BOUNDS FOR THE NORMALIZED DETERMINANT OF HADAMARD PRODUCT OF TWO POSITIVE OPERATORS IN HILBERT SPACES

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Abstract. For positive invertible operators A on a Hilbert space H and a fixed unit vector $x \in H$, define the normalized determinant by $\Delta_x(A) := \exp \langle \ln Ax, x\rangle$. In this paper we obtain upper and lower bounds for the determinant $\Delta_x(A \circ B)$ of the Hadamard product of two operators under some natural assumptions such as $0 < m_1 \leq A \leq M_1$ and $0 < m_2 \leq B \leq M_2$, where m_i, M_i $(i = 1, 2)$ are constants.

1. Introduction

Let $B(H)$ be the space of all bounded linear operators on a Hilbert space H, and I stands for the identity operator on H. An operator A in $B(H)$ is said to be positive (in symbol: $A \geq 0$) if $\langle Ax, x\rangle \geq 0$ for all $x \in H$. In particular, $A > 0$ means that A is positive and invertible. For a pair A, B of selfadjoint operators the order relation $A \geq B$ means as usual that $A - B$ is positive.

In 1998, Fujii et al. [5], [6], introduced the *normalized determinant* $\Delta_x(A)$ for positive invertible operators A on a Hilbert space H and a fixed unit vector $x \in H$, namely $\|x\| = 1$, defined by $\Delta_x(A) := \exp \langle \ln Ax, x\rangle$ and discussed it as a continuous geometric mean and observed some inequalities around the determinant from this point of view.

Some of the fundamental properties of normalized determinant are as follows, [5].

For each unit vector $x \in H$, see also [10], we have:

(i) *continuity*: the map $A \to \Delta_x(A)$ is norm continuous;
(ii) *bounds*: $\langle A^{-1}x, x\rangle^{-1} \leq \Delta_x(A) \leq \langle Ax, x\rangle$;
(iii) *continuous mean*: $\langle A^p x, x\rangle^{1/p} \downarrow \Delta_x(A)$ for $p \downarrow 0$ and $\langle A^p x, x\rangle^{1/p} \uparrow \Delta_x(A)$ for $p \uparrow 0$;
(iv) *power equality:* $\Delta_x(A^t) = \Delta_x(A)^t$ for all $t > 0$;
(v) *homogeneity*: $\Delta_x(tA) = t\Delta_x(A)$ and $\Delta_x(tI) = t$ for all $t > 0$;
(vi) *monotonicity*: $0 < A \leq B$ implies $\Delta_x(A) \leq \Delta_x(B)$;
(vii) *multiplicativity*: $\Delta_x(AB) = \Delta_x(A)\Delta_x(B)$ for commuting A and B;
(viii) *Ky Fan type inequality*: $\Delta_x((1-\alpha)A + \alpha B) \geq \Delta_x(A)^{1-\alpha}\Delta_x(B)^\alpha$ for $0 < \alpha < 1$.

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We define the logarithmic mean of two positive numbers a, b by

$$L(a,b) := \begin{cases} \frac{b-a}{\ln b - \ln a} \text{ if } b \neq a, \\ \\ a \text{ if } b = a. \end{cases}$$

In [5] the authors obtained the following additive reverse inequality for the operator A which satisfy the condition $0 < mI \leq A \leq MI$, where m, M are positive numbers,

$$0 \leq \langle Ax, x\rangle - \Delta_x(A) \leq L(m,M)\left[\ln L(m,M) + \frac{M \ln m - m \ln M}{M-m} - 1\right] \tag{1.1}$$

for all $x \in H$, $\|x\| = 1$.

The famous *Young inequality* for scalars says that if $a, b > 0$ and $\nu \in [0,1]$, then

$$a^{1-\nu} b^{\nu} \leq (1-\nu) a + \nu b \tag{1.2}$$

with equality if and only if $a = b$. The inequality (1.2) is also called *ν-weighted arithmetic-geometric mean inequality*.

We recall that *Specht's ratio* is defined by [14]

$$S(h) := \begin{cases} \dfrac{h^{\frac{1}{h-1}}}{e \ln\left(h^{\frac{1}{h-1}}\right)} \text{ if } h \in (0,1) \cup (1,\infty) \\ \\ 1 \text{ if } h = 1. \end{cases} \tag{1.3}$$

It is well known that $\lim_{h \to 1} S(h) = 1$, $S(h) = S\left(\frac{1}{h}\right) > 1$ for $h > 0$, $h \neq 1$. The function is decreasing on $(0,1)$ and increasing on $(1,\infty)$.

In [6], the authors obtained the following multiplicative reverse inequality as well

$$1 \leq \frac{\langle Ax, x\rangle}{\Delta_x(A)} \leq S\left(\frac{M}{m}\right) \tag{1.4}$$

for $0 < mI \leq A \leq MI$ and $x \in H$, $\|x\| = 1$.

Since $0 < M^{-1}I \leq A^{-1} \leq m^{-1}I$, then by (1.4) for A^{-1} we get

$$1 \leq \frac{\langle A^{-1}x, x\rangle}{\Delta_x(A^{-1})} \leq S\left(\frac{m^{-1}}{M^{-1}}\right) = S\left(\left(\frac{m}{M}\right)^{-1}\right) = S\left(\frac{M}{m}\right),$$

which is equivalent to

$$1 \leq \frac{\Delta_x(A)}{\langle A^{-1}x, x\rangle^{-1}} \leq S\left(\frac{M}{m}\right) \tag{1.5}$$

for $x \in H$, $\|x\| = 1$.

We consider the *Kantorovich's constant* defined by

$$K(h) := \frac{(h+1)^2}{4h}, \ h > 0. \tag{1.6}$$

The function K is decreasing on $(0,1)$ and increasing on $[1,\infty)$, $K(h) \geq 1$ for any $h > 0$ and $K(h) = K\left(\frac{1}{h}\right)$ for any $h > 0$.

The following multiplicative refinement and reverse of Young inequality in terms of Kantorovich's constant holds

$$K^r\left(\frac{b}{a}\right) a^{1-\nu} b^{\nu} \leq (1-\nu) a + \nu b \leq K^R\left(\frac{b}{a}\right) a^{1-\nu} b^{\nu} \tag{1.7}$$

where a, $b > 0$, $\nu \in [0,1]$, $r = \min\{1-\nu, \nu\}$ and $R = \max\{1-\nu, \nu\}$.

The first inequality in (1.7) was obtained by Zuo et al. in [18] while the second by Liao et al. [13].

Recall the *geometric operator mean* for the positive operators $A,\, B>0$

$$A\#_t B:=A^{1/2}(A^{-1/2}BA^{-1/2})^t A^{1/2}$$

where $t\in[0,1]$ and

$$A\#B:=A^{1/2}(A^{-1/2}BA^{-1/2})^{1/2}A^{1/2}.$$

By the definitions of $\#$ and $\otimes$ we have

$$A\#B=B\#A \text{ and } (A\#B)\otimes(B\#A)=(A\otimes B)\,\#\,(B\otimes A)\,.$$

In 2007, S. Wada [16] obtained the following *Callebaut type inequalities* for tensorial product

$$\begin{aligned}(A\#B)\otimes(A\#B)&\leq\frac{1}{2}\left[(A\#_\alpha B)\otimes(A\#_{1-\alpha}B)+(A\#_{1-\alpha}B)\otimes(A\#_\alpha B)\right]\\&\leq\frac{1}{2}\left(A\otimes B+B\otimes A\right)\end{aligned}\tag{1.8}$$

for $A,\, B>0$ and $\alpha\in[0,1]\,.$

Recall that the *Hadamard product* of A and B in $B(H)$ is defined to be the operator $A\circ B\in B(H)$ satisfying

$$\langle (A\circ B)\, e_j, e_j\rangle=\langle Ae_j,e_j\rangle\,\langle Be_j,e_j\rangle$$

for all $j\in\mathbb{N}$, where $\{e_j\}_{j\in\mathbb{N}}$ is an *orthonormal basis* for the separable Hilbert space H.

It is known that, see [4], we have the representation

$$A\circ B=\mathcal{U}^*\,(A\otimes B)\,\mathcal{U}\tag{1.9}$$

where $\mathcal{U}:H\to H\otimes H$ is the isometry defined by $\mathcal{U}e_j=e_j\otimes e_j$ for all $j\in\mathbb{N}$.

If f *is super-multiplicative (sub-multiplicative)* on $[0,\infty)\,,$ then also [9, p. 173]

$$f\,(A\circ B)\geq(\leq)\,f\,(A)\circ f\,(B)\text{ for all } A,\ B\geq 0.\tag{1.10}$$

We recall the following elementary inequalities for the Hadamard product

$$A^{1/2}\circ B^{1/2}\leq\left(\frac{A+B}{2}\right)\circ 1\text{ for } A,\ B\geq 0$$

and *Fiedler inequality*

$$A\circ A^{-1}\geq 1\text{ for } A>0.\tag{1.11}$$

As extension of Kadison's Schwarz inequality on the Hadamard product, Ando [1] showed that

$$A\circ B\leq\left(A^2\circ 1\right)^{1/2}\left(B^2\circ 1\right)^{1/2}\ \text{ for } A,\ B\geq 0$$

and Aujla and Vasudeva [3] gave an alternative upper bound

$$A\circ B\leq\left(A^2\circ B^2\right)^{1/2}\ \text{ for } A,\ B\geq 0.$$

It has been shown in [11] that $\left(A^2\circ 1\right)^{1/2}\left(B^2\circ 1\right)^{1/2}$ and $\left(A^2\circ B^2\right)^{1/2}$ are incomparable for 2-square positive definite matrices A and B.

Motivated by the above results, we establish in this paper the following upper and lower bounds for the determinant $\Delta_x (A \circ B)$

$$
\begin{aligned}
&(m_1 m_2)^{\frac{M_1 M_2 - \langle (A\circ B)x,x\rangle}{M_1 M_2 - m_1 m_2}} (M_1 M_2)^{\frac{\langle (A\circ B)x,x\rangle - m_1 m_2}{M_1 M_2 - m_1 m_2}} \\
&\le K\left(\frac{M_1 M_2}{m_1 m_2}\right)^{\left(\frac{1}{2} - \frac{1}{M_1 M_2 - m_1 m_2}\left\langle \mathcal{U}^*\left(\left|A\otimes B - \frac{m_1 m_2 + M_1 M_2}{2}\right|\right)\mathcal{U}x,x\right\rangle\right)} \\
&\times (m_1 m_2)^{\frac{M_1 M_2 - \langle (A\circ B)x,x\rangle}{M_1 M_2 - m_1 m_2}} (M_1 M_2)^{\frac{\langle (A\circ B)x,x\rangle - m_1 m_2}{M_1 M_2 - m_1 m_2}} \\
&\le \exp\left[\langle \mathcal{U}^* (\ln (A\otimes B))\mathcal{U}x,x\rangle\right] \\
&\le \Delta_x (A\circ B) \\
&\le K\left(\frac{M_1 M_2}{m_1 m_2}\right)^{\left(\frac{1}{2} + \frac{1}{M_1 M_2 - m_1 m_2}\left\langle \left|A\circ B - \frac{m_1 m_2 + M_1 M_2}{2}\right|x,x\right\rangle\right)} \\
&\times (m_1 m_2)^{\frac{M_1 M_2 - \langle (A\circ B)x,x\rangle}{M_1 M_2 - m_1 m_2}} (M_1 M_2)^{\frac{\langle (A\circ B)x,x\rangle - m_1 m_2}{M_1 M_2 - m_1 m_2}} \\
&\le K\left(\frac{M_1 M_2}{m_1 m_2}\right)(m_1 m_2)^{\frac{M_1 M_2 - \langle (A\circ B)x,x\rangle}{M_1 M_2 - m_1 m_2}} (M_1 M_2)^{\frac{\langle (A\circ B)x,x\rangle - m_1 m_2}{M_1 M_2 - m_1 m_2}} \\
&\le K\left(\frac{M_1 M_2}{m_1 m_2}\right)\exp\langle \mathcal{U}^* (\ln (A\otimes B))\mathcal{U}x,x\rangle ,
\end{aligned}
$$

provided that $0 < m_1 \le A \le M_1$ and $0 < m_2 \le B \le M_2$.

2. Main Results

We start to the following operator inequalities involved positive operators and positive linear maps:

Theorem 1. *Assume that the selfadjoint operator P satisfies the condition $0 < m \le P \le M$ for some constants, m, M and Φ a unital positive linear map from $B(H)$ into $B(K)$. Then*

$$
\begin{aligned}
&\ln m \frac{M-\Phi(P)}{M-m} + \ln M \frac{\Phi(P)-m}{M-m} \\
&\le \ln K\left(\frac{M}{m}\right)\left(\frac{1}{2} - \frac{1}{M-m}\left|\Phi(P) - \frac{m+M}{2}\right|\right) \\
&+ \ln m \frac{M-\Phi(P)}{M-m} + \ln M \frac{\Phi(P)-m}{M-m} \\
&\le \ln \Phi(P) \\
&\le \ln K\left(\frac{M}{m}\right)\left(\frac{1}{2} + \frac{1}{M-m}\left|\Phi(P) - \frac{m+M}{2}\right|\right) \\
&+ \ln m \frac{M-\Phi(P)}{M-m} + \ln M \frac{\Phi(P)-m}{M-m}
\end{aligned} \tag{2.1}
$$

and

$$
\begin{aligned}
(2.2)\qquad & \ln m\frac{M-\Phi(P)}{M-m}+\ln M\frac{\Phi(P)-m}{M-m} \\
& \le \ln K\left(\frac{M}{m}\right)\left(\frac{1}{2}-\frac{1}{M-m}\Phi\left(\left|P-\frac{m+M}{2}\right|\right)\right) \\
& + \ln m\frac{M-\Phi(P)}{M-m}+\ln M\frac{\Phi(P)-m}{M-m} \\
& \le \Phi(\ln P) \\
& \le \ln K\left(\frac{M}{m}\right)\left(\frac{1}{2}+\frac{1}{M-m}\Phi\left(\left|T-\frac{m+M}{2}\right|\right)\right) \\
& + \ln m\frac{M-\Phi(P)}{M-m}+\ln M\frac{\Phi(P)-m}{M-m}.
\end{aligned}
$$

Proof. Assume that $0<a<b$. If we take $\nu=\frac{t-a}{b-a}\in[0,1]$ for $t\in[a,b]$ and observe that

$$
\begin{aligned}
r &= \min\left\{\frac{t-a}{b-a},\frac{b-t}{b-a}\right\}=\frac{1}{2}-\frac{1}{b-a}\left|t-\frac{a+b}{2}\right|, \\
R &= \max\left\{\frac{t-a}{b-a},\frac{b-t}{b-a}\right\}=\frac{1}{2}+\frac{1}{b-a}\left|t-\frac{a+b}{2}\right|
\end{aligned}
$$

and

$$
(1-\nu)a+\nu b=\frac{b-t}{b-a}a+\frac{t-a}{b-a}b=t.
$$

By utilizing (1.7) we get

$$
K^{\frac{1}{2}-\frac{1}{b-a}\left|t-\frac{a+b}{2}\right|}\left(\frac{b}{a}\right)a^{\frac{b-t}{b-a}}b^{\frac{t-a}{b-a}}\le t\le K^{\frac{1}{2}+\frac{1}{b-a}\left|t-\frac{a+b}{2}\right|}\left(\frac{b}{a}\right)a^{\frac{b-t}{b-a}}b^{\frac{t-a}{b-a}} \tag{2.3}
$$

for all $t\in[a,b]$.

If we take the logarithm in (2.3), then we get

$$
\begin{aligned}
(2.4)\qquad & \ln a\frac{b-t}{b-a}+\ln b\frac{t-a}{b-a} \\
& \le \ln K\left(\frac{b}{a}\right)\left(\frac{1}{2}-\frac{1}{b-a}\left|t-\frac{a+b}{2}\right|\right)+\ln a\frac{b-t}{b-a}+\ln b\frac{t-a}{b-a} \\
& \le \ln t \\
& \le \ln K\left(\frac{b}{a}\right)\left(\frac{1}{2}+\frac{1}{b-a}\left|t-\frac{a+b}{2}\right|\right)+\ln a\frac{b-t}{b-a}+\ln b\frac{t-a}{b-a}
\end{aligned}
$$

for all $t\in[a,b]$.

By utilizing the continuous functional calculus for selfadjoint operators T with spectra $\operatorname{Sp}(T)\subseteq[a,b]$, we obtain from (2.4) that

$$
\begin{aligned}
(2.5)\qquad & \ln a\frac{b-T}{b-a}+\ln b\frac{T-a}{b-a} \\
& \le \ln K\left(\frac{b}{a}\right)\left(\frac{1}{2}-\frac{1}{b-a}\left|T-\frac{a+b}{2}\right|\right)+\ln a\frac{b-T}{b-a}+\ln b\frac{T-a}{b-a} \\
& \le \ln T \\
& \le \ln K\left(\frac{b}{a}\right)\left(\frac{1}{2}+\frac{1}{b-a}\left|T-\frac{a+b}{2}\right|\right)+\ln a\frac{b-T}{b-a}+\ln b\frac{T-a}{b-a}.
\end{aligned}
$$

Now if $0 < m \leq P \leq M$, then $0 < m \leq \Phi(P) \leq M$ and by (2.5) we get for $T = \Phi(P)$, $a = m$ and $b = M$ the inequality (2.1). If we take $T = P$, $a = m$ and $b = M$ in (2.5) and then apply Φ we also obtain (2.2). □

Corollary 1. *With the assumptions of Theorem 1 we have the chain of inequalities*

$$\begin{aligned}
&\ln m \frac{M - \Phi(P)}{M - m} + \ln M \frac{\Phi(P) - m}{M - m} \\
&\leq \ln K\left(\frac{M}{m}\right)\left(\frac{1}{2} - \frac{1}{M - m}\Phi\left(\left|P - \frac{m + M}{2}\right|\right)\right) \\
&+ \ln m \frac{M - \Phi(P)}{M - m} + \ln M \frac{\Phi(P) - m}{M - m} \\
&\leq \Phi(\ln P) \leq \ln \Phi(P) \\
&\leq \ln K\left(\frac{M}{m}\right)\left(\frac{1}{2} + \frac{1}{M - m}\left|\Phi(P) - \frac{m + M}{2}\right|\right) \\
&+ \ln m \frac{M - \Phi(P)}{M - m} + \ln M \frac{\Phi(P) - m}{M - m} \\
&\leq \ln K\left(\frac{M}{m}\right) + \ln m \frac{M - \Phi(P)}{M - m} + \ln M \frac{\Phi(P) - m}{M - m} \\
&\leq \ln K\left(\frac{M}{m}\right) + \Phi(\ln P).
\end{aligned} \tag{2.6}$$

Proof. Third inequality follows by Jensen's operator inequality for the operator concave function $\ln$. The fifth inequality follows by the fact that

$$\left|\Phi(P) - \frac{m + M}{2}\right| \leq \frac{1}{2}(M - m),$$

while the last inequality follows by the fact that

$$\ln m \frac{M - \Phi(P)}{M - m} + \ln M \frac{\Phi(P) - m}{M - m} \leq \Phi(\ln P)$$

from the first part of (2.6). □

Theorem 2. *Assume that* $0 < m_1 \le A \le M_1$ *and* $0 < m_2 \le B \le M_2$*, then*

$$
\begin{aligned}
(2.7)\qquad & (m_1 m_2)^{\frac{M_1M_2 - \langle (A\circ B)x,x\rangle}{M_1M_2 - m_1m_2}} (M_1M_2)^{\frac{\langle (A\circ B)x,x\rangle - m_1m_2}{M_1M_2 - m_1m_2}} \\
& \le K\left(\frac{M_1M_2}{m_1m_2}\right)^{\left(\frac{1}{2} - \frac{1}{M_1M_2 - m_1m_2}\left\langle \mathcal{U}^*\left(\left|A\otimes B - \frac{m_1m_2 + M_1M_2}{2}\right|\right)\mathcal{U}x,x\right\rangle\right)} \\
& \times (m_1 m_2)^{\frac{M_1M_2 - \langle (A\circ B)x,x\rangle}{M_1M_2 - m_1m_2}} (M_1M_2)^{\frac{\langle (A\circ B)x,x\rangle - m_1m_2}{M_1M_2 - m_1m_2}} \\
& \le \exp\left[\langle \mathcal{U}^*\left(\ln\left(A\otimes B\right)\right)\mathcal{U}x,x\rangle\right] \\
& \le \Delta_x\left(A\circ B\right) \\
& \le K\left(\frac{M_1M_2}{m_1m_2}\right)^{\left(\frac{1}{2} + \frac{1}{M_1M_2 - m_1m_2}\left\langle \left|A\circ B - \frac{m_1m_2 + M_1M_2}{2}\right|x,x\right\rangle\right)} \\
& \times (m_1 m_2)^{\frac{M_1M_2 - \langle (A\circ B)x,x\rangle}{M_1M_2 - m_1m_2}} (M_1M_2)^{\frac{\langle (A\circ B)x,x\rangle - m_1m_2}{M_1M_2 - m_1m_2}} \\
& \le K\left(\frac{M_1M_2}{m_1m_2}\right)(m_1 m_2)^{\frac{M_1M_2 - \langle (A\circ B)x,x\rangle}{M_1M_2 - m_1m_2}} (M_1M_2)^{\frac{\langle (A\circ B)x,x\rangle - m_1m_2}{M_1M_2 - m_1m_2}} \\
& \le K\left(\frac{M_1M_2}{m_1m_2}\right)\exp\langle \mathcal{U}^*\left(\ln\left(A\otimes B\right)\right)\mathcal{U}x,x\rangle,
\end{aligned}
$$

for $x \in H$, $\|x\| = 1$.

Proof. Since $0 < m_1 \le A \le M_1$ and $0 < m_2 \le B \le M_2$, then $0 < m_1m_2 \le P = A\otimes B \le M_1M_2$. From (2.6) for $m = m_1m_2$, $M = M_1M_2$, $\Phi(P) = \mathcal{U}^*(A\otimes B)\mathcal{U} = A\circ B$ we get

$$
\begin{aligned}
(2.8)\qquad & \ln(m_1m_2)\frac{M_1M_2 - A\circ B}{M_1M_2 - m_1m_2} + \ln(M_1M_2)\frac{A\circ B - m_1m_2}{M_1M_2 - m_1m_2} \\
& \le \ln K\left(\frac{M_1M_2}{m_1m_2}\right) \\
& \times\left(\frac{1}{2} - \frac{1}{M_1M_2 - m_1m_2}\mathcal{U}^*\left(\left|A\otimes B - \frac{m_1m_2 + M_1M_2}{2}\right|\right)\mathcal{U}\right) \\
& + \ln(m_1m_2)\frac{M_1M_2 - A\circ B}{M_1M_2 - m_1m_2} + \ln(M_1M_2)\frac{A\circ B - m_1m_2}{M_1M_2 - m_1m_2} \\
& \le \mathcal{U}^*\left(\ln\left(A\otimes B\right)\right)\mathcal{U} \le \ln\left(A\circ B\right) \\
& \le \ln K\left(\frac{M_1M_2}{m_1m_2}\right)\left(\frac{1}{2} + \frac{1}{M_1M_2 - m_1m_2}\left|A\circ B - \frac{m_1m_2 + M_1M_2}{2}\right|\right) \\
& + \ln(m_1m_2)\frac{M_1M_2 - A\circ B}{M_1M_2 - m_1m_2} + \ln(M_1M_2)\frac{A\circ B - m_1m_2}{M_1M_2 - m_1m_2} \\
& \le \ln K\left(\frac{M_1M_2}{m_1m_2}\right) \\
& + \ln(m_1m_2)\frac{M_1M_2 - A\circ B}{M_1M_2 - m_1m_2} + \ln(M_1M_2)\frac{A\circ B - m_1m_2}{M_1M_2 - m_1m_2} \\
& \le \ln K\left(\frac{M_1M_2}{m_1m_2}\right) + \mathcal{U}^*\left(\ln\left(A\otimes B\right)\right)\mathcal{U}.
\end{aligned}
$$

If we take the inner product for $x \in H$, $\|x\| = 1$, then we get

$$\begin{aligned}
&\ln(m_1m_2)\frac{M_1M_2-\langle(A\circ B)x,x\rangle}{M_1M_2-m_1m_2}+\ln(M_1M_2)\frac{\langle(A\circ B)x,x\rangle-m_1m_2}{M_1M_2-m_1m_2}\\
&\leq \ln K\left(\frac{M_1M_2}{m_1m_2}\right)\\
&\times\left(\frac{1}{2}-\frac{1}{M_1M_2-m_1m_2}\left\langle\mathcal{U}^*\left(\left|A\otimes B-\frac{m_1m_2+M_1M_2}{2}\right|\right)\mathcal{U}x,x\right\rangle\right)\\
&+\ln(m_1m_2)\frac{M_1M_2-A\circ B}{M_1M_2-m_1m_2}+\ln(M_1M_2)\frac{A\circ B-m_1m_2}{M_1M_2-m_1m_2}\\
&\leq\langle\mathcal{U}^*(\ln(A\otimes B))\mathcal{U}x,x\rangle\leq\langle\ln(A\circ B)x,x\rangle\\
&\leq \ln K\left(\frac{M_1M_2}{m_1m_2}\right)\\
&\times\left(\frac{1}{2}+\frac{1}{M_1M_2-m_1m_2}\left\langle\left|A\circ B-\frac{m_1m_2+M_1M_2}{2}\right|x,x\right\rangle\right)\\
&+\ln(m_1m_2)\frac{M_1M_2-\langle(A\circ B)x,x\rangle}{M_1M_2-m_1m_2}+\ln(M_1M_2)\frac{\langle(A\circ B)x,x\rangle-m_1m_2}{M_1M_2-m_1m_2}\\
&\leq \ln K\left(\frac{M_1M_2}{m_1m_2}\right)\\
&+\ln(m_1m_2)\frac{M_1M_2-\langle(A\circ B)x,x\rangle}{M_1M_2-m_1m_2}+\ln(M_1M_2)\frac{\langle(A\circ B)x,x\rangle-m_1m_2}{M_1M_2-m_1m_2}\\
&\leq \ln K\left(\frac{M_1M_2}{m_1m_2}\right)+\langle\mathcal{U}^*(\ln(A\otimes B))\mathcal{U}x,x\rangle,
\end{aligned}$$

namely

$$\begin{aligned}
(2.9)\quad &\ln\left[(m_1m_2)^{\frac{M_1M_2-\langle(A\circ B)x,x\rangle}{M_1M_2-m_1m_2}}(M_1M_2)^{\frac{\langle(A\circ B)x,x\rangle-m_1m_2}{M_1M_2-m_1m_2}}\right]\\
&\leq\ln\left[K\left(\frac{M_1M_2}{m_1m_2}\right)\right]^{\left(\frac{1}{2}-\frac{1}{M_1M_2-m_1m_2}\left\langle\mathcal{U}^*\left(\left|A\otimes B-\frac{m_1m_2+M_1M_2}{2}\right|\right)\mathcal{U}x,x\right\rangle\right)}\\
&+\ln\left[(m_1m_2)^{\frac{M_1M_2-\langle(A\circ B)x,x\rangle}{M_1M_2-m_1m_2}}(M_1M_2)^{\frac{\langle(A\circ B)x,x\rangle-m_1m_2}{M_1M_2-m_1m_2}}\right]\\
&\leq\langle\mathcal{U}^*(\ln(A\otimes B))\mathcal{U}x,x\rangle\leq\langle\ln(A\circ B)x,x\rangle\\
&\leq\ln\left[K\left(\frac{M_1M_2}{m_1m_2}\right)\right]^{\left(\frac{1}{2}+\frac{1}{M_1M_2-m_1m_2}\left\langle\left|A\circ B-\frac{m_1m_2+M_1M_2}{2}\right|x,x\right\rangle\right)}\\
&+\ln(m_1m_2)\frac{M_1M_2-\langle(A\circ B)x,x\rangle}{M_1M_2-m_1m_2}+\ln(M_1M_2)\frac{\langle(A\circ B)x,x\rangle-m_1m_2}{M_1M_2-m_1m_2}\\
&\leq\ln K\left(\frac{M_1M_2}{m_1m_2}\right)\\
&+\ln\left[(m_1m_2)^{\frac{M_1M_2-\langle(A\circ B)x,x\rangle}{M_1M_2-m_1m_2}}(M_1M_2)^{\frac{\langle(A\circ B)x,x\rangle-m_1m_2}{M_1M_2-m_1m_2}}\right]\\
&\leq\ln K\left(\frac{M_1M_2}{m_1m_2}\right)+\langle\mathcal{U}^*(\ln(A\otimes B))\mathcal{U}x,x\rangle,
\end{aligned}$$

for $x \in H$, $\|x\| = 1$.

If we take the exponential in (2.9), then we get (2.7). $\square$

3. Connection to Oppenheim's Inequalities

In the finite dimensional case, if we consider the matrices $A = (a_{ij})$, $B = (b_{ij}) \in \mathbb{M}_n(\mathbb{C})$, then $A \circ B$ has an associated matrix $A \circ B = (a_{ij} b_{ij})$ in $\mathbb{M}_n(\mathbb{C})$.

Recall Hadamard determinant inequality [17, p. 218] for $A \geq 0$

$$\det A \leq \det(A \circ 1) \ \ (= \prod_{i=1}^{n} a_{ii})$$

and Oppenheim's inequality [17, p. 242] for $A,\ B \geq 0$

$$\det A \det B \leq \det(A \circ B) \leq \det(A \circ 1) \det(B \circ 1) \ \ \left(= \prod_{i=1}^{n} a_{ii} b_{ii}\right).$$

In the recent paper [10] the authors obtained the following Oppenheim's type inequalities

$$\frac{1}{S(h_1) S(h_2)} \leq \frac{\Delta_x(A \circ B)}{\Delta_x(A \circ 1) \Delta_x(1 \circ B)} \leq S(h_1 h_2) \tag{3.1}$$

for $x \in H$, $\|x\| = 1$, provided that $0 < m_1 \leq A \leq M_1$ and $0 < m_2 \leq B \leq M_2$.

We have the following inequalities:

Proposition 1. *With the assumptions of Theorem 2 we have the determinant inequalities*

$$\frac{1}{K(h_1) K(h_2)} \leq \frac{\Delta_x(A \circ B)}{\Delta_x(A \circ 1) \Delta_x(1 \circ B)} \leq K(h_1 h_2) \tag{3.2}$$

where $h_1 = \frac{M_1}{m_1} > 1$, $h_2 = \frac{M_2}{m_2} > 1$.

Proof. By the properties of the tensorial product, we have that

$$A \otimes B = (A \otimes 1)(1 \otimes B)$$

where $A \otimes 1$ and $1 \otimes B$ are commutative operators.

Therefore

$$\ln(A \otimes B) = \ln[(A \otimes 1)(1 \otimes B)] = \ln(A \otimes 1) + \ln(1 \otimes B)$$

and

$$\begin{aligned} \mathcal{U}^*(\ln(A \otimes B))\mathcal{U} &= \mathcal{U}^*[\ln(A \otimes 1) + \ln(1 \otimes B)]\mathcal{U} \\ &= \mathcal{U}^*(\ln(A \otimes 1))\mathcal{U} + \mathcal{U}^*(\ln(1 \otimes B))\mathcal{U}. \end{aligned}$$

Using Jensen's operator inequality for the operator concave function ln, we also have

$$\mathcal{U}^*(\ln(A \otimes 1))\mathcal{U} \leq \ln(\mathcal{U}^*(A \otimes 1)\mathcal{U}) = \ln(A \circ 1)$$

and

$$\mathcal{U}^*(\ln(1 \otimes B))\mathcal{U} \leq \ln(\mathcal{U}^*((1 \otimes B))\mathcal{U}) = \ln(1 \circ B).$$

These imply for $x \in H$, $\|x\| = 1$ that

$$\begin{aligned} \exp\langle \mathcal{U}^*(\ln(A \otimes B))\mathcal{U}x, x\rangle &\leq \exp[\langle \ln(A \circ 1)x, x\rangle + \langle \ln(1 \circ B)x, x\rangle] \\ &= \exp[\langle \ln(A \circ 1)x, x\rangle] \exp[\langle \ln(1 \circ B)x, x\rangle] \\ &= \Delta_x(A \circ 1)\Delta_x(1 \circ B) \end{aligned}$$

and by the second part of (2.7) we derive the second inequality in (3.2).

From (2.6) we have

$$\ln \Phi(P) \leq \ln K\left(\frac{M}{m}\right) + \Phi(\ln P)$$

provided that $0 < m \leq P \leq M$.

Now, if we take in this inequality $0 < m_1 \leq P = A \otimes 1 \leq M_1$, then we get for $\Phi(P) = \mathcal{U}^*(A \otimes 1)\mathcal{U} = A \circ 1$ that

$$\ln(A \circ 1) \leq \ln K\left(\frac{M_1}{m_1}\right) + \mathcal{U}^*(\ln(A \otimes 1))\mathcal{U}$$

while for $0 < m_2 \leq P = 1 \otimes B \leq M_2$

$$\ln(1 \circ B) \leq \ln K\left(\frac{M_2}{m_2}\right) + \mathcal{U}^*(\ln(1 \otimes B))\mathcal{U},$$

which gives, by addition, that

$$\begin{aligned} &\ln(A \circ 1) + \ln(1 \circ B) - \ln\left[K\left(\frac{M_1}{m_1}\right) K\left(\frac{M_2}{m_2}\right)\right] \\ &\leq \mathcal{U}^*(\ln(A \otimes 1))\mathcal{U} + \mathcal{U}^*(\ln(1 \otimes B))\mathcal{U} = \mathcal{U}^*(\ln(A \otimes B))\mathcal{U}. \end{aligned}$$

By taking the inner product for $x \in H$, $\|x\| = 1$ we get that

$$\begin{aligned} &\langle \ln(A \circ 1)x, x\rangle + \langle \ln(1 \circ B)x, x\rangle - \ln\left[K\left(\frac{M_1}{m_1}\right) K\left(\frac{M_2}{m_2}\right)\right] \\ &\leq \langle \mathcal{U}^*(\ln(A \otimes B))\mathcal{U}x, x\rangle \end{aligned}$$

and by taking the exponential, we derive

$$\frac{\exp\langle (A \circ 1)x, x\rangle \exp\langle \ln(1 \circ B)x, x\rangle}{K(h_1)K(h_2)} \leq \exp\langle \mathcal{U}^*(\ln(A \otimes B))\mathcal{U}x, x\rangle$$

for $x \in H$, $\|x\| = 1$ and by the third inequality in (2.7) we obtain the first part of (3.2). $\square$

Remark 1. *Since $K(h) \geq S(h)$ for $h > 0$ (see for instance* [8, p. 4]*), then the bounds for the ratio*

$$\frac{\Delta_x(A \circ B)}{\Delta_x(A \circ 1)\Delta_x(1 \circ B)}$$

provided by (3.1) are better than the ones from (3.2).

Lemma 1. *For all $h_1, h_2 \in (1, \infty)$ or $h_1, h_2 \in (0, 1)$ we have*

$$K(h_1 h_2) \geq K(h_1)K(h_2). \tag{3.3}$$

If $h_1 \in (1, \infty)$ and $h_2 \in (0, 1)$ or $h_2 \in (1, \infty)$ and $h_1 \in (0, 1)$ then the sign of inequality reverses in (3.3).

Proof. We have for $h_1, h_2 \in (0,\infty)$ that

$$\begin{aligned} K(h_1h_2) - K(h_1)K(h_2) &= \frac{(h_1h_2+1)^2}{4h_1h_2} - \frac{(h_1+1)^2}{4h_1}\frac{(h_2+1)^2}{4h_2} \\ &= \frac{1}{16h_1h_2}\left[4(h_1h_2+1)^2 - (h_1+1)^2(h_2+1)^2\right] \\ &= \frac{1}{16h_1h_2}\left[2(h_1h_2+1) + (h_1+1)(h_2+1)\right] \\ &\quad\times\left[2(h_1h_2+1) - (h_1+1)(h_2+1)\right]. \end{aligned}$$

Observe that

$$\begin{aligned} 2(h_1h_2+1) - (h_1+1)(h_2+1) &= 2h_1h_2 + 2 - h_1h_2 - h_1 - h_2 - 1 \\ &= h_1h_2 + 1 - h_1 - h_2 = (h_1-1)(h_2-1), \end{aligned}$$

which shows that the sign of $K(h_1h_2) - K(h_1)K(h_2)$ is the same with the one for $(h_1-1)(h_2-1)$, and this proves the lemma. □

Corollary 2. *With the assumptions of Theorem 2 we have the deteminant inequalities*

$$\frac{1}{K(h_1h_2)} \le \frac{\Delta_x(A\circ B)}{\Delta_x(A\circ 1)\Delta_x(1\circ B)} \le K(h_1h_2). \tag{3.4}$$

The proof follows by (3.2) and (3.3).

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