q (|nx - k_0|), \tag{43}$$

$\forall \ k_0 \in [\lceil na \rceil, \lfloor nb \rfloor] \cap \mathbb{Z}$.

We can choose $k_0 \in [\lceil na \rceil, \lfloor nb \rfloor] \cap \mathbb{Z}$, such that $|nx - k_0| < 1$.

Notice that $|nx - k_0|$ could be $\lesseqqgtr \frac{\ln q}{2}$. If $0 \leq |nx - k_0| < \frac{\ln q}{2}$, by concavity of ψ_q over $\mathbb{R}$, we can choose $z \in [\frac{\ln q}{2}, +\infty)$ such that $\psi_q (|nx - k_0|) = \psi_q (z)$. If $|nx - k_0| \geq \frac{\ln q}{2}$ we just set $z := |nx - k_0|$. Next, we can choose large enough $\lambda_q > 1$, and such that $\lambda_q > z_0 > 0$ where $\psi (z_0) = \psi_q (0)$. Clearly, it is $z \leq z_0 < \lambda_q$.

Since ψ_q is decreaasing over $[\frac{\ln q}{2}, +\infty)$ we get that $\psi_q (|nx - k_0|) \geq \psi_q (\lambda_q)$.

Consequently,

$$\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \psi_q (|nx - k|) > \psi_q (\lambda_q),$$

and

$$\frac{1}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \psi_q (|nx - k|)} < \frac{1}{\psi_q (\lambda_q)}, \tag{44}$$

$\forall \ q > 0$, $q \neq 1$.

If $nx - k > 0$, by (44), we get

$$\frac{1}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \psi_q (nx - k)} < \frac{1}{\psi_q (\lambda_q)}, \ \ \forall \ q > 0, \ q \neq 1. \tag{45}$$

We have also that

$$\frac{1}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \psi_{\frac{1}{q}} (|nx - k|)} < \frac{1}{\psi_{\frac{1}{q}} \left(\lambda_{\frac{1}{q}}\right)}, \ \ \forall \ q > 0, \ q \neq 1. \tag{46}$$

Let now $nx-k\leq 0$, then

$$\frac{1}{\sum\limits_{k=\lceil na\rceil}^{\lfloor nb\rfloor}\psi_q(nx-k)}\overset{(12)}{=}\frac{1}{\sum\limits_{k=\lceil na\rceil}^{\lfloor nb\rfloor}\psi_{\frac{1}{q}}(-(nx-k))}\overset{(46)}{<}\frac{1}{\psi_{\frac{1}{q}}\left(\lambda_{\frac{1}{q}}\right)}, \tag{47}$$

$\forall\ q>0$, $q\neq 1$.

Consequently, it holds

$$\frac{1}{\sum\limits_{k=\lceil na\rceil}^{\lfloor nb\rfloor}\psi_q(nx-k)}<\max\left\{\frac{1}{\psi_q(\lambda_q)},\frac{1}{\psi_{\frac{1}{q}}\left(\lambda_{\frac{1}{q}}\right)}\right\}, \tag{48}$$

$\forall\ q>0$, $q\neq 1$.

The claim is proved. ■

We make

Remark 5 *(i) We also notice for $q>1$ that*

$$1-\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}\psi_q(nb-k)=\sum_{k=-\infty}^{\lceil na\rceil-1}\psi_q(nb-k)+\sum_{k=\lfloor nb\rfloor+1}^{\infty}\psi_q(nb-k)$$
$$>\psi_q(nb-\lfloor nb\rfloor-1) \tag{49}$$

(call $\varepsilon:=nb-\lfloor nb\rfloor$, $0\leq\varepsilon<1$)

$$=\psi_q(\varepsilon-1)=\psi_q(-(1-\varepsilon))=\psi_{\frac{1}{q}}(1-\varepsilon)$$

($0<\frac{1}{q}<1$ and $0<1-\varepsilon<1$)

($\psi_{\frac{1}{q}}$ is decreasing on $[0,+\infty)$).

$$\geq\psi_{\frac{1}{q}}(1)>0.$$

Therefore

$$\lim_{n\to\infty}\left(1-\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}\psi_q(nb-k)\right)>0,\quad q>1. \tag{50}$$

(ii) Let now $0<q<1$, then we work as in (i), and we have

$$1-\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}\psi_q(nb-k)>\psi_{\frac{1}{q}}(1-\varepsilon) \tag{51}$$

($\varepsilon:=nb-\lfloor nb\rfloor$, $0\leq\varepsilon<1$).

That is $\frac{1}{q}>1$ and choose $\lambda:0<1-\varepsilon<1<\lambda$, where $\lambda>\frac{\ln\frac{1}{q}}{2}=-\frac{\ln q}{2}$.

First assume that $1-\varepsilon \in [-\frac{\ln q}{2}, +\infty)$. *Hence*

$$\psi_{\frac{1}{q}}(1-\varepsilon) > \psi_{\frac{1}{q}}(\lambda) > 0, \tag{52}$$

by $\psi_{\frac{1}{q}}$ *being decreasing on* $[-\frac{\ln q}{2}, +\infty)$.

If $0 < 1-\varepsilon < -\frac{\ln q}{2}$, *then we use the concavity-bell shape of* ψ_q.

So, there exists $z_\varepsilon \in \left(-\frac{\ln q}{2}, +\infty\right)$ *such that* $\psi_{\frac{1}{q}}(1-\varepsilon) = \psi_{\frac{1}{q}}(z_\varepsilon)$. *We also consider* $z_0 \in \left(-\frac{\ln q}{2}, +\infty\right)$ *such that* $\psi_{\frac{1}{q}}(z_0) = \psi_{\frac{1}{q}}(0)$. *Clearly it holds* $-\frac{\ln q}{2} < z_\varepsilon \leq z_0$ *and we choose* $\lambda : z_0 < \lambda$. *Therefore, it holds* $\psi_{\frac{1}{q}}(1-\varepsilon) \geq \psi_{\frac{1}{q}}(0) \geq \psi_{\frac{1}{q}}(\lambda) > 0$, *by* $\psi_{\frac{1}{q}}$ *being decreasing on* $[-\frac{\ln q}{2}, +\infty)$.

Again it holds

$$\lim_{n\to\infty}\left(1-\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}\psi_q(nb-k)\right) > 0, \quad 0<q<1. \tag{53}$$

(iii) Similarly, $(q>0)$

$$\begin{aligned}1-\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}\psi_q(na-k) &= \sum_{k=-\infty}^{\lceil na\rceil-1}\psi_q(na-k)+\sum_{k=\lfloor nb\rfloor+1}^{\infty}\psi_q(na-k)\\ &> \psi_q(na-\lceil na\rceil+1)\\ &(\textit{call}\ \ \eta := \lceil na\rceil - na,\ 0\leq\eta<1)\\ &= \psi_q(1-\eta), \quad \textit{etc.}\end{aligned} \tag{54}$$

Acting as in (i), (ii) we derive that

$$\lim_{n\to+\infty}\left(1-\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}\psi_q(na-k)\right) > 0. \tag{55}$$

Conclusion: *We have that*

$$\lim_{n\to+\infty}\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}\psi_q(nx-k) \neq 1, \quad \textit{for at least some } x\in[a,b]. \tag{56}$$

Let $(X, \|\cdot\|)$ be a Banach space.

Definition 6 *Let* $f \in C([a,b], X)$ *and* $n \in \mathbb{N} : \lceil na\rceil \leq \lfloor nb\rfloor$. *We introduce and define the* X*-valued linear neural network operators*

$$H_n(f,x) := \frac{\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} f\left(\frac{k}{n}\right)\psi_q(nx-k)}{\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}\psi_q(nx-k)}, \quad x\in[a,b];\ q>0,\ q\neq 1. \tag{57}$$

For large enough n we always obtain $\lceil na \rceil \leq \lfloor nb \rfloor$. Also $a \leq \frac{k}{n} \leq b$, iff $\lceil na \rceil \leq k \leq \lfloor nb \rfloor$. The same H_n is used for real valued functions. We study here the pointwise and uniform convergence of $H_n(f,x)$ to $f(x)$ with rates.

For convenience, also we call

$$H_n^*(f,x) := \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} f\left(\frac{k}{n}\right) \psi_q(nx-k), \tag{58}$$

(the same H_n^* can be defined for real valued functions) that is

$$H_n(f,x) := \frac{H_n^*(f,x)}{\sum\limits_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \psi_q(nx-k)}. \tag{59}$$

So that

$$H_n(f,x) - f(x) = \frac{H_n^*(f,x)}{\sum\limits_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \psi_q(nx-k)} - f(x) = \tag{60}$$

$$\frac{H_n^*(f,x) - f(x)\left(\sum\limits_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \psi_q(nx-k)\right)}{\sum\limits_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \psi_q(nx-k)}.$$

Consequently, we derive that

$$\|H_n(f,x) - f(x)\| \leq \Phi(q) \left\| H_n^*(f,x) - f(x)\left(\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \psi_q(nx-k)\right) \right\| =$$

$$\Phi(q) \left\| \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \left(f\left(\frac{k}{n}\right) - f(x)\right) \psi_q(nx-k) \right\|, \tag{61}$$

where $\Phi(q)$ as in (40).

We will estimate the right hand side of the last quantity.

For that we need, for $f \in C([a,b], X)$ the first modulus of continuity

$$\omega_1(f,\delta) := \sup_{\substack{x,y \in [a,b] \\ |x-y| \leq \delta}} \|f(x) - f(y)\|, \quad \delta > 0. \tag{62}$$

Similarly, it is defined ω_1 for $f \in C_{uB}(\mathbb{R}, X)$ (uniformly continuous and bounded functions from $\mathbb{R}$ into X), for $f \in C_B(\mathbb{R}, X)$ (continuous and bounded X-valued), and for $f \in C_u(\mathbb{R}, X)$ (uniformly continuous).

The fact $f \in C([a,b], X)$ or $f \in C_u(\mathbb{R}, X)$, is equivalent to $\lim_{\delta \to 0} \omega_1(f, \delta) = 0$, see [11].

We make

Definition 7 *When $f \in C_{uB}(\mathbb{R}, X)$, or $f \in C_B(\mathbb{R}, X)$, we define*

$$\overline{H_n}(f, x) := \sum_{k=-\infty}^{\infty} f\left(\frac{k}{n}\right) \psi_q(nx - k), \tag{63}$$

$n \in \mathbb{N}$, $x \in \mathbb{R}$, the X-valued quasi-interpolation neural network operator.

We give

Remark 8 *We have that*

$$\left\| f\left(\frac{k}{n}\right) \right\| \leq \|f\|_{\infty, \mathbb{R}} < +\infty,$$

and

$$\left\| f\left(\frac{k}{n}\right) \right\| \psi_q(nx - k) \leq \|f\|_{\infty, \mathbb{R}} \psi_q(nx - k) \tag{64}$$

and

$$\sum_{k=-\lambda}^{\lambda} \left\| f\left(\frac{k}{n}\right) \right\| \psi_q(nx - k) \leq \|f\|_{\infty, \mathbb{R}} \left(\sum_{k=-\lambda}^{\lambda} \psi_q(nx - k) \right),$$

and finally

$$\sum_{k=-\infty}^{\infty} \left\| f\left(\frac{k}{n}\right) \right\| \psi_q(nx - k) \leq \|f\|_{\infty, \mathbb{R}}, \tag{65}$$

a convergent series in $\mathbb{R}$.

So, the series $\sum_{k=-\infty}^{\infty} \left\| f\left(\frac{k}{n}\right) \right\| \psi_q(nx - k)$ is absolutely convergent in X, hence it is convergent in X and $\overline{H_n}(f, x) \in X$. We denote by $\|f\|_\infty := \sup_{x \in [a,b]} \|f(x)\|$, for $f \in C([a,b], X)$, similarly it is defined for $f \in C_B(\mathbb{R}, X)$.

3 Main Results

We present a set of X-valued neural network approximations to a function given with rates.

Theorem 9 *Let $f \in C([a,b], X)$, $0 < \alpha < 1$, $n \in \mathbb{N} : n^{1-\alpha} > 2$, $q > 0$, $q \neq 1$, $x \in [a,b]$. Then*

i)

$$\|H_n(f, x) - f(x)\| \leq \Phi(q) \left[\omega_1\left(f, \frac{1}{n^\alpha}\right) + 2\|f\|_\infty Q e^{-2n^{(1-\alpha)}} \right] =: \tau, \tag{66}$$

where Q as in (30),

and

ii)

$$\left\| H_n(f) - f \right\|_\infty \leq \tau. \tag{67}$$

We get that $\lim_{n\to\infty} H_n(f) = f$, *pointwise and uniformly.*

Proof. We see that

$$\left\| \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \left(f\left(\frac{k}{n}\right) - f(x) \right) \psi_q(nx-k) \right\| \leq$$

$$\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \left\| f\left(\frac{k}{n}\right) - f(x) \right\| \psi_q(nx-k) =$$

$$\sum_{\substack{\begin{cases} k=\lceil na \rceil \\ \left|\frac{k}{n}-x\right| \leq \frac{1}{n^\alpha} \end{cases}}}^{\lfloor nb \rfloor} \left\| f\left(\frac{k}{n}\right) - f(x) \right\| \psi_q(nx-k) +$$

$$\sum_{\substack{\begin{cases} k=\lceil na \rceil \\ \left|\frac{k}{n}-x\right| > \frac{1}{n^\alpha} \end{cases}}}^{\lfloor nb \rfloor} \left\| f\left(\frac{k}{n}\right) - f(x) \right\| \psi_q(nx-k) \leq \tag{68}$$

$$\sum_{\substack{\begin{cases} k=\lceil na \rceil \\ \left|\frac{k}{n}-x\right| \leq \frac{1}{n^\alpha} \end{cases}}}^{\lfloor nb \rfloor} \omega_1\left(f, \left|\frac{k}{n}-x\right|\right) \psi_q(nx-k) +$$

$$2\|f\|_\infty \sum_{\substack{\begin{cases} k=\lceil na \rceil \\ |k-nx| > n^{1-\alpha} \end{cases}}}^{\lfloor nb \rfloor} \psi_q(nx-k) \leq$$

$$\omega_1\left(f, \frac{1}{n^\alpha}\right) \sum_{\substack{\begin{cases} k=-\infty \\ \left|\frac{k}{n}-x\right| \leq \frac{1}{n^\alpha} \end{cases}}}^{\infty} \psi_q(nx-k) +$$

$$2\|f\|_\infty \sum_{\substack{\begin{cases} k=-\infty \\ |k-nx| > n^{1-\alpha} \end{cases}}}^{\infty} \psi_q(nx-k) \underset{\text{(by Theorem 3)}}{\leq}$$

$$\omega_1\left(f, \frac{1}{n^\alpha}\right) + 2\|f\|_\infty Qe^{-2n^{(1-\alpha)}} \tag{69}$$

That is

$$\left\|\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}\left(f\left(\frac{k}{n}\right) - f(x)\right)\psi_q(nx-k)\right\| \leq$$

$$\omega_1\left(f, \frac{1}{n^\alpha}\right) + 2\|f\|_\infty Qe^{-2n^{(1-\alpha)}}. \tag{70}$$

Using the last equality we derive (66). ■

Next we give

Theorem 10 *Let $f \in C_B(\mathbb{R}, X)$, $0 < \alpha < 1$, $q > 0$, $q \neq 1$, $n \in \mathbb{N} : n^{1-\alpha} > 2$, $x \in \mathbb{R}$. Then*

i)

$$\left\|\overline{H}_n(f,x) - f(x)\right\| \leq \omega_1\left(f, \frac{1}{n^\alpha}\right) + 2\|f\|_\infty Qe^{-2n^{(1-\alpha)}} =: \gamma, \tag{71}$$

and

ii)

$$\left\|\overline{H}_n(f) - f\right\|_\infty \leq \gamma. \tag{72}$$

For $f \in C_{uB}(\mathbb{R}, X)$ we get $\lim_{n\to\infty}\overline{H}_n(f) = f$, pointwise and uniformly.

Proof. We observe that

$$\left\|\overline{H}_n(f,x) - f(x)\right\| \overset{(24)}{=} \left\|\sum_{k=-\infty}^{\infty} f\left(\frac{k}{n}\right)\psi_q(nx-k) - f(x)\sum_{k=-\infty}^{\infty}\psi_q(nx-k)\right\| =$$

$$\left\|\sum_{k=-\infty}^{\infty}\left(f\left(\frac{k}{n}\right) - f(x)\right)\psi_q(nx-k)\right\| \leq$$

$$\sum_{k=-\infty}^{\infty}\left\|f\left(\frac{k}{n}\right) - f(x)\right\|\psi_q(nx-k) =$$

$$\sum_{\begin{cases} k = -\infty \\ \left|\frac{k}{n} - x\right| \leq \frac{1}{n^\alpha}\end{cases}}^{\infty}\left\|f\left(\frac{k}{n}\right) - f(x)\right\|\psi_q(nx-k) +$$

$$\sum_{\begin{cases} k = -\infty \\ \left|\frac{k}{n} - x\right| > \frac{1}{n^\alpha}\end{cases}}^{\infty}\left\|f\left(\frac{k}{n}\right) - f(x)\right\|\psi_q(nx-k) \leq \tag{73}$$

$$\sum_{\begin{cases} k=-\infty \\ \left|\frac{k}{n}-x\right| \leq \frac{1}{n^{\alpha}} \end{cases}}^{\infty} \omega_1\left(f,\left|\frac{k}{n}-x\right|\right) \psi_q(nx-k) +$$

$$2\|f\|_{\infty} \sum_{\begin{cases} k=-\infty \\ \left|\frac{k}{n}-x\right| > \frac{1}{n^{\alpha}} \end{cases}}^{\infty} \psi_q(nx-k) \leq$$

$$\omega_1\left(f,\frac{1}{n^{\alpha}}\right) \sum_{\begin{cases} k=-\infty \\ \left|\frac{k}{n}-x\right| \leq \frac{1}{n^{\alpha}} \end{cases}}^{\infty} \psi_q(nx-k) + 2\|f\|_{\infty} Qe^{-2n^{(1-\alpha)}} \leq$$

$$\omega_1\left(f,\frac{1}{n^{\alpha}}\right) + 2\|f\|_{\infty} Qe^{-2n^{(1-\alpha)}}, \tag{74}$$

proving the claim. ■

We need the X-valued Taylor's formula in an appropiate form:

Theorem 11 *([10], [12]) Let $N \in \mathbb{N}$, and $f \in C^N([a,b],X)$, where $[a,b] \subset \mathbb{R}$ and X is a Banach space. Let any $x,y \in [a,b]$. Then*

$$f(x) = \sum_{i=0}^{N} \frac{(x-y)^i}{i!} f^{(i)}(y) + \frac{1}{(N-1)!} \int_y^x (x-t)^{N-1} \left(f^{(N)}(t) - f^{(N)}(y)\right) dt. \tag{75}$$

The derivatives $f^{(i)}$, $i \in \mathbb{N}$, are defined like the numerical ones, see [23], p. 83. The integral $\int_y^x$ in (75) is of Bochner type, see [21].

By [12], [19] we have that: if $f \in C([a,b],X)$, then $f \in L_{\infty}([a,b],X)$ and $f \in L_1([a,b],X)$.

In the next we discuss high order neural network X-valued approximation by using the smoothness of f.

Theorem 12 *Let $f \in C^N([a,b],X)$, $n,N \in \mathbb{N}$, $q>0$, $q \neq 1$, $0<\alpha<1$, $x \in [a,b]$ and $n^{1-\alpha} > 2$. Then*

i)

$$\|H_n(f,x) - f(x)\| \leq \Phi(q) \left\{ \sum_{j=1}^{N} \frac{\|f^{(j)}(x)\|}{j!} \left[\frac{1}{n^{\alpha j}} + (b-a)^j Qe^{-2n^{(1-\alpha)}}\right] + \right. \tag{76}$$

$$\left. \left[\omega_1\left(f^{(N)},\frac{1}{n^{\alpha}}\right) \frac{1}{n^{\alpha N} N!} + \frac{2\|f^{(N)}\|_{\infty} (b-a)^N}{N!} Qe^{-2n^{(1-\alpha)}}\right]\right\},$$

ii) assume further $f^{(j)}(x_0) = 0$, $j = 1, ..., N$, for some $x_0 \in [a, b]$, it holds

$$\|H_n(f, x_0) - f(x_0)\| \leq \Phi(q) \cdot$$

$$\left\{ \omega_1 \left(f^{(N)}, \frac{1}{n^\alpha} \right) \frac{1}{n^{\alpha N} N!} + \frac{2 \left\| f^{(N)} \right\|_\infty (b-a)^N}{N!} Q e^{-2n^{(1-\alpha)}} \right\}, \tag{77}$$

and

iii)

$$\|H_n(f) - f\|_\infty \leq \Phi(q) \left\{ \sum_{j=1}^{N} \frac{\left\| f^{(j)} \right\|_\infty}{j!} \left[\frac{1}{n^{\alpha j}} + (b-a)^j Q e^{-2n^{(1-\alpha)}} \right] + \right.$$

$$\left. \left[\omega_1 \left(f^{(N)}, \frac{1}{n^\alpha} \right) \frac{1}{n^{\alpha N} N!} + 2 \left\| f^{(N)} \right\|_\infty (b-a)^N Q e^{-2n^{(1-\alpha)}} \right] \right\}. \tag{78}$$

Again we obtain $\lim_{n \to \infty} H_n(f) = f$, pointwise and uniformly.

Proof. It is lengthy, and as similar to [15] is omitted. ■

All integrals from now on are of Bochner type [21].

We need

Definition 13 *([12]) Let $[a, b] \subset \mathbb{R}$, X be a Banach space, $\alpha > 0$; $m = \lceil \alpha \rceil \in \mathbb{N}$, ($\lceil \cdot \rceil$ is the ceiling of the number), $f : [a, b] \to X$. We assume that $f^{(m)} \in L_1([a, b], X)$. We call the Caputo-Bochner left fractional derivative of order α:*

$$\left(D_{*a}^\alpha f \right)(x) := \frac{1}{\Gamma(m - \alpha)} \int_a^x (x - t)^{m - \alpha - 1} f^{(m)}(t) \, dt, \quad \forall \, x \in [a, b]. \tag{79}$$

If $\alpha \in \mathbb{N}$, we set $D_{*a}^\alpha f := f^{(m)}$ the ordinary X-valued derivative (defined similar to numerical one, see [23], p. 83), and also set $D_{*a}^0 f := f$.

By [12], $\left(D_{*a}^\alpha f \right)(x)$ exists almost everywhere in $x \in [a, b]$ and $D_{*a}^\alpha f \in L_1([a, b], X)$.

If $\left\| f^{(m)} \right\|_{L_\infty([a,b],X)} < \infty$, then by [12], $D_{*a}^\alpha f \in C([a, b], X)$, hence $\left\| D_{*a}^\alpha f \right\| \in C([a, b])$.

We mention

Definition 14 *([10]) Let $[a, b] \subset \mathbb{R}$, X be a Banach space, $\alpha > 0$, $m := \lceil \alpha \rceil$. We assume that $f^{(m)} \in L_1([a, b], X)$, where $f : [a, b] \to X$. We call the Caputo-Bochner right fractional derivative of order α:*

$$\left(D_{b-}^\alpha f \right)(x) := \frac{(-1)^m}{\Gamma(m - \alpha)} \int_x^b (z - x)^{m - \alpha - 1} f^{(m)}(z) \, dz, \quad \forall \, x \in [a, b]. \tag{80}$$

We observe that $\left(D_{b-}^m f \right)(x) = (-1)^m f^{(m)}(x)$, for $m \in \mathbb{N}$, and $\left(D_{b-}^0 f \right)(x) = f(x)$.

By [10], $\left(D_{b-}^{\alpha}f\right)(x)$ exists almost everywhere on $[a,b]$ and $\left(D_{b-}^{\alpha}f\right) \in L_1([a,b],X)$.

If $\left\|f^{(m)}\right\|_{L_\infty([a,b],X)} < \infty$, and $\alpha \notin \mathbb{N}$, by [10], $D_{b-}^{\alpha}f \in C([a,b],X)$, hence $\left\|D_{b-}^{\alpha}f\right\| \in C([a,b])$.

We make

Remark 15 *([11]) Let $f \in C^{n-1}([a,b])$, $f^{(n)} \in L_\infty([a,b])$, $n = \lceil\nu\rceil$, $\nu > 0$, $\nu \notin \mathbb{N}$. Then*

$$\left\|D_{*a}^{\nu}f(x)\right\| \leq \frac{\left\|f^{(n)}\right\|_{L_\infty([a,b],X)}}{\Gamma(n-\nu+1)}(x-a)^{n-\nu}, \quad \forall\, x \in [a,b]. \tag{81}$$

Thus we observe

$$\omega_1\left(D_{*a}^{\nu}f,\delta\right) = \sup_{\substack{x,y\in[a,b]\\|x-y|\leq\delta}}\left\|D_{*a}^{\nu}f(x) - D_{*a}^{\nu}f(y)\right\| \leq \tag{82}$$

$$\sup_{\substack{x,y\in[a,b]\\|x-y|\leq\delta}}\left(\frac{\left\|f^{(n)}\right\|_{L_\infty([a,b],X)}}{\Gamma(n-\nu+1)}(x-a)^{n-\nu} + \frac{\left\|f^{(n)}\right\|_{L_\infty([a,b],X)}}{\Gamma(n-\nu+1)}(y-a)^{n-\nu}\right)$$

$$\leq \frac{2\left\|f^{(n)}\right\|_{L_\infty([a,b],X)}}{\Gamma(n-\nu+1)}(b-a)^{n-\nu}.$$

Consequently

$$\omega_1\left(D_{*a}^{\nu}f,\delta\right) \leq \frac{2\left\|f^{(n)}\right\|_{L_\infty([a,b],X)}}{\Gamma(n-\nu+1)}(b-a)^{n-\nu}. \tag{83}$$

Similarly, let $f \in C^{m-1}([a,b])$, $f^{(m)} \in L_\infty([a,b])$, $m = \lceil\alpha\rceil$, $\alpha > 0$, $\alpha \notin \mathbb{N}$, then

$$\omega_1\left(D_{b-}^{\alpha}f,\delta\right) \leq \frac{2\left\|f^{(m)}\right\|_{L_\infty([a,b],X)}}{\Gamma(m-\alpha+1)}(b-a)^{m-\alpha}. \tag{84}$$

So for $f \in C^{m-1}([a,b])$, $f^{(m)} \in L_\infty([a,b])$, $m = \lceil\alpha\rceil$, $\alpha > 0$, $\alpha \notin \mathbb{N}$, we find

$$\sup_{x_0\in[a,b]}\omega_1\left(D_{*x_0}^{\alpha}f,\delta\right)_{[x_0,b]} \leq \frac{2\left\|f^{(m)}\right\|_{L_\infty([a,b],X)}}{\Gamma(m-\alpha+1)}(b-a)^{m-\alpha}, \tag{85}$$

and

$$\sup_{x_0\in[a,b]}\omega_1\left(D_{x_0-}^{\alpha}f,\delta\right)_{[a,x_0]} \leq \frac{2\left\|f^{(m)}\right\|_{L_\infty([a,b],X)}}{\Gamma(m-\alpha+1)}(b-a)^{m-\alpha}. \tag{86}$$

By [12] we get that $D_{*x_0}^{\alpha}f \in C([x_0,b],X)$, and by [10] we obtain that $D_{x_0-}^{\alpha}f \in C([a,x_0],X)$.

We present the following X-valued fractional approximation result by neural networks.

Theorem 16 *Let $\alpha > 0$, $q > 0$, $q \neq 1$, $N = \lceil \alpha \rceil$, $\alpha \notin \mathbb{N}$, $f \in C^N([a,b], X)$, $0 < \beta < 1$, $x \in [a,b]$, $n \in \mathbb{N} : n^{1-\beta} > 2$. Then*

i)

$$\left\| H(f,x) - \sum_{j=1}^{N-1} \frac{f^{(j)}(x)}{j!} H_n\left((\cdot - x)^j\right)(x) - f(x) \right\| \leq$$

$$\frac{\Phi(q)}{\Gamma(\alpha+1)} \left\{ \frac{\left(\omega_1\left(D_{x-}^{\alpha} f, \frac{1}{n^\beta}\right)_{[a,x]} + \omega_1\left(D_{*x}^{\alpha} f, \frac{1}{n^\beta}\right)_{[x,b]}\right)}{n^{\alpha\beta}} + \right.$$

$$\left. Qe^{-2n^{(1-\beta)}} \left(\left\| D_{x-}^{\alpha} f \right\|_{\infty,[a,x]} (x-a)^{\alpha} + \left\| D_{*x}^{\alpha} f \right\|_{\infty,[x,b]} (b-x)^{\alpha} \right) \right\}, \qquad (87)$$

ii) if $f^{(j)}(x) = 0$, for $j = 1, ..., N-1$, we have

$$\| H_n(f,x) - f(x) \| \leq \frac{\Phi(q)}{\Gamma(\alpha+1)}$$

$$\left\{ \frac{\left(\omega_1\left(D_{x-}^{\alpha} f, \frac{1}{n^\beta}\right)_{[a,x]} + \omega_1\left(D_{*x}^{\alpha} f, \frac{1}{n^\beta}\right)_{[x,b]}\right)}{n^{\alpha\beta}} + \right.$$

$$\left. Qe^{-2n^{(1-\beta)}} \left(\left\| D_{x-}^{\alpha} f \right\|_{\infty,[a,x]} (x-a)^{\alpha} + \left\| D_{*x}^{\alpha} f \right\|_{\infty,[x,b]} (b-x)^{\alpha} \right) \right\}, \qquad (88)$$

iii)

$$\| H_n(f,x) - f(x) \| \leq \Phi(q)$$

$$\left\{ \sum_{j=1}^{N-1} \frac{\left\| f^{(j)}(x) \right\|}{j!} \left\{ \frac{1}{n^{\beta j}} + (b-a)^j Qe^{-2n^{(1-\beta)}} \right\} + \right.$$

$$\frac{1}{\Gamma(\alpha+1)} \left\{ \frac{\left(\omega_1\left(D_{x-}^{\alpha} f, \frac{1}{n^\beta}\right)_{[a,x]} + \omega_1\left(D_{*x}^{\alpha} f, \frac{1}{n^\beta}\right)_{[x,b]}\right)}{n^{\alpha\beta}} + \right.$$

$$\left.\left. Qe^{-2n^{(1-\beta)}} \left(\left\| D_{x-}^{\alpha} f \right\|_{\infty,[a,x]} (x-a)^{\alpha} + \left\| D_{*x}^{\alpha} f \right\|_{\infty,[x,b]} (b-x)^{\alpha} \right) \right\} \right\}, \qquad (89)$$

$\forall\, x \in [a,b]$,

and

iv)

$$\| H_n f - f \|_{\infty} \leq \Phi(q)$$

$$\left\{ \sum_{j=1}^{N-1} \frac{\left\| f^{(j)} \right\|_{\infty}}{j!} \left\{ \frac{1}{n^{\beta j}} + (b-a)^j Qe^{-2n^{(1-\beta)}} \right\} + \right.$$

$$\frac{1}{\Gamma(\alpha+1)}\left\{\frac{\left(\sup_{x\in[a,b]}\omega_1\left(D_{x-}^{\alpha}f,\frac{1}{n^{\beta}}\right)_{[a,x]}+\sup_{x\in[a,b]}\omega_1\left(D_{*x}^{\alpha}f,\frac{1}{n^{\beta}}\right)_{[x,b]}\right)}{n^{\alpha\beta}}+\right.$$

$$\left.Qe^{-2n^{(1-\beta)}}(b-a)^{\alpha}\left(\sup_{x\in[a,b]}\left\|D_{x-}^{\alpha}f\right\|_{\infty,[a,x]}+\sup_{x\in[a,b]}\left\|D_{*x}^{\alpha}f\right\|_{\infty,[x,b]}\right)\right\}\Bigg\}. \quad (90)$$

Above, when $N=1$ the sum $\sum_{j=1}^{N-1}\cdot=0$.

As we see here we obtain X-valued fractionally type pointwise and uniform convergence with rates of $H_n\to I$ the unit operator, as $n\to\infty$.

Proof. The proof is very lengtly and similar to [15], as such is omitted. ■

Next we apply Theorem 16 for $N=1$.

Theorem 17 *Let $0<\alpha,\beta<1$, $q>0$, $q\neq1$, $f\in C^1([a,b],X)$, $x\in[a,b]$, $n\in\mathbb{N}:n^{1-\beta}>2$. Then*

i)

$$\|H_n(f,x)-f(x)\|\leq$$

$$\frac{\Phi(q)}{\Gamma(\alpha+1)}\left\{\frac{\left(\omega_1\left(D_{x-}^{\alpha}f,\frac{1}{n^{\beta}}\right)_{[a,x]}+\omega_1\left(D_{*x}^{\alpha}f,\frac{1}{n^{\beta}}\right)_{[x,b]}\right)}{n^{\alpha\beta}}+\right.$$

$$\left.Qe^{-2n^{(1-\beta)}}\left(\left\|D_{x-}^{\alpha}f\right\|_{\infty,[a,x]}(x-a)^{\alpha}+\left\|D_{*x}^{\alpha}f\right\|_{\infty,[x,b]}(b-x)^{\alpha}\right)\right\}, \quad (91)$$

and

ii)

$$\|H_nf-f\|_{\infty}\leq\frac{\Phi(q)}{\Gamma(\alpha+1)}$$

$$\left\{\frac{\left(\sup_{x\in[a,b]}\omega_1\left(D_{x-}^{\alpha}f,\frac{1}{n^{\beta}}\right)_{[a,x]}+\sup_{x\in[a,b]}\omega_1\left(D_{*x}^{\alpha}f,\frac{1}{n^{\beta}}\right)_{[x,b]}\right)}{n^{\alpha\beta}}+\right.$$

$$\left.(b-a)^{\alpha}Qe^{-2n^{(1-\beta)}}\left(\sup_{x\in[a,b]}\left\|D_{x-}^{\alpha}f\right\|_{\infty,[a,x]}+\sup_{x\in[a,b]}\left\|D_{*x}^{\alpha}f\right\|_{\infty,[x,b]}\right)\right\}. \quad (92)$$

When $\alpha=\frac{1}{2}$ we derive

Corollary 18 *Let $0<\beta<1$, $q>0$, $q\neq1$, $f\in C^1([a,b],X)$, $x\in[a,b]$, $n\in\mathbb{N}:n^{1-\beta}>2$. Then*

i)

$$\|H_n(f,x)-f(x)\|\leq$$

$$\frac{2\Phi(q)}{\sqrt{\pi}}\left\{\frac{\left(\omega_1\left(D^{\frac{1}{2}}_{x-}f,\frac{1}{n^{\beta}}\right)_{[a,x]}+\omega_1\left(D^{\frac{1}{2}}_{*x}f,\frac{1}{n^{\beta}}\right)_{[x,b]}\right)}{n^{\frac{\beta}{2}}}+\right.$$

$$\left.Qe^{-2n^{(1-\beta)}}\left(\left\|D^{\frac{1}{2}}_{x-}f\right\|_{\infty,[a,x]}\sqrt{(x-a)}+\left\|D^{\frac{1}{2}}_{*x}f\right\|_{\infty,[x,b]}\sqrt{(b-x)}\right)\right\}, \tag{93}$$

and
ii)

$$\|H_n f-f\|_{\infty}\leq\frac{2\Phi(q)}{\sqrt{\pi}}$$

$$\left\{\frac{\left(\sup_{x\in[a,b]}\omega_1\left(D^{\frac{1}{2}}_{x-}f,\frac{1}{n^{\beta}}\right)_{[a,x]}+\sup_{x\in[a,b]}\omega_1\left(D^{\frac{1}{2}}_{*x}f,\frac{1}{n^{\beta}}\right)_{[x,b]}\right)}{n^{\frac{\beta}{2}}}+\right.$$

$$\left.\sqrt{(b-a)}Qe^{-2n^{(1-\beta)}}\left(\sup_{x\in[a,b]}\left\|D^{\frac{1}{2}}_{x-}f\right\|_{\infty,[a,x]}+\sup_{x\in[a,b]}\left\|D^{\frac{1}{2}}_{*x}f\right\|_{\infty,[x,b]}\right)\right\}<\infty. \tag{94}$$

We make

Remark 19 *Some convergence analysis follows based on Corollary 18.*

Let $0<\beta<1$, $\lambda>0$, $f\in C^1([a,b],X)$, $x\in[a,b]$, $n\in\mathbb{N}:n^{1-\beta}>2$. *We elaborate on (94). Assume that*

$$\omega_1\left(D^{\frac{1}{2}}_{x-}f,\frac{1}{n^{\beta}}\right)_{[a,x]}\leq\frac{R_1}{n^{\beta}}, \tag{95}$$

and

$$\omega_1\left(D^{\frac{1}{2}}_{*x}f,\frac{1}{n^{\beta}}\right)_{[x,b]}\leq\frac{R_2}{n^{\beta}}, \tag{96}$$

$\forall\ x\in[a,b]$, $\forall\ n\in\mathbb{N}$, *where* $R_1,R_2>0$.

Then it holds

$$\frac{\left[\sup_{x\in[a,b]}\omega_1\left(D^{\frac{1}{2}}_{x-}f,\frac{1}{n^{\beta}}\right)_{[a,x]}+\sup_{x\in[a,b]}\omega_1\left(D^{\frac{1}{2}}_{*x}f,\frac{1}{n^{\beta}}\right)_{[x,b]}\right]}{n^{\frac{\beta}{2}}}\leq$$

$$\frac{\frac{(R_1+R_2)}{n^{\beta}}}{n^{\frac{\beta}{2}}}=\frac{(R_1+R_2)}{n^{\frac{3\beta}{2}}}=\frac{R}{n^{\frac{3\beta}{2}}}, \tag{97}$$

where $R:=R_1+R_2>0$.

The other summand of the right hand side of (94), for large enough n, converges to zero at the speed $e^{-2n^{(1-\beta)}}$, so it is about $Ae^{-2n^{(1-\beta)}}$, where $A > 0$ is a constant.

Then, for large enough $n \in \mathbb{N}$, by (94), (97) and the above comment, we obtain that

$$\|H_n f - f\|_\infty \leq \frac{B}{n^{\frac{3\beta}{2}}}, \tag{98}$$

where $B > 0$, converging to zero at the high speed of $\frac{1}{n^{\frac{3\beta}{2}}}$.

In Theorem 9, for $f \in C([a,b], X)$ and for large enough $n \in \mathbb{N}$, the speed is $\frac{1}{n^\beta}$. So by (98), $\|H_n f - f\|_\infty$ converges much faster to zero. The last comes because we assumed differentiability of f. Notice that in Corollary 18 no initial condition is assumed.

References

[1] G.A. Anastassiou, *Rate of convergence of some neural network operators to the unit-univariate case*, J. Math. Anal. Appl, 212 (1997), 237-262.

[2] G.A. Anastassiou, *Quantitative Approximations*, Chapman & Hall / CRC, Boca Raton, New York, 2001.

[3] G.A. Anastassiou, *Univariate hyperbolic tangent neural network approximation*, Mathematics and Computer Modelling, 53 (2011), 1111-1132.

[4] G.A. Anastassiou, *Multivariate hyperbolic tangent neural network approximation*, Computers and Mathematics, 61 (2011), 809-821.

[5] G.A. Anastassiou, *Multivariate sigmoidal neural network approximation*, Neural Networks, 24 (2011), 378-386.

[6] G.A. Anastassiou, *Inteligent Systems: Approximation by Artificial Neural Networks*, Intelligent Systems Reference Library, Vol. 19, Springer, Heidelberg, 2011.

[7] G.A. Anastassiou, *Univariate sigmoidal neural network approximation*, J. of Computational Analysis and Applications, Vol. 14, No. 4, 2012, 659-690.

[8] G.A. Anastassiou, *Fractional neural network approximation*, Computers and Mathematics with Applications, 64 (2012), 1655-1676.

[9] G.A. Anastassiou, *Intelligent Systems II: Complete Approximation by Neural Network Operators*, Springer, Heidelberg, New York, 2016.

[10] G.A. Anastassiou, *Strong Right Fractional Calculus for Banach space valued functions*, ‘Revista Proyecciones, Vol. 36, No. 1 (2017), 149-186.

[11] G.A. Anastassiou, *Vector fractional Korovkin type Approximations*, Dynamic Systems and Applications, 26(2017), 81-104.

[12] G.A. Anastassiou, *A strong Fractional Calculus Theory for Banach space valued functions*, Nonlinear Functional Analysis and Applications (Korea), 22(3) (2017), 495-524.

[13] G.A. Anastassiou, *Nonlinearity: Ordinary and Fractional Approximations by Sublinear and Max-Product Operators*, Springer, Heidelberg, New York, 2018.

[14] G.A. Anastassiou, *General sigmoid based Banach spacevalued neural network approximation*, J. of Computational Analysis and Applications, accepted, 2022.

[15] G.A. Anastassiou, S. Karateke, *Parametrized hyperbolic tangent induced Banach space valued ordinary and fractional neural network approximation*, submitted, 2022.

[16] Z. Chen and F. Cao, *The approximation operators with sigmoidal functions*, Computers and Mathematics with Applications, 58 (2009), 758-765.

[17] S.A. El-Shehawy, E.A-B. Abdel-Salam, *The* q-deformed hyperbolic Secant family, Intern. J. of Applied Math. & Stat., 29 (5) (2012), 51-62.

[18] S. Haykin, *Neural Networks: A Comprehensive Foundation* (2 ed.), Prentice Hall, New York, 1998.

[19] M. Kreuter, *Sobolev Spaces of Vector-valued functions*, Ulm Univ., Master Thesis in Math., Ulm, Germany, 2015.

[20] W. McCulloch and W. Pitts, *A logical calculus of the ideas immanent in nervous activity*, Bulletin of Mathematical Biophysis, 7 (1943), 115-133.

[21] J. Mikusinski, *The Bochner integral*, Academic Press, New York, 1978.

[22] T.M. Mitchell, *Machine Learning*, WCB-McGraw-Hill, New York, 1997.

[23] G.E. Shilov, *Elementary Functional Analysis*, Dover Publications, Inc., New York, 1996.