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REFINEMENTS AND REVERSES OF SOME INEQUALITIES FOR THE NORMALIZED DETERMINANTS OF SEQUENCES OF POSITIVE OPERATORS IN HILBERT SPACES

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ABSTRACT. For positive invertible operators A on a Hilbert space H and a fixed unit vector $x \in H$, define the normalized determinant by $\Delta_x(A) := \exp \langle \ln Ax, x\rangle$. In this paper we prove among others that, if $0 < m \leq A_j \leq M$ and $p_j \geq 0$, $j \in \{1, ..., n\}$ with $\sum_{j=1}^n p_j = 1$, then for $p \in (\infty, 0) \cup (1, \infty)$

$$1 \leq \exp\left(\gamma_p(m, M)\left[\sum_{j=1}^n p_j \left\langle A_j^p x, x\right\rangle - \left(\sum_{j=1}^n p_j \langle A_j x, x\rangle\right)^p\right]\right)$$
$$\leq \frac{\sum_{j=1}^n p_j \langle A_j x, x\rangle}{\prod_{j=1}^n [\Delta_x(A_j)]^{p_j}}$$
$$\leq \exp\left(\Gamma_p(m, M)\left[\sum_{j=1}^n p_j \left\langle A_j^p x, x\right\rangle - \left(\sum_{j=1}^n p_j \langle A_j x, x\rangle\right)^p\right]\right)$$

for $x \in H$, $\|x\| = 1$, where

$$\gamma_p(m, M) := \begin{cases} \frac{M^{-p}}{p(p-1)} & \text{for } p \in (1, \infty), \\ \frac{m^{-p}}{p(p-1)} & \text{for } p \in (-\infty, 0) \end{cases}$$

and

$$\Gamma_p(m, M) := \begin{cases} \frac{m^{-p}}{p(p-1)} & \text{for } p \in (1, \infty), \\ \frac{M^{-p}}{p(p-1)} & \text{for } p \in (-\infty, 0). \end{cases}$$

1. Introduction

Let $B(H)$ be the space of all bounded linear operators on a Hilbert space H, and I stands for the identity operator on H. An operator A in $B(H)$ is said to be positive (in symbol: $A \geq 0$) if $\langle Ax, x\rangle \geq 0$ for all $x \in H$. In particular, $A > 0$ means that A is positive and invertible. For a pair A, B of selfadjoint operators the order relation $A \geq B$ means as usual that $A - B$ is positive.

In 1998, Fujii et al. [5], [6], introduced the *normalized determinant* $\Delta_x(A)$ for positive invertible operators A on a Hilbert space H and a fixed unit vector $x \in H$, namely $\|x\| = 1$, defined by $\Delta_x(A) := \exp \langle \ln Ax, x\rangle$ and discussed it as a continuous geometric mean and observed some inequalities around the determinant from this point of view.

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RGMIA

Some of the fundamental properties of normalized determinant are as follows, [5].

For each unit vector $x \in H$, see also [7], we have:

(i) *continuity*: the map $A \to \Delta_x(A)$ is norm continuous;
(ii) *bounds*: $\langle A^{-1}x, x\rangle^{-1} \le \Delta_x(A) \le \langle Ax, x\rangle$;
(iii) *continuous mean*: $\langle A^p x, x\rangle^{1/p} \downarrow \Delta_x(A)$ for $p \downarrow 0$ and $\langle A^p x, x\rangle^{1/p} \uparrow \Delta_x(A)$ for $p \uparrow 0$;
(iv) *power equality:* $\Delta_x(A^t) = \Delta_x(A)^t$ for all $t > 0$;
(v) *homogeneity*: $\Delta_x(tA) = t\Delta_x(A)$ and $\Delta_x(tI) = t$ for all $t > 0$;
(vi) *monotonicity*: $0 < A \le B$ implies $\Delta_x(A) \le \Delta_x(B)$;
(vii) *multiplicativity*: $\Delta_x(AB) = \Delta_x(A)\Delta_x(B)$ for commuting A and B;
(viii) *Ky Fan type inequality*: $\Delta_x((1-\alpha)A + \alpha B) \ge \Delta_x(A)^{1-\alpha}\Delta_x(B)^{\alpha}$ for $0 < \alpha < 1$.

We define the logarithmic mean of two positive numbers a, b by

$$L(a,b) := \begin{cases} \frac{b-a}{\ln b - \ln a} \text{ if } b \ne a, \\ \\ a \text{ if } b = a. \end{cases}$$

In [5] the authors obtained the following additive reverse inequality for the operator A which satisfy the condition $0 < mI \le A \le MI$, where m, M are positive numbers,

$$0 \le \langle Ax, x\rangle - \Delta_x(A) \le L(m,M)\left[\ln L(m,M) + \frac{M\ln m - m\ln M}{M-m} - 1\right] \tag{1.1}$$

for all $x \in H$, $\|x\| = 1$.

We recall that *Specht's ratio* is defined by [11]

$$S(h) := \begin{cases} \frac{h^{\frac{1}{h-1}}}{e\ln\left(h^{\frac{1}{h-1}}\right)} \text{ if } h \in (0,1) \cup (1,\infty), \\ \\ 1 \text{ if } h = 1. \end{cases} \tag{1.2}$$

It is well known that $\lim_{h\to 1} S(h) = 1$, $S(h) = S\left(\frac{1}{h}\right) > 1$ for $h > 0$, $h \ne 1$. The function is decreasing on $(0,1)$ and increasing on $(1,\infty)$.

In [6], the authors obtained the following multiplicative reverse inequality as well

$$1 \le \frac{\langle Ax, x\rangle}{\Delta_x(A)} \le S\left(\frac{M}{m}\right) \tag{1.3}$$

for $0 < mI \le A \le MI$ and $x \in H$, $\|x\| = 1$.

In this paper we obtain several refinements and reverses for the normalized determinant of a sequence of operators that have the spectra in a positive interval $[m, M]$. For this purpose we used some Jensen's type inequalities for twice differentiable functions obtained by the author in [3].

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2. Inequalities for $p \in (-\infty, 0) \cup (1, \infty)$

Assume that $A > 0$. For a vector $y \neq 0$ we can extend the normalized determinant as $\tilde{\Delta}_y(A) := \exp \langle \ln Ay, y \rangle$. We observe that

$$\tilde{\Delta}_y(A) := \exp \langle \ln Ay, y \rangle = \exp \left(\|y\|^2 \left\langle \ln A \frac{y}{\|y\|}, \frac{y}{\|y\|} \right\rangle \right)$$
$$= \left[\exp \left(\left\langle \ln A \frac{y}{\|y\|}, \frac{y}{\|y\|} \right\rangle \right) \right]^{\|y\|^2} = \left[\Delta_{\frac{y}{\|y\|}}(A) \right]^{\|y\|^2}$$

for any $y \neq 0$.

Theorem 1. *Assume that A_j are operators such that $0 < m \leq A_j \leq M$, $j \in \{1, ..., n\}$. Define*

$$\gamma_p(m, M) := \begin{cases} \frac{M^{-p}}{p(p-1)} \text{ for } p \in (1, \infty), \\ \frac{m^{-p}}{p(p-1)} \text{ for } p \in (-\infty, 0) \end{cases}$$

and

$$\Gamma_p(m, M) := \begin{cases} \frac{m^{-p}}{p(p-1)} \text{ for } p \in (1, \infty), \\ \frac{M^{-p}}{p(p-1)} \text{ for } p \in (-\infty, 0). \end{cases}$$

Then

$$1 \leq \exp \left(\gamma_p(m, M) \left[\sum_{j=1}^n \left\langle A_j^p x_j, x_j \right\rangle - \left(\sum_{j=1}^n \langle A_j x_j, x_j \rangle \right)^p \right] \right) \tag{2.1}$$
$$\leq \frac{\sum_{j=1}^n \langle A_j x_j, x_j \rangle}{\prod_{j=1}^n \tilde{\Delta}_{x_j}(A_j)}$$
$$\leq \exp \left(\Gamma_p(m, M) \left[\sum_{j=1}^n \left\langle A_j^p x_j, x_j \right\rangle - \left(\sum_{j=1}^n \langle A_j x_j, x_j \rangle \right)^p \right] \right),$$

for each $x_j \in H$, $j \in \{1, ..., n\}$ with $\sum_{j=1}^n \|x_j\|^2 = 1$.

Proof. Let A_j be positive definite operators with $\operatorname{Sp}(A_j) \subseteq [m, M] \subset (0, \infty)$ $j \in \{1, ..., n\}$. If f is a twice differentiable function on (m, M) and for $p \in (-\infty, 0) \cup (1, \infty)$ we have for some $\gamma < \Gamma$ that

$$\gamma \leq g(t) := \frac{t^{2-p}}{p(p-1)} f''(t) \leq \Gamma \text{ for any } t \in (m, M), \tag{2.2}$$

then, see [3],

$$\gamma \left[\sum_{j=1}^n \left\langle A_j^p x_j, x_j \right\rangle - \left(\sum_{j=1}^n \langle A_j x_j, x_j \rangle \right)^p \right]$$

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$$\leq \sum_{j=1}^{n} \langle f(A_j) x_j, x_j \rangle - f\left(\sum_{j=1}^{n} \langle A_j x_j, x_j \rangle\right)$$
$$\leq \Gamma \left[\sum_{j=1}^{n} \left\langle A_j^p x_j, x_j \right\rangle - \left(\sum_{j=1}^{n} \langle A_j x_j, x_j \rangle\right)^p\right]$$

for each $x_j \in H$, $j \in \{1, ..., n\}$ with $\sum_{j=1}^{n} \|x_j\|^2 = 1$.

We consider the convex function $f(t) = -\ln t$, $t \in [m, M] \subset (0, \infty)$. Then

$$g(t) = \frac{t^{2-p}}{p(p-1)} \frac{1}{t^2} = \frac{1}{p(p-1) t^p}.$$

For $p \in (1, \infty)$, we have

$$\sup_{t \in [m,M]} g(t) = \frac{m^{-p}}{p(p-1)} \text{ and } \inf_{t \in [m,M]} g(t) = \frac{M^{-p}}{p(p-1)}$$

and for $p \in (-\infty, 0)$

$$\sup_{t \in [m,M]} g(t) = \sup_{t \in [m,M]} \frac{t^{-p}}{p(p-1)} = \frac{M^{-p}}{p(p-1)}$$

and

$$\inf_{t \in [m,M]} g(t) = \inf_{t \in [m,M]} \frac{t^{-p}}{p(p-1)} = \frac{m^{-p}}{p(p-1)}.$$

Therefore by (**??**) we get

$$0 \leq \gamma_p(m, M) \left[\sum_{j=1}^{n} \left\langle A_j^p x_j, x_j \right\rangle - \left(\sum_{j=1}^{n} \langle A_j x_j, x_j \rangle\right)^p\right] \tag{2.3}$$
$$\leq \ln\left(\sum_{j=1}^{n} \langle A_j x_j, x_j \rangle\right) - \sum_{j=1}^{n} \langle \ln A_j x_j, x_j \rangle$$
$$\leq \Gamma_p(m, M) \left[\sum_{j=1}^{n} \left\langle A_j^p x_j, x_j \right\rangle - \left(\sum_{j=1}^{n} \langle A_j x_j, x_j \rangle\right)^p\right],$$

where $\gamma_p(m, M)$ and $\Gamma_p(m, M)$ are given above.

If we take the exponential in (2.3), then we get

$$1 \leq \exp\left(\gamma_p(m, M) \left[\sum_{j=1}^{n} \left\langle A_j^p x_j, x_j \right\rangle - \left(\sum_{j=1}^{n} \langle A_j x_j, x_j \rangle\right)^p\right]\right) \tag{2.4}$$
$$\leq \frac{\exp \ln\left(\sum_{j=1}^{n} \langle A_j x_j, x_j \rangle\right)}{\exp\left(\sum_{j=1}^{n} \langle \ln A_j x_j, x_j \rangle\right)}$$
$$\leq \exp\left(\Gamma_p(m, M) \left[\sum_{j=1}^{n} \left\langle A_j^p x_j, x_j \right\rangle - \left(\sum_{j=1}^{n} \langle A_j x_j, x_j \rangle\right)^p\right]\right),$$

for each $x_j \in H$, $j \in \{1, ..., n\}$ with $\sum_{j=1}^{n} \|x_j\|^2 = 1$.

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Since

$$\exp\left(\sum_{j=1}^{n}\langle\ln A_jx_j,x_j\rangle\right)=\prod_{j=1}^{n}\exp\langle\ln A_jx_j,x_j\rangle=\prod_{j=1}^{n}\Delta_{x_j}(A_j),$$

hence by (2.4) we derive (2.1). □

Corollary 1. *With the assumptions of Theorem 1, define*

$$\tilde{\gamma}_p(m,M):=\begin{cases}\frac{m^p}{p(p-1)} & \text{for } p\in(1,\infty),\\[2ex] \frac{M^p}{p(p-1)} & \text{for } p\in(-\infty,0)\end{cases}$$

and

$$\tilde{\Gamma}_p(m,M):=\begin{cases}\frac{M^p}{p(p-1)} & \text{for } p\in(1,\infty),\\[2ex] \frac{m^p}{p(p-1)} & \text{for } p\in(-\infty,0).\end{cases}$$

Then

$$1\le\exp\left(\tilde{\gamma}_p(m,M)\left[\sum_{j=1}^{n}\left\langle A_j^{-p}x_j,x_j\right\rangle-\left(\sum_{j=1}^{n}\left\langle A_j^{-1}x_j,x_j\right\rangle\right)^p\right]\right)\tag{2.5}$$
$$\le\frac{\prod_{j=1}^{n}\tilde{\Delta}_{x_j}(A_j)}{\left(\sum_{j=1}^{n}\left\langle A_j^{-1}x_j,x_j\right\rangle\right)^{-1}}$$
$$\le\exp\left(\tilde{\Gamma}_p(m,M)\left[\sum_{j=1}^{n}\left\langle A_j^{-p}x_j,x_j\right\rangle-\left(\sum_{j=1}^{n}\left\langle A_j^{-1}x_j,x_j\right\rangle\right)^p\right]\right),$$

for each $x_j\in H$, $j\in\{1,...,n\}$ with $\sum_{j=1}^{n}\|x_j\|^2=1$.

The proof follows by Theorem 1 written for $M^{-1}\le A_j^{-1}\le m^{-1}$, $j\in\{1,...,n\}$.

Remark 1. *Assume that A_j are operators such that $0<m\le A_j\le M$, $j\in\{1,...,n\}$. If we take $p=2$ in (2.1), then we get*

$$1\le\exp\left(\frac{1}{2M^2}\left[\sum_{j=1}^{n}\left\langle A_j^2x_j,x_j\right\rangle-\left(\sum_{j=1}^{n}\langle A_jx_j,x_j\rangle\right)^2\right]\right)\tag{2.6}$$
$$\le\frac{\sum_{j=1}^{n}\langle A_jx_j,x_j\rangle}{\prod_{j=1}^{n}\tilde{\Delta}_{x_j}(A_j)}$$
$$\le\exp\left(\frac{1}{2m^2}\left[\sum_{j=1}^{n}\left\langle A_j^2x_j,x_j\right\rangle-\left(\sum_{j=1}^{n}\langle A_jx_j,x_j\rangle\right)^2\right]\right)$$

RGMIA

and from (2.5),

$$1 \le \exp\left(\frac{m^2}{2}\left[\sum_{j=1}^{n}\left\langle A_j^{-2}x_j, x_j\right\rangle - \left(\sum_{j=1}^{n}\left\langle A_j^{-1}x_j, x_j\right\rangle\right)^2\right]\right) \tag{2.7}$$
$$\le \frac{\prod_{j=1}^{n}\tilde{\Delta}_{x_j}(A_j)}{\left(\sum_{j=1}^{n}\left\langle A_j^{-1}x_j, x_j\right\rangle\right)^{-1}}$$
$$\le \exp\left(\frac{M^2}{2}\left[\sum_{j=1}^{n}\left\langle A_j^{-2}x_j, x_j\right\rangle - \left(\sum_{j=1}^{n}\left\langle A_j^{-1}x_j, x_j\right\rangle\right)^2\right]\right),$$

for each $x_j \in H$, $j \in \{1, ..., n\}$ with $\sum_{j=1}^{n}\|x_j\|^2 = 1$.

If we take $p = -1$ in (2.1), then we get

$$1 \le \exp\left(\frac{m}{2}\left[\sum_{j=1}^{n}\left\langle A_j^{-1}x_j, x_j\right\rangle - \left(\sum_{j=1}^{n}\langle A_jx_j, x_j\rangle\right)^{-1}\right]\right) \tag{2.8}$$
$$\le \frac{\sum_{j=1}^{n}\langle A_jx_j, x_j\rangle}{\prod_{j=1}^{n}\tilde{\Delta}_{x_j}(A_j)}$$
$$\le \exp\left(\frac{M}{2}\left[\sum_{j=1}^{n}\left\langle A_j^{-1}x_j, x_j\right\rangle - \left(\sum_{j=1}^{n}\langle A_jx_j, x_j\rangle\right)^{-1}\right]\right)$$

and from (2.5)

$$1 \le \exp\left(\frac{1}{2M}\left[\sum_{j=1}^{n}\langle A_jx_j, x_j\rangle - \left(\sum_{j=1}^{n}\left\langle A_j^{-1}x_j, x_j\right\rangle\right)^{-1}\right]\right) \tag{2.9}$$
$$\le \frac{\prod_{j=1}^{n}\tilde{\Delta}_{x_j}(A_j)}{\left(\sum_{j=1}^{n}\left\langle A_j^{-1}x_j, x_j\right\rangle\right)^{-1}}$$
$$\le \exp\left(\frac{1}{2m}\left[\sum_{j=1}^{n}\langle A_jx_j, x_j\rangle - \left(\sum_{j=1}^{n}\left\langle A_j^{-1}x_j, x_j\right\rangle\right)^{-1}\right]\right),$$

for each $x_j \in H$, $j \in \{1, ..., n\}$ with $\sum_{j=1}^{n}\|x_j\|^2 = 1$.

The case of normalized determinant is as follows:

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Corollary 2. *Assume that $p_j \geq 0$ with $\sum_{j=1}^{n} p_j = 1$, then*

$$1 \leq \exp\left(\gamma_p(m,M)\left[\sum_{j=1}^{n} p_j \left\langle A_j^p x, x\right\rangle - \left(\sum_{j=1}^{n} p_j \left\langle A_j x, x\right\rangle\right)^p\right]\right) \tag{2.10}$$
$$\leq \frac{\sum_{j=1}^{n} p_j \left\langle A_j x, x\right\rangle}{\prod_{j=1}^{n} \left[\Delta_x(A_j)\right]^{p_j}}$$
$$\leq \exp\left(\Gamma_p(m,M)\left[\sum_{j=1}^{n} p_j \left\langle A_j^p x, x\right\rangle - \left(\sum_{j=1}^{n} p_j \left\langle A_j x, x\right\rangle\right)^p\right]\right),$$

and

$$1 \leq \exp\left(\tilde{\gamma}_p(m,M)\left[\sum_{j=1}^{n} p_j \left\langle A_j^{-p} x, x\right\rangle - \left(\sum_{j=1}^{n} p_j \left\langle A_j^{-1} x, x\right\rangle\right)^p\right]\right) \tag{2.11}$$
$$\leq \frac{\prod_{j=1}^{n} \left[\Delta_x(A_j)\right]^{p_j}}{\left(\sum_{j=1}^{n} p_j \left\langle A_j^{-1} x, x\right\rangle\right)^{-1}}$$
$$\leq \exp\left(\tilde{\Gamma}_p(m,M)\left[\sum_{j=1}^{n} p_j \left\langle A_j^{-p} x, x\right\rangle - \left(\sum_{j=1}^{n} p_j \left\langle A_j^{-1} x, x\right\rangle\right)^p\right]\right)$$

for $x \in H$, $\|x\| = 1$.

The proof follows by (2.1) and (2.5) by taking $x_j = \sqrt{p_j}x$, $x \in H$, $\|x\| = 1$ and observing that

$$\prod_{j=1}^{n} \tilde{\Delta}_{x_j}(A_j) = \prod_{j=1}^{n} \exp\left\langle \ln A_j \sqrt{p_j}x, \sqrt{p_j}x\right\rangle = \prod_{j=1}^{n} \exp\left[p_j \left\langle \ln A_j x, x\right\rangle\right]$$
$$= \prod_{j=1}^{n} \exp\left[\left\langle \ln A_j x, x\right\rangle\right]^{p_j} = \prod_{j=1}^{n} \left[\Delta_x(A_j)\right]^{p_j}.$$

If we take $p = 2$ in (2.10), then we get

$$1 \leq \exp\left(\frac{1}{2M^2}\left[\sum_{j=1}^{n} p_j \left\langle A_j^2 x, x\right\rangle - \left(\sum_{j=1}^{n} \left\langle A_j x, x\right\rangle\right)^2\right]\right) \tag{2.12}$$
$$\leq \frac{\sum_{j=1}^{n} p_j \left\langle A_j x, x\right\rangle}{\prod_{j=1}^{n} \left[\Delta_x(A_j)\right]^{p_j}}$$
$$\leq \exp\left(\frac{1}{2m^2}\left[\sum_{j=1}^{n} p_j \left\langle A_j^2 x, x\right\rangle - \left(\sum_{j=1}^{n} \left\langle A_j x, x\right\rangle\right)^2\right]\right)$$

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and from (2.11),

$$1 \leq \exp\left(\frac{m^2}{2}\left[\sum_{j=1}^{n}\left\langle A_j^{-2}x_j, x_j\right\rangle - \left(\sum_{j=1}^{n}\left\langle A_j^{-1}x_j, x_j\right\rangle\right)^2\right]\right) \tag{2.13}$$
$$\leq \frac{\prod_{j=1}^{n}\left[\Delta_x\left(A_j\right)\right]^{p_j}}{\left(\sum_{j=1}^{n} p_j\left\langle A_j^{-1}x, x\right\rangle\right)^{-1}}$$
$$\leq \exp\left(\frac{M^2}{2}\left[\sum_{j=1}^{n}\left\langle A_j^{-2}x_j, x_j\right\rangle - \left(\sum_{j=1}^{n}\left\langle A_j^{-1}x_j, x_j\right\rangle\right)^2\right]\right),$$

for each $x \in H$, $\|x\| = 1$.

If we take $p = -1$ in (2.10), then we get

$$1 \leq \exp\left(\frac{m}{2}\left[\sum_{j=1}^{n} p_j\left\langle A_j^{-1}x, x\right\rangle - \left(\sum_{j=1}^{n} p_j\left\langle A_j x, x\right\rangle\right)^{-1}\right]\right) \tag{2.14}$$
$$\leq \frac{\sum_{j=1}^{n} p_j\left\langle A_j x, x\right\rangle}{\prod_{j=1}^{n}\left[\Delta_x\left(A_j\right)\right]^{p_j}}$$
$$\leq \exp\left(\frac{M}{2}\left[\sum_{j=1}^{n} p_j\left\langle A_j^{-1}x, x\right\rangle - \left(\sum_{j=1}^{n} p_j\left\langle A_j x, x\right\rangle\right)^{-1}\right]\right)$$

and from (2.11)

$$1 \leq \exp\left(\frac{1}{2M}\left[\sum_{j=1}^{n} p_j\left\langle A_j x, x\right\rangle - \left(\sum_{j=1}^{n} p_j\left\langle A_j^{-1}x, x\right\rangle\right)^{-1}\right]\right) \tag{2.15}$$
$$\leq \frac{\prod_{j=1}^{n}\left[\Delta_x\left(A_j\right)\right]^{p_j}}{\left(\sum_{j=1}^{n} p_j\left\langle A_j^{-1}x, x\right\rangle\right)^{-1}}$$
$$\leq \exp\left(\frac{1}{2m}\left[\sum_{j=1}^{n} p_j\left\langle A_j x, x\right\rangle - \left(\sum_{j=1}^{n} p_j\left\langle A_j^{-1}x, x\right\rangle\right)^{-1}\right]\right),$$

for $x \in H$, $\|x\| = 1$.

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The case of two operators is as follows. Assume that $0 < m \leq A$, $B \leq M$, and $t \in [0,1]$. Then

$$\begin{aligned}(2.16)\qquad & 1 \leq \exp\left\{\gamma_p(m,M)\right.\\ & \times\left.[\langle[(1-t)A^p + tB^p]x,x\rangle - (\langle[(1-t)A+tB]x,x\rangle)^p]\right\}\\ & \leq \frac{\langle[(1-t)A+tB]x,x\rangle}{[\Delta_x(A)]^{(1-t)}[\Delta_x(B)]^t}\\ & \leq \exp\left\{\Gamma_p(m,M)\right.\\ & \times\left.[\langle[(1-t)A^p + tB^p]x,x\rangle - (\langle[(1-t)A+tB]x,x\rangle)^p]\right\}\end{aligned}$$

and

$$\begin{aligned}(2.17)\qquad & 1 \leq \exp\left\{\tilde{\gamma}_p(m,M)\right.\\ & \times\left.\left[\left\langle\left[(1-t)A^{-p} + tB^{-p}\right]x,x\right\rangle - \left(\left\langle\left[(1-t)A^{-1}+tB^{-1}\right]x,x\right\rangle\right)^p\right]\right\}\\ & \leq \frac{[\Delta_x(A)]^{(1-t)}[\Delta_x(B)]^t}{(\langle[(1-t)A^{-1}+tB^{-1}]x,x\rangle)^{-1}}\\ & \leq \exp\left\{\tilde{\Gamma}_p(m,M)\right.\\ & \times\left.\left[\left\langle\left[(1-t)A^{-p} + tB^{-p}\right]x,x\right\rangle - \left(\left\langle\left[(1-t)A^{-1}+tB^{-1}\right]x,x\right\rangle\right)^p\right]\right\}\end{aligned}$$

If $B = A$, then we get

$$\begin{aligned}(2.18)\qquad & 1 \leq \exp\left\{\left(\gamma_p(m,M)\left[\langle A^p x,x\rangle - \langle Ax,x\rangle^p\right]\right)\right\}\\ & \leq \frac{\langle Ax,x\rangle}{\Delta_x(A)} \leq \exp\left\{\Gamma_p(m,M)\left[\langle A^p x,x\rangle - \langle Ax,x\rangle^p\right]\right\}\end{aligned}$$

and

$$\begin{aligned}(2.19)\qquad & 1 \leq \exp\left\{\tilde{\gamma}_p(m,M)\left[\left\langle A^{-p}x,x\right\rangle - \left\langle A^{-1}x,x\right\rangle^p\right]\right\}\\ & \leq \frac{\Delta_x(A)}{(\langle A^{-1}x,x\rangle)^{-1}} \leq \exp\left\{\tilde{\Gamma}_p(m,M)\left[\left\langle A^{-p}x,x\right\rangle - \left\langle A^{-1}x,x\right\rangle^p\right]\right\}\end{aligned}$$

for $x \in H$, $\|x\| = 1$.

For $p = 2$ we have

$$\begin{aligned}(2.20)\qquad & 1 \leq \exp\left\{\left(\frac{1}{2M^2}\left[\left\langle A^2x,x\right\rangle - \langle Ax,x\rangle^2\right]\right)\right\}\\ & \leq \frac{\langle Ax,x\rangle}{\Delta_x(A)} \leq \exp\left\{\frac{1}{2m^2}\left[\left\langle A^2x,x\right\rangle - \langle Ax,x\rangle^2\right]\right\}\end{aligned}$$

and

$$\begin{aligned}(2.21)\qquad & 1 \leq \exp\left\{\frac{m^2}{2}\left[\left\langle A^{-2}x,x\right\rangle - \left\langle A^{-1}x,x\right\rangle^2\right]\right\}\\ & \leq \frac{\Delta_x(A)}{(\langle A^{-1}x,x\rangle)^{-1}} \leq \exp\left\{\frac{M^2}{2}\left[\left\langle A^{-2}x,x\right\rangle - \left\langle A^{-1}x,x\right\rangle^2\right]\right\}\end{aligned}$$

for $x \in H$, $\|x\| = 1$.

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For $p=-1$ we derive

$$1 \le \exp\left\{\left(\frac{m}{2}\left[\left\langle A^{-1}x,x\right\rangle - \langle Ax,x\rangle^{-1}\right]\right)\right\} \le \frac{\langle Ax,x\rangle}{\Delta_x(A)} \le \exp\left\{\frac{M}{2}\left[\left\langle A^{-1}x,x\right\rangle - \langle Ax,x\rangle^{-1}\right]\right\} \tag{2.22}$$

and

$$1 \le \exp\left\{\frac{1}{2M}\left[\langle Ax,x\rangle - \left\langle A^{-1}x,x\right\rangle^{-1}\right]\right\} \le \frac{\Delta_x(A)}{\left(\left\langle A^{-1}x,x\right\rangle\right)^{-1}} \le \exp\left\{\frac{1}{2m}\left[\langle Ax,x\rangle - \left\langle A^{-1}x,x\right\rangle^{-1}\right]\right\} \tag{2.23}$$

for $x \in H$, $\|x\| = 1$.

We observe that the above inequalities (2.18)-(2.23) provide refinements and reverse of the fundamental bounds for the normalized determinant incorporated in (ii) from the introduction.

It is well known that, see for instance [4, p. 28],

$$\left\langle A^2x,x\right\rangle - \langle Ax,x\rangle^2 \le \frac{1}{4}(M-m)^2 \tag{2.24}$$

for $x \in H$, $\|x\| = 1$.

Then by (2.20) we get

$$\frac{\langle Ax,x\rangle}{\Delta_x(A)} \le \exp\left\{\frac{1}{2m^2}\left[\left\langle A^2x,x\right\rangle - \langle Ax,x\rangle^2\right]\right\} \le \exp\left\{\frac{1}{8}\left(\frac{M}{m}-1\right)^2\right\} \tag{2.25}$$

for $x \in H$, $\|x\| = 1$.

Since $0 < M^{-1} \le A^{-1} \le m^{-1}$, hence

$$\left\langle A^{-2}x,x\right\rangle - \left\langle A^{-1}x,x\right\rangle^2 \le \frac{1}{4}\left(\frac{M-m}{mM}\right)^2$$

and by (2.21) we get

$$\begin{aligned} \frac{\Delta_x(A)}{\left(\left\langle A^{-1}x,x\right\rangle\right)^{-1}} &\le \exp\left\{\frac{M^2}{2}\left[\left\langle A^{-2}x,x\right\rangle - \left\langle A^{-1}x,x\right\rangle^2\right]\right\} \\ &\le \exp\left\{\frac{1}{8}\left(\frac{M}{m}-1\right)^2\right\} \end{aligned} \tag{2.26}$$

for $x \in H$, $\|x\| = 1$.

We also use the well known inequality, see for instance [4, p. 28],

$$\left\langle A^{-1}x,x\right\rangle - \langle Ax,x\rangle^{-1} \le \frac{\left(\sqrt{M}-\sqrt{m}\right)^2}{mM} \tag{2.27}$$

for $x \in H$, $\|x\| = 1$.

Then by (2.22) we obtain

$$\frac{\langle Ax,x\rangle}{\Delta_x(A)} \le \exp\left\{\frac{M}{2}\left[\left\langle A^{-1}x,x\right\rangle - \langle Ax,x\rangle^{-1}\right]\right\} \le \exp\left\{\frac{1}{2}\left(\sqrt{\frac{M}{m}}-1\right)^2\right\} \tag{2.28}$$

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for $x \in H$, $\|x\| = 1$.

Also, by (2.23) we derive

$$\frac{\Delta_x(A)}{(\langle A^{-1}x, x\rangle)^{-1}} \leq \exp\left\{\frac{1}{2m}\left[\langle Ax, x\rangle - \langle A^{-1}x, x\rangle^{-1}\right]\right\} \leq \exp\left\{\frac{1}{2}\left(\sqrt{\frac{M}{m}} - 1\right)^2\right\}.$$

These inequalities provide simple upper bounds related to the fundamental inequalities incorporated in (ii).

3. Related Results

We also have the following reverse of Ky Fan's inequality (viii):

Theorem 2. *Assume that $A_j > 0$ and $p_j \geq 0$, $j \in \{1, ..., n\}$ with $\sum_{j=1}^n p_j = 1$, then we have the following reverse of Ky Fan's inequality*

$$(3.1) \quad 1 \leq \frac{\Delta_x\left(\sum_{j=1}^n p_j A_j\right)}{\prod_{j=1}^n [\Delta_x(A_j)]^{p_j}} \leq \frac{\sum_{j=1}^n p_j \langle A_j x, x\rangle}{\prod_{j=1}^n [\Delta_x(A_j)]^{p_j}} \leq \exp\left(\Gamma_p(m, M)\left[\sum_{j=1}^n p_j \langle A_j^p x, x\rangle - \left(\sum_{j=1}^n p_j \langle A_j x, x\rangle\right)^p\right]\right)$$

for $x \in H$, $\|x\| = 1$.

In particular, we have

$$(3.2) \quad 1 \leq \frac{\Delta_x\left(\sum_{j=1}^n p_j A_j\right)}{\prod_{j=1}^n [\Delta_x(A_j)]^{p_j}} \leq \frac{\sum_{j=1}^n p_j \langle A_j x, x\rangle}{\prod_{j=1}^n [\Delta_x(A_j)]^{p_j}} \leq \exp\left(\frac{1}{2m^2}\left[\sum_{j=1}^n p_j \langle A_j^2 x, x\rangle - \left(\sum_{j=1}^n p_j \langle A_j x, x\rangle\right)^2\right]\right)$$

and

$$(3.3) \quad 1 \leq \frac{\Delta_x\left(\sum_{j=1}^n p_j A_j\right)}{\prod_{j=1}^n [\Delta_x(A_j)]^{p_j}} \leq \frac{\sum_{j=1}^n p_j \langle A_j x, x\rangle}{\prod_{j=1}^n [\Delta_x(A_j)]^{p_j}} \leq \exp\left(\frac{M}{2}\left[\sum_{j=1}^n p_j \langle A_j^{-1} x, x\rangle - \left(\sum_{j=1}^n p_j \langle A_j x, x\rangle\right)^{-1}\right]\right)$$

for $x \in H$, $\|x\| = 1$.

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Proof. Observe that, by Jensen's inequality for the concave function ln

$$\sum_{j=1}^{n} p_j \langle A_j x, x\rangle = \left\langle \left(\sum_{j=1}^{n} p_j A_j\right) x, x\right\rangle = \exp\left(\ln\left\langle \left(\sum_{j=1}^{n} p_j A_j\right) x, x\right\rangle\right)$$
$$\geq \exp\left\langle \ln\left(\sum_{j=1}^{n} p_j A_j\right) x, x\right\rangle = \Delta_x\left(\sum_{j=1}^{n} p_j A_j\right)$$

for $x \in H$, $\|x\| = 1$.

By the second inequality in (ii) from introduction and (2.10) we then get

$$\frac{\Delta_x\left(\sum_{j=1}^{n} p_j A_j\right)}{\prod_{j=1}^{n}[\Delta_x(A_j)]^{p_j}} \leq \frac{\sum_{j=1}^{n} p_j \langle A_j x, x\rangle}{\prod_{j=1}^{n}[\Delta_x(A_j)]^{p_j}}$$
$$\leq \exp\left(\Gamma_p(m,M)\left[\sum_{j=1}^{n} p_j \langle A_j^p x, x\rangle - \left(\sum_{j=1}^{n} p_j \langle A_j x, x\rangle\right)^p\right]\right)$$

for $x \in H$, $\|x\| = 1$.

If we use Ky Fan's type inequality (viii) and a standard induction argument we also have

$$1 \leq \frac{\Delta_x\left(\sum_{j=1}^{n} p_j A_j\right)}{\prod_{j=1}^{n}[\Delta_x(A_j)]^{p_j}}$$

for $x \in H$, $\|x\| = 1$.

These prove the desired result (3.1). □

Assume that $0 < m \leq A_j \leq M$, $j \in \{1,...,n\}$ and $x_j \in H$, $j \in \{1,...,n\}$ with $\sum_{j=1}^{n} \|x_j\|^2 = 1$. As in [4, p. 6], if we put

$$\widetilde{A} := \begin{pmatrix} A_1 & . & . & . & 0 \\ & . & & & \\ & & . & & \\ & & & . & \\ 0 & . & . & . & A_n \end{pmatrix} \text{ and } \widetilde{x} = \begin{pmatrix} x_1 \\ . \\ . \\ . \\ x_n \end{pmatrix}$$

then we have $Sp\left(\widetilde{A}\right) \subseteq [m, M]$, $\|\widetilde{x}\|_2 = 1$ and

$$\left\langle f\left(\widetilde{A}\right)\widetilde{x}, \widetilde{x}\right\rangle = \sum_{j=1}^{n} \langle f(A_j) x_j, x_j\rangle, \left\langle \widetilde{A}\widetilde{x}, \widetilde{x}\right\rangle = \sum_{j=1}^{n} \langle A_j x_j, x_j\rangle$$

for any continuous function f on $[m, M]$.

Therefore, by (2.24) and (2.27) we derive

$$\sum_{j=1}^{n} \left\langle A_j^2 x_j, x_j\right\rangle - \left(\sum_{j=1}^{n} \langle A_j x_j, x_j\rangle\right)^2 \leq \frac{1}{4}(M-m)^2 \tag{3.4}$$

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and

$$\sum_{j=1}^{n}\left\langle A_j x_j, x_j\right\rangle-\left(\sum_{j=1}^{n}\left\langle A_j^{-1} x_j, x_j\right\rangle\right)^{-1} \leq \frac{\left(\sqrt{M}-\sqrt{m}\right)^2}{mM} \tag{3.5}$$

provided that A_j are operators such that $0<m \leq A_j \leq M$, $j \in\{1, \ldots, n\}$ and $x_j \in H$, $j \in\{1, \ldots, n\}$ with $\sum_{j=1}^n\left\|x_j\right\|^2=1$.

Using (3.4) and (3.5) we get

$$\sum_{j=1}^{n} p_j\left\langle A_j^2 x, x\right\rangle-\left(\sum_{j=1}^{n} p_j\left\langle A_j x, x\right\rangle\right)^2 \leq \frac{1}{4}(M-m)^2 \tag{3.6}$$

and

$$\sum_{j=1}^{n} p_j\left\langle A_j x, x\right\rangle-\left(\sum_{j=1}^{n} p_j\left\langle A_j^{-1} x, x\right\rangle\right)^{-1} \leq \frac{\left(\sqrt{M}-\sqrt{m}\right)^2}{mM} \tag{3.7}$$

provided that A_j are operators such that $0<m \leq A_j \leq M$, $j \in\{1, \ldots, n\}$, $p_j \geq 0$, $j \in\{1, \ldots, n\}$ with $\sum_{j=1}^n p_j=1$ and $x \in H$, $\|x\|=1$.

Corollary 3. *With the assumptions of Theorem 2 we have the following reverses of Ky Fan's inequality*

$$\begin{aligned}
1 &\leq \frac{\Delta_x\left(\sum_{j=1}^n p_j A_j\right)}{\prod_{j=1}^{n}\left[\Delta_x\left(A_j\right)\right]^{p_j}} \leq \frac{\sum_{j=1}^n p_j\left\langle A_j x, x\right\rangle}{\prod_{j=1}^{n}\left[\Delta_x\left(A_j\right)\right]^{p_j}} \\
&\leq \exp\left(\frac{1}{2m^2}\left[\sum_{j=1}^{n} p_j\left\langle A_j^2 x, x\right\rangle-\left(\sum_{j=1}^{n} p_j\left\langle A_j x, x\right\rangle\right)^2\right]\right) \\
&\leq \exp\left\{\frac{1}{8}\left(\frac{M}{m}-1\right)^2\right\}
\end{aligned} \tag{3.8}$$

and

$$\begin{aligned}
1 &\leq \frac{\Delta_x\left(\sum_{j=1}^n p_j A_j\right)}{\prod_{j=1}^{n}\left[\Delta_x\left(A_j\right)\right]^{p_j}} \leq \frac{\sum_{j=1}^n p_j\left\langle A_j x, x\right\rangle}{\prod_{j=1}^{n}\left[\Delta_x\left(A_j\right)\right]^{p_j}} \\
&\leq \exp\left(\frac{M}{2}\left[\sum_{j=1}^{n} p_j\left\langle A_j^{-1} x, x\right\rangle-\left(\sum_{j=1}^{n} p_j\left\langle A_j x, x\right\rangle\right)^{-1}\right]\right) \\
&\leq \exp\left\{\frac{1}{2}\left(\sqrt{\frac{M}{m}}-1\right)^2\right\}
\end{aligned} \tag{3.9}$$

for all $x \in H$, $\|x\|=1$.

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4. Inequalities for $p \in (0,1)$

We also have:

Theorem 3. *Assume that A_j are operators such that $0 < m \leq A_j \leq M$, $j \in \{1,...,n\}$. Then for $p \in (0,1)$*

$$1 \leq \exp\left(\frac{1}{p(1-p)M^p}\left[\left(\sum_{j=1}^{n}\langle A_j x_j, x_j\rangle\right)^p - \sum_{j=1}^{n}\langle A_j^p x_j, x_j\rangle\right]\right) \tag{4.1}$$
$$\leq \frac{\sum_{j=1}^{n}\langle A_j x_j, x_j\rangle}{\prod_{j=1}^{n}\tilde{\Delta}_{x_j}(A_j)}$$
$$\leq \left(\frac{1}{p(1-p)m^p}\left[\left(\sum_{j=1}^{n}\langle A_j x_j, x_j\rangle\right)^p - \sum_{j=1}^{n}\langle A_j^p x_j, x_j\rangle\right]\right)$$

for each $x_j \in H$, $j \in \{1,...,n\}$ with $\sum_{j=1}^{n}\|x_j\|^2 = 1$.

In particular,

$$1 \leq \exp\left(\frac{4}{M^{1/2}}\left[\left(\sum_{j=1}^{n}\langle A_j x_j, x_j\rangle\right)^{1/2} - \sum_{j=1}^{n}\left\langle A_j^{1/2} x_j, x_j\right\rangle\right]\right) \tag{4.2}$$
$$\leq \frac{\sum_{j=1}^{n}\langle A_j x_j, x_j\rangle}{\prod_{j=1}^{n}\tilde{\Delta}_{x_j}(A_j)}$$
$$\leq \exp\left(\frac{4}{m^{1/2}}\left[\left(\sum_{j=1}^{n}\langle A_j x_j, x_j\rangle\right)^{1/2} - \sum_{j=1}^{n}\left\langle A_j^{1/2} x_j, x_j\right\rangle\right]\right)$$

for each $x_j \in H$, $j \in \{1,...,n\}$ with $\sum_{j=1}^{n}\|x_j\|^2 = 1$.

Proof. If the following condition is satisfied

$$\delta \leq h(t) := \frac{t^{2-p}}{p(1-p)}f''(t) \leq \Delta \text{ for any } t \in (m,M) \tag{4.3}$$

and for some $\delta < \Delta$, where $p \in (0,1)$, then for $p \in (0,1)$, we also have [3]

$$\delta\left[\left(\sum_{j=1}^{n}\langle A_j x_j, x_j\rangle\right)^p - \sum_{j=1}^{n}\langle A_j^p x_j, x_j\rangle\right] \tag{4.4}$$
$$\leq \sum_{j=1}^{n}\langle f(A_j)x_j, x_j\rangle - f\left(\sum_{j=1}^{n}\langle A_j x_j, x_j\rangle\right)$$
$$\leq \Delta\left[\left(\sum_{j=1}^{n}\langle A_j x_j, x_j\rangle\right)^p - \sum_{j=1}^{n}\langle A_j^p x_j, x_j\rangle\right]$$

for each $x_j \in H$, $j \in \{1,...,n\}$ with $\sum_{j=1}^{n}\|x_j\|^2 = 1$.

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If we take $f(t) = -\ln t$, then

$$\begin{aligned} h(t) &= \frac{t^{2-p}}{p(1-p)} \frac{1}{t^2} \\ &= \frac{1}{p(1-p)t^p} \in \left[\frac{1}{p(1-p)M^p}, \frac{1}{p(1-p)m^p}\right] \end{aligned}$$

and by (4.4) we get

$$\begin{aligned} 0 &\le \frac{1}{p(1-p)M^p}\left[\left(\sum_{j=1}^n \langle A_j x_j, x_j\rangle\right)^p - \sum_{j=1}^n \left\langle A_j^p x_j, x_j\right\rangle\right] \\ &\le \ln\left(\sum_{j=1}^n \langle A_j x_j, x_j\rangle\right) - \sum_{j=1}^n \langle \ln A_j x_j, x_j\rangle \\ &\le \frac{1}{p(1-p)m^p}\left[\left(\sum_{j=1}^n \langle A_j x_j, x_j\rangle\right)^p - \sum_{j=1}^n \left\langle A_j^p x_j, x_j\right\rangle\right] \end{aligned}$$

for each $x_j \in H$, $j \in \{1, ..., n\}$ with $\sum_{j=1}^n \|x_j\|^2 = 1$, which implies, by taking the exponential, the desired result (4.1). □

Corollary 4. *Assume that A_j are operators such that $0 < m \le A_j \le M$, $j \in \{1, ..., n\}$ and $p_j \ge 0$, $j \in \{1, ..., n\}$ with $\sum_{j=1}^n p_j = 1$. Then for $p \in (0, 1)$*

$$\begin{aligned} 1 &\le \exp\left(\frac{1}{p(1-p)M^p}\left[\left(\sum_{j=1}^n p_j \langle A_j x, x\rangle\right)^p - \sum_{j=1}^n p_j \left\langle A_j^p x, x\right\rangle\right]\right) \\ &\le \frac{\sum_{j=1}^n p_j \langle A_j x, x\rangle}{\prod_{j=1}^n [\Delta_x(A_j)]^{p_j}} \\ &\le \exp\left(\frac{1}{p(1-p)m^p}\left[\left(\sum_{j=1}^n p_j \langle A_j x, x\rangle\right)^p - \sum_{j=1}^n p_j \left\langle A_j^p x, x\right\rangle\right]\right) \end{aligned} \tag{4.5}$$

for each $x \in H$ with $\|x\| = 1$.

In particular,

$$\begin{aligned} 1 &\le \exp\left(\frac{4}{M^{1/2}}\left[\left(\sum_{j=1}^n p_j \langle A_j x, x\rangle\right)^{1/2} - \sum_{j=1}^n p_j \left\langle A_j^{1/2} x, x\right\rangle\right]\right) \\ &\le \frac{\sum_{j=1}^n p_j \langle A_j x, x\rangle}{\prod_{j=1}^n [\Delta_x(A_j)]^{p_j}} \\ &\le \exp\left(\frac{4}{m^{1/2}}\left[\left(\sum_{j=1}^n p_j \langle A_j x, x\rangle\right)^{1/2} - \sum_{j=1}^n p_j \left\langle A_j^{1/2} x, x\right\rangle\right]\right) \end{aligned} \tag{4.6}$$

for each $x \in H$ with $\|x\| = 1$.

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Remark 2. *If we write the above inequalities for A_j^{-1}, then, under the same assumptions,*

$$1 \le \exp\left(\frac{m^p}{p(1-p)}\left[\left(\sum_{j=1}^n \left\langle A_j^{-1}x_j, x_j\right\rangle\right)^p - \sum_{j=1}^n \left\langle A_j^{-p}x_j, x_j\right\rangle\right]\right) \le \frac{\prod_{j=1}^n \tilde{\Delta}_{x_j}(A_j)}{\left(\sum_{j=1}^n \left\langle A_j^{-1}x_j, x_j\right\rangle\right)^{-1}} \le \exp\left(\frac{M^p}{p(1-p)}\left[\left(\sum_{j=1}^n \left\langle A_j^{-1}x_j, x_j\right\rangle\right)^p - \sum_{j=1}^n \left\langle A_j^{-p}x_j, x_j\right\rangle\right]\right) \tag{4.7}$$

for each $x_j \in H$, $j \in \{1, ..., n\}$ with $\sum_{j=1}^n \|x_j\|^2 = 1$.

In particular,

$$1 \le \exp\left(4m^{1/2}\left[\left(\sum_{j=1}^n \left\langle A_j^{-1}x_j, x_j\right\rangle\right)^{1/2} - \sum_{j=1}^n \left\langle A_j^{-1/2}x_j, x_j\right\rangle\right]\right) \le \frac{\prod_{j=1}^n \tilde{\Delta}_{x_j}(A_j)}{\left(\sum_{j=1}^n \left\langle A_j^{-1}x_j, x_j\right\rangle\right)^{-1}} \le \exp\left(4M^{1/2}\left[\left(\sum_{j=1}^n \left\langle A_j x_j, x_j\right\rangle\right)^{1/2} - \sum_{j=1}^n \left\langle A_j^{1/2}x_j, x_j\right\rangle\right]\right) \tag{4.8}$$

for each $x_j \in H$, $j \in \{1, ..., n\}$ with $\sum_{j=1}^n \|x_j\|^2 = 1$.

Also, if $p_j \ge 0$, $j \in \{1, ..., n\}$ with $\sum_{j=1}^n p_j = 1$

$$1 \le \exp\left(\frac{m^p}{p(1-p)}\left[\left(\sum_{j=1}^n p_j \left\langle A_j^{-1}x, x\right\rangle\right)^p - \sum_{j=1}^n p_j \left\langle A_j^{-p}x, x\right\rangle\right]\right) \le \frac{\prod_{j=1}^n \left[\Delta_x(A_j)\right]^{p_j}}{\left(\sum_{j=1}^n p_j \left\langle A_j^{-1}x, x\right\rangle\right)^{-1}} \le \exp\left(\frac{M^p}{p(1-p)}\left[\left(\sum_{j=1}^n p_j \left\langle A_j^{-1}x, x\right\rangle\right)^p - \sum_{j=1}^n p_j \left\langle A_j^{-p}x, x\right\rangle\right]\right) \tag{4.9}$$

for each $x \in H$ with $\|x\| = 1$.

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In particular,

$$1 \le \exp\left(4m^{1/2}\left[\left(\sum_{j=1}^{n} p_j \left\langle A_j^{-1}x, x\right\rangle\right)^{1/2} - \sum_{j=1}^{n} p_j \left\langle A_j^{-1/2}x, x\right\rangle\right]\right) \tag{4.10}$$
$$\le \frac{\prod_{j=1}^{n} [\Delta_x(A_j)]^{p_j}}{\left(\sum_{j=1}^{n} p_j \left\langle A_j^{-1}x, x\right\rangle\right)^{-1}}$$
$$\le \exp\left(4M^{1/2}\left[\left(\sum_{j=1}^{n} p_j \left\langle A_j^{-1}x, x\right\rangle\right)^{1/2} - \sum_{j=1}^{n} p_j \left\langle A_j^{-1/2}x, x\right\rangle\right]\right)$$

for each $x \in H$ with $\|x\| = 1$.

Similar particular inequalities may be stated, however we state only the case of one operators, namely, for the operator A satisfying the condition $0 < m \le A \le M$,

$$1 \le \exp\left(\frac{1}{p(1-p)M^p}\left[\langle Ax, x\rangle^p - \langle A^p x, x\rangle\right]\right) \tag{4.11}$$
$$\le \frac{\langle Ax, x\rangle}{\Delta_x(A)} \le \exp\left(\frac{1}{p(1-p)m^p}\left[\langle Ax, x\rangle^p - \langle A^p x, x\rangle\right]\right)$$

and

$$1 \le \exp\left(\frac{m^p}{p(1-p)}\left[\left\langle A^{-1}x, x\right\rangle^p - \left\langle A^{-p}x, x\right\rangle\right]\right) \tag{4.12}$$
$$\le \frac{\Delta_x(A)}{\langle A^{-1}x, x\rangle^{-1}} \le \exp\left(\frac{M^p}{p(1-p)}\left[\left\langle A^{-1}x, x\right\rangle^p - \left\langle A^{-p}x, x\right\rangle\right]\right)$$

for each $x \in H$ with $\|x\| = 1$, where $p \in (0, 1)$.

For $p = 1/2$ we get

$$1 \le \exp\left(\frac{4}{M^{1/2}}\left[\langle Ax, x\rangle^{1/2} - \left\langle A^{1/2}x, x\right\rangle\right]\right) \tag{4.13}$$
$$\le \frac{\langle Ax, x\rangle}{\Delta_x(A)} \le \exp\left(\frac{4}{m^{1/2}}\left[\langle Ax, x\rangle^{1/2} - \left\langle A^{1/2}x, x\right\rangle\right]\right)$$

and

$$1 \le \exp\left(4m^{1/2}\left[\left\langle A^{-1}x, x\right\rangle^{1/2} - \left\langle A^{-1/2}x, x\right\rangle\right]\right) \tag{4.14}$$
$$\le \frac{\Delta_x(A)}{\langle A^{-1}x, x\rangle^{-1}} \le \exp\left(4M^{1/2}\left[\left\langle A^{-1}x, x\right\rangle^{1/2} - \left\langle A^{-1/2}x, x\right\rangle\right]\right)$$

for each $x \in H$ with $\|x\| = 1$.

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RGMIA

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