SOME INEQUALITIES FOR THE NORMALIZED DETERMINANT OF HADAMARD PRODUCT OF TWO POSITIVE OPERATORS IN HILBERT SPACES

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Abstract. For positive invertible operators A on a Hilbert space H and a fixed unit vector $x \in H$, define the normalized determinant by $\Delta_x(A) := \exp \langle \ln Ax, x \rangle$. In this paper we obtain some inequalities for the determinant $\Delta_x(A \circ B)$ of the Hadamard product of two operators under some natural assumptions such as $0 < m_1 \leq A \leq M_1$ and $0 < m_2 \leq B \leq M_2$, where m_i, M_i $(i = 1, 2)$ are constants.

1. Introduction

Let $B(H)$ be the space of all bounded linear operators on a Hilbert space H, and I stands for the identity operator on H. An operator A in $B(H)$ is said to be positive (in symbol: $A \geq 0$) if $\langle Ax, x \rangle \geq 0$ for all $x \in H$. In particular, $A > 0$ means that A is positive and invertible. For a pair A, B of selfadjoint operators the order relation $A \geq B$ means as usual that $A - B$ is positive.

In 1998, Fujii et al. [6], [7], introduced the *normalized determinant* $\Delta_x(A)$ for positive invertible operators A on a Hilbert space H and a fixed unit vector $x \in H$, namely $\|x\| = 1$, defined by $\Delta_x(A) := \exp \langle \ln Ax, x \rangle$ and discussed it as a continuous geometric mean and observed some inequalities around the determinant from this point of view.

Some of the fundamental properties of normalized determinant are as follows, [6].

For each unit vector $x \in H$, see also [11], we have:

(i) *continuity*: the map $A \to \Delta_x(A)$ is norm continuous;
(ii) *bounds*: $\left\langle A^{-1}x, x \right\rangle^{-1} \leq \Delta_x(A) \leq \langle Ax, x \rangle$;
(iii) *continuous mean*: $\langle A^p x, x \rangle^{1/p} \downarrow \Delta_x(A)$ for $p \downarrow 0$ and $\langle A^p x, x \rangle^{1/p} \uparrow \Delta_x(A)$ for $p \uparrow 0$;
(iv) *power equality:* $\Delta_x(A^t) = \Delta_x(A)^t$ for all $t > 0$;
(v) *homogeneity*: $\Delta_x(tA) = t\Delta_x(A)$ and $\Delta_x(tI) = t$ for all $t > 0$;
(vi) *monotonicity*: $0 < A \leq B$ implies $\Delta_x(A) \leq \Delta_x(B)$;
(vii) *multiplicativity*: $\Delta_x(AB) = \Delta_x(A)\Delta_x(B)$ for commuting A and B;
(viii) *Ky Fan type inequality*: $\Delta_x((1 - \alpha) A + \alpha B) \geq \Delta_x(A)^{1-\alpha}\Delta_x(B)^{\alpha}$ for $0 < \alpha < 1$.

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We define the logarithmic mean of two positive numbers a, b by

$$L(a,b) := \begin{cases} \frac{b-a}{\ln b - \ln a} \text{ if } b \neq a, \\ \\ a \text{ if } b = a. \end{cases}$$

In [6] the authors obtained the following additive reverse inequality for the operator A which satisfy the condition $0 < mI \leq A \leq MI$, where m, M are positive numbers,

$$0 \leq \langle Ax, x\rangle - \Delta_x(A) \leq L(m,M) \left[\ln L(m,M) + \frac{M \ln m - m \ln M}{M - m} - 1 \right] \tag{1.1}$$

for all $x \in H$, $\|x\| = 1$.

The famous *Young inequality* for scalars says that if $a, b > 0$ and $\nu \in [0,1]$, then

$$a^{1-\nu} b^{\nu} \leq (1-\nu) a + \nu b \tag{1.2}$$

with equality if and only if $a = b$. The inequality (1.2) is also called *ν-weighted arithmetic-geometric mean inequality.*

We recall that *Specht's ratio* is defined by [15]

$$S(h) := \begin{cases} \frac{h^{\frac{1}{h-1}}}{e \ln\left(h^{\frac{1}{h-1}}\right)} \text{ if } h \in (0,1) \cup (1,\infty) \\ \\ 1 \text{ if } h = 1. \end{cases} \tag{1.3}$$

It is well known that $\lim_{h\to 1} S(h) = 1$, $S(h) = S\left(\frac{1}{h}\right) > 1$ for $h > 0$, $h \neq 1$. The function is decreasing on $(0,1)$ and increasing on $(1,\infty)$.

In [7], the authors obtained the following multiplicative reverse inequality as well

$$1 \leq \frac{\langle Ax, x\rangle}{\Delta_x(A)} \leq S\left(\frac{M}{m}\right) \tag{1.4}$$

for $0 < mI \leq A \leq MI$ and $x \in H$, $\|x\| = 1$.

Since $0 < M^{-1}I \leq A^{-1} \leq m^{-1}I$, then by (1.4) for A^{-1} we get

$$1 \leq \frac{\langle A^{-1}x, x\rangle}{\Delta_x(A^{-1})} \leq S\left(\frac{m^{-1}}{M^{-1}}\right) = S\left(\left(\frac{m}{M}\right)^{-1}\right) = S\left(\frac{M}{m}\right),$$

which is equivalent to

$$1 \leq \frac{\Delta_x(A)}{\langle A^{-1}x, x\rangle^{-1}} \leq S\left(\frac{M}{m}\right) \tag{1.5}$$

for $x \in H$, $\|x\| = 1$.

In [7], the authors obtained the following multiplicative reverse inequality as well

$$1 \leq \frac{\langle Ax, x\rangle}{\Delta_x(A)} \leq S\left(\frac{M}{m}\right) \tag{1.6}$$

for $0 < mI \leq A \leq MI$ and $x \in H$, $\|x\| = 1$.

Since $0 < M^{-1}I \leq A^{-1} \leq m^{-1}I$, then by (1.4) for A^{-1} we get

$$1 \leq \frac{\langle A^{-1}x, x\rangle}{\Delta_x(A^{-1})} \leq S\left(\frac{m^{-1}}{M^{-1}}\right) = S\left(\left(\frac{m}{M}\right)^{-1}\right) = S\left(\frac{M}{m}\right),$$

which is equivalent to

$$1 \leq \frac{\Delta_x(A)}{\langle A^{-1}x, x\rangle^{-1}} \leq S\left(\frac{M}{m}\right) \tag{1.7}$$

for $x \in H$, $\|x\| = 1$.

We consider the *Kantorovich's constant* defined by

$$K(h) := \frac{(h+1)^2}{4h}, \; h > 0. \tag{1.8}$$

The function K is decreasing on $(0,1)$ and increasing on $[1,\infty)$, $K(h) \geq 1$ for any $h > 0$ and $K(h) = K\left(\frac{1}{h}\right)$ for any $h > 0$.

The following multiplicative refinement and reverse of Young inequality in terms of Kantorovich's constant holds

$$K^r\left(\frac{b}{a}\right)a^{1-\nu}b^\nu \leq (1-\nu)a + \nu b \leq K^R\left(\frac{b}{a}\right)a^{1-\nu}b^\nu \tag{1.9}$$

where a, $b > 0$, $\nu \in [0,1]$, $r = \min\{1-\nu, \nu\}$ and $R = \max\{1-\nu, \nu\}$.

The first inequality in (1.9) was obtained by Zuo et al. in [19] while the second by Liao et al. [14].

Recall the *geometric operator mean* for the positive operators A, $B > 0$

$$A\#_t B := A^{1/2}(A^{-1/2}BA^{-1/2})^t A^{1/2}$$

where $t \in [0,1]$ and

$$A\#B := A^{1/2}(A^{-1/2}BA^{-1/2})^{1/2}A^{1/2}.$$

By the definitions of $\#$ and $\otimes$ we have

$$A\#B = B\#A \text{ and } (A\#B)\otimes(B\#A) = (A\otimes B)\#(B\otimes A).$$

In 2007, S. Wada [17] obtained the following *Callebaut type inequalities* for tensorial product

$$\begin{aligned}(A\#B)\otimes(A\#B) &\leq \frac{1}{2}\left[(A\#_\alpha B)\otimes(A\#_{1-\alpha}B) + (A\#_{1-\alpha}B)\otimes(A\#_\alpha B)\right] \\ &\leq \frac{1}{2}(A\otimes B + B\otimes A)\end{aligned} \tag{1.10}$$

for A, $B > 0$ and $\alpha \in [0,1]$.

Recall that the *Hadamard product* of A and B in $B(H)$ is defined to be the operator $A \circ B \in B(H)$ satisfying

$$\langle (A\circ B)e_j, e_j\rangle = \langle Ae_j, e_j\rangle \langle Be_j, e_j\rangle$$

for all $j \in \mathbb{N}$, where $\{e_j\}_{j\in\mathbb{N}}$ is an *orthonormal basis* for the separable Hilbert space H.

It is known that, see [5], we have the representation

$$A \circ B = \mathcal{U}^*(A\otimes B)\mathcal{U} \tag{1.11}$$

where $\mathcal{U} : H \to H \otimes H$ is the isometry defined by $\mathcal{U}e_j = e_j \otimes e_j$ for all $j \in \mathbb{N}$.

If f *is super-multiplicative (sub-multiplicative)* on $[0,\infty)$, then also [10, p. 173]

$$f(A\circ B) \geq (\leq) f(A) \circ f(B) \text{ for all } A, \; B \geq 0. \tag{1.12}$$

We recall the following elementary inequalities for the Hadamard product

$$A^{1/2} \circ B^{1/2} \leq \left(\frac{A+B}{2}\right) \circ 1 \text{ for } A,\ B \geq 0$$

and *Fiedler inequality*

$$A \circ A^{-1} \geq 1 \text{ for } A > 0. \tag{1.13}$$

As extension of Kadison's Schwarz inequality on the Hadamard product, Ando [1] showed that

$$A \circ B \leq \left(A^2 \circ 1\right)^{1/2} \left(B^2 \circ 1\right)^{1/2} \text{ for } A,\ B \geq 0$$

and Aujla and Vasudeva [3] gave an alternative upper bound

$$A \circ B \leq \left(A^2 \circ B^2\right)^{1/2} \text{ for } A,\ B \geq 0.$$

It has been shown in [12] that $\left(A^2 \circ 1\right)^{1/2} \left(B^2 \circ 1\right)^{1/2}$ and $\left(A^2 \circ B^2\right)^{1/2}$ are incomparable for 2-square positive definite matrices A and B.

Motivated by the above results, in this paper we obtain some inequalities for the determinant $\Delta_x (A \circ B)$ of the Hadamard product of two operators under some natural assumptions such as $0 < m_1 \leq A \leq M_1$ and $0 < m_2 \leq B \leq M_2$, where m_i, M_i $(i = 1, 2)$ are constants.

2. Multiplicative Inequalities

We have the following result for general convex functions [4]:

Lemma 1. *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a convex function on the interval I, $a, b \in \mathring{I}$, the interior of I, with $a < b$ and $\nu \in [0, 1]$. Then*

$$\begin{aligned} &\nu(1-\nu)(b-a)\left[f'_+((1-\nu)a+\nu b) - f'_-((1-\nu)a+\nu b)\right] \\ &\leq (1-\nu) f(a) + \nu f(b) - f((1-\nu)a+\nu b) \\ &\leq \nu(1-\nu)(b-a)\left[f'_-(b) - f'_+(a)\right]. \end{aligned} \tag{2.1}$$

In particular, we have

$$\begin{aligned} \frac{1}{4}(b-a)\left[f'_+\left(\frac{a+b}{2}\right) - f'_-\left(\frac{a+b}{2}\right)\right] &\leq \frac{f(a)+f(b)}{2} - f\left(\frac{a+b}{2}\right) \\ &\leq \frac{1}{4}(b-a)\left[f'_-(b) - f'_+(a)\right]. \end{aligned} \tag{2.2}$$

The constant $\frac{1}{4}$ is best possible in both inequalities from (2.2).

Corollary 1. *If the function $f : I \subset \mathbb{R} \to \mathbb{R}$ is a differentiable convex function on $\mathring{I}$, then for any a, $b \in \mathring{I}$ and $\nu \in [0, 1]$ we have*

$$\begin{aligned} 0 &\leq (1-\nu) f(a) + \nu f(b) - f((1-\nu)a+\nu b) \\ &\leq \nu(1-\nu)(b-a)\left[f'(b) - f'(a)\right]. \end{aligned} \tag{2.3}$$

Proof. If $a < b$, then the inequality (2.3) follows by (2.1). If $b < a$, then by (2.1) we get

$$\begin{aligned} 0 &\leq (1-\nu) f(b) + \nu f(a) - f((1-\nu)b+\nu a) \\ &\leq \nu(1-\nu)(b-a)\left[f'(b) - f'(a)\right] \end{aligned} \tag{2.4}$$

for any $\nu \in [0, 1]$. If we replace ν by $1-\nu$ in (2.4), then we get (2.3). □

Corollary 2. *For any a, $b>0$ and $\nu\in[0,1]$ we have*

$$0\leq(1-\nu)\,a+\nu b-a^{1-\nu}b^{\nu}\leq\nu\,(1-\nu)\,(a-b)\,(\ln a-\ln b) \tag{2.5}$$

and

$$\begin{aligned}(1-\nu)\ln a+\nu\ln b&\leq\ln((1-\nu)\,a+\nu b)\\&\leq(1-\nu)\ln a+\nu\ln b+\nu\,(1-\nu)\,\frac{(b-a)^2}{ab}.\end{aligned} \tag{2.6}$$

Proof. If we write the inequality (2.3) for the convex function $f:\mathbb{R}\to(0,\infty)$, $f(x)=\exp(x)$, then we have

$$\begin{aligned}0&\leq(1-\nu)\exp(x)+\nu\exp(y)-\exp((1-\nu)\,x+\nu y)\\&\leq\nu\,(1-\nu)\,(x-y)\,[\exp(x)-\exp(y)]\end{aligned} \tag{2.7}$$

for any x, $y\in\mathbb{R}$ and $\nu\in[0,1]$.

Let a, $b>0$. If we take $x=\ln a$, $y=\ln b$ in (2.7), then we get the desired inequality (2.5).

Now, if we write the inequality (2.3) for the convex function $f:(0,\infty)\to\mathbb{R}$, $f(x)=-\ln x$, then we get

$$0\leq\ln((1-\nu)\,a+\nu b)-(1-\nu)\ln a-\nu\ln b\leq\nu\,(1-\nu)\,\frac{(b-a)^2}{ab}$$

for a, $b>0$ and $\nu\in[0,1]$. □

We start to the following operator inequalities involving positive operators and positive linear maps:

Theorem 1. *Assume that the selfadjoint operator P satisfies the condition $0<m\leq P\leq M$ for some constants, m, M and Φ a unital positive linear map from $B(H)$ into $B(K)$. Then*

$$\begin{aligned}&\ln m\frac{M-\Phi(P)}{M-m}+\ln M\frac{\Phi(P)-m}{M-m}\\&\leq\ln\Phi(P)\\&\leq\ln m\frac{M-\Phi(P)}{M-m}+\ln M\frac{\Phi(P)-m}{M-m}+\frac{1}{mM}\,(M-\Phi(P))\,(\Phi(P)-m)\\&\leq\ln m\frac{M-\Phi(P)}{M-m}+\ln M\frac{\Phi(P)-m}{M-m}+\frac{1}{4mM}\,(M-m)^2\end{aligned} \tag{2.8}$$

and

$$\begin{aligned}&\ln m\frac{M-\Phi(P)}{M-m}+\ln M\frac{\Phi(P)-m}{M-m}\\&\leq\Phi(\ln P)\\&\leq\ln m\frac{M-\Phi(P)}{M-m}+\ln M\frac{\Phi(P)-m}{M-m}+\frac{1}{mM}\Phi\,[(M-T)\,(T-m)]\\&\leq\ln m\frac{M-\Phi(P)}{M-m}+\ln M\frac{\Phi(P)-m}{M-m}+\frac{1}{4mM}\,(M-m)^2.\end{aligned} \tag{2.9}$$

Proof. Assume that $0<a<b$. We take $\nu=\frac{t-a}{b-a}\in[0,1]$ for $t\in[a,b]$ and observe that

$$(1-\nu)\,a+\nu b=\frac{b-t}{b-a}a+\frac{t-a}{b-a}b=t$$

and

$$\nu(1-\nu)=\frac{(b-t)(t-a)}{(b-a)^2}.$$

From (2.6) we get

$$\begin{aligned}
&\frac{b-t}{b-a}\ln a+\frac{t-a}{b-a}\ln b \\
&\le \ln t\le \frac{b-t}{b-a}\ln a+\frac{t-a}{b-a}\ln b+\frac{1}{ab}(b-t)(t-a) \\
&\le \frac{b-t}{b-a}\ln a+\frac{t-a}{b-a}\ln b+\frac{1}{4ab}(b-a)^2
\end{aligned}\tag{2.10}$$

for all $t\in[a,b]$.

By utilizing the continuous functional calculus for selfadjoint operators T with spectra $\operatorname{Sp}(T)\subseteq[a,b]$, we obtain from (2.10) that

$$\begin{aligned}
&\ln a\frac{b-T}{b-a}+\ln b\frac{T-a}{b-a} \\
&\le \ln T\le \ln a\frac{b-T}{b-a}+\ln b\frac{T-a}{b-a}+\frac{1}{ab}(b-T)(T-a) \\
&\le \ln a\frac{b-T}{b-a}+\ln b\frac{T-a}{b-a}+\frac{1}{4ab}(b-a)^2.
\end{aligned}\tag{2.11}$$

Now if $0<m\le P\le M$, then $0<m\le\Phi(P)\le M$ and by (2.11) we get for $T=\Phi(P)$, $a=m$ and $b=M$ the inequality (2.8). If we take $T=P$, $a=m$ and $b=M$ in (2.11) and then apply Φ we also obtain (2.9). $\square$

Corollary 3. *With the assumptions of Theorem 1 we have the chain of inequalities*

$$\begin{aligned}
&\ln m\frac{M-\Phi(P)}{M-m}+\ln M\frac{\Phi(P)-m}{M-m} \\
&\le \Phi(\ln P)\le\ln\Phi(P) \\
&\le \ln m\frac{M-\Phi(P)}{M-m}+\ln M\frac{\Phi(P)-m}{M-m}+\frac{1}{mM}(M-\Phi(P))(\Phi(P)-m) \\
&\le \ln m\frac{M-\Phi(P)}{M-m}+\ln M\frac{\Phi(P)-m}{M-m}+\frac{m}{4M}\left(\frac{M}{m}-1\right)^2 \\
&\le \frac{m}{4M}\left(\frac{M}{m}-1\right)^2+\Phi(\ln P).
\end{aligned}\tag{2.12}$$

Proof. Second inequality follows by Jensen's operator inequality for the operator concave function ln, while the last inequality follows by the fact that

$$\ln m\frac{M-\Phi(P)}{M-m}+\ln M\frac{\Phi(P)-m}{M-m}\le\Phi(\ln P)$$

from the first part of (2.12). $\square$

We have the following inequalities for the determinant $\Delta_x(A\circ B)$ for $x\in H$, $\|x\|=1$.

Theorem 2. *Assume that* $0 < m_1 \leq A \leq M_1$ *and* $0 < m_2 \leq B \leq M_2$*, then for* $m = m_1 m_2$*,* $M = M_1 M_2$*,*

$$\begin{aligned}
&m^{\frac{M - A\circ B}{M-m}} M^{\frac{A\circ B - m}{M-m}} \\
&\leq \exp\left[\langle \mathcal{U}^* \left(\ln\left(A\otimes B\right)\right)\mathcal{U}x, x\rangle\right] \\
&\leq \Delta_x \left(A \circ B\right) \\
&\leq m^{\frac{M - A\circ B}{M-m}} M^{\frac{A\circ B - m}{M-m}} \\
&\times \exp\left(\frac{1}{mM}\left\langle \left(M - A\circ B\ \right)\left(A\circ B\ - m\right)x, x\right\rangle\right) \\
&\leq \exp\left[\frac{m}{4M}\left(\frac{M}{m} - 1\right)^2\right] \exp\left\langle \mathcal{U}^*\left(\ln A \otimes B\right)\mathcal{U}x, x\right\rangle
\end{aligned} \tag{2.13}$$

for $x \in H$*,* $\|x\| = 1$*.*

Proof. Since $0 < m_1 \leq A \leq M_1$ and $0 < m_2 \leq B \leq M_2$, then $0 < m_1 m_2 = m \leq P = A \otimes B \leq M = M_1 M_2$. From (2.12) for $\Phi(P) = \mathcal{U}^*(A \otimes B)\mathcal{U} = A \circ B$ we get

$$\begin{aligned}
&\ln m \frac{M - A\circ B}{M - m} + \ln M \frac{A\circ B\ - m}{M - m} \\
&\leq \mathcal{U}^*\left(\ln A \otimes B\right)\mathcal{U} \leq \ln\left(A\circ B\right) \\
&\leq \ln m \frac{M - A\circ B}{M - m} + \ln M \frac{A\circ B\ - m}{M - m} \\
&+ \frac{1}{mM}\left(M - A\circ B\ \right)\left(A\circ B\ - m\right) \\
&\leq \ln m \frac{M - A\circ B}{M - m} + \ln M \frac{A\circ B\ - m}{M - m} + \frac{m}{4M}\left(\frac{M}{m} - 1\right)^2 \\
&\leq \frac{m}{4M}\left(\frac{M}{m} - 1\right)^2 + \mathcal{U}^*\left(\ln A \otimes B\right)\mathcal{U}.
\end{aligned} \tag{2.14}$$

If we take the inner product for $x \in H$, $\|x\| = 1$, then we get

$$\begin{aligned}
&\ln m \frac{M - \langle (A\circ B)x, x\rangle}{M - m} + \ln M \frac{\langle (A\circ B)x, x\rangle\ - m}{M - m} \\
&\leq \langle \mathcal{U}^*\left(\ln A \otimes B\right)\mathcal{U}x, x\rangle \leq \langle \ln\left(A\circ B\right)x, x\ \rangle \\
&\leq \ln m \frac{M - \langle (A\circ B)x, x\rangle}{M - m} + \ln M \frac{\langle (A\circ B)x, x\rangle - m}{M - m} \\
&+ \frac{1}{mM}\left\langle \left(M - A\circ B\ \right)\left(A\circ B\ - m\right)x, x\right\rangle \\
&\leq \ln m \frac{M - \langle (A\circ B)x, x\rangle}{M - m} + \ln M \frac{\langle (A\circ B)x, x\rangle\ - m}{M_1 M_2 - m} \\
&+ \frac{m}{4M}\left(\frac{M}{m} - 1\right)^2 \\
&\leq \frac{m}{4M}\left(\frac{M}{m} - 1\right)^2 + \left\langle \mathcal{U}^*\left(\ln A \otimes B\right)\mathcal{U}x, x\right\rangle,
\end{aligned}$$

namely

$$\begin{aligned}
&\ln\left(m^{\frac{M-A\circ B}{M-m}}M^{\frac{A\circ B-m}{M-m}}\right)\\
&\leq\langle\mathcal{U}^{*}(\ln A\otimes B)\,\mathcal{U}x,x\rangle\leq\langle\ln(A\circ B)\,x,x\ \rangle\\
&\ln\left(m^{\frac{M-A\circ B}{M-m}}M^{\frac{A\circ B-m}{M-m}}\right)+\frac{1}{mM}\langle(M-A\circ B\ )\,(A\circ B\ -m)\,x,x\rangle\\
&\leq\ln\left(m^{\frac{M-A\circ B}{M-m}}M^{\frac{A\circ B-m}{M-m}}\right)+\frac{m}{4M}\left(\frac{M}{m}-1\right)^{2}\\
&\leq\frac{m}{4}\left(\frac{M}{m}-1\right)^{2}+\langle\mathcal{U}^{*}(\ln A\otimes B)\,\mathcal{U}x,x\rangle
\end{aligned}$$

and by taking the exponential, we get (2.13). □

3. Additive Inequalities

We start to the following operator inequalities involving positive operators and positive linear maps:

Theorem 3. *Assume that the selfadjoint operator P satisfies the condition $0<m\leq P\leq M$ for some constants, m, M and Φ a unital positive linear map from $B(H)$ into $B(K)$. Then*

$$\begin{aligned}
&\ln\left(m^{\frac{M-\Phi(P)}{M-m}}M^{\frac{\Phi(P)-m}{M-m}}\right)\\
&\leq\Phi(\ln P)\leq\ln\Phi(P)\\
&\leq\ln\left[m^{\frac{M-\Phi(P)}{M-m}}M^{\frac{\Phi(P)-m}{M-m}}+(M-\Phi(P))\,(\Phi(P)-m)\left(\frac{\ln M-\ln m}{M-m}\right)\right]\\
&\leq\ln\left[m^{\frac{M-\Phi(P)}{M-m}}M^{\frac{\Phi(P)-m}{M-m}}+\frac{1}{4}(M-m)\,(\ln M-\ln m)\right].
\end{aligned}\tag{3.1}$$

Proof. From (2.5) we get

$$a^{1-\nu}b^{\nu}\leq(1-\nu)\,a+\nu b\leq a^{1-\nu}b^{\nu}+\nu(1-\nu)\,(a-b)\,(\ln a-\ln b)\tag{3.2}$$

for $0<a<b$ and $\nu\in[0,1]$.

Assume that $0<a<b$. We take $\nu=\frac{t-a}{b-a}\in[0,1]$ for $t\in[a,b]$ and observe that

$$(1-\nu)\,a+\nu b=\frac{b-t}{b-a}a+\frac{t-a}{b-a}b=t$$

and

$$\nu(1-\nu)=\frac{(b-t)\,(t-a)}{(b-a)^{2}}.$$

By (3.2) we get

$$\begin{aligned}
a^{\frac{b-t}{b-a}}b^{\frac{t-a}{b-a}}&\leq t\leq a^{\frac{b-t}{b-a}}b^{\frac{t-a}{b-a}}+(b-t)\,(t-a)\left(\frac{\ln a-\ln b}{a-b}\right)\\
&\leq a^{\frac{b-t}{b-a}}b^{\frac{t-a}{b-a}}+\frac{1}{4}(a-b)\,(\ln a-\ln b)
\end{aligned}\tag{3.3}$$

for all $t\in[a,b]$.

If we take the ln in (3.3) we get

$$\begin{aligned}(3.4)\qquad & \frac{b-t}{b-a}\ln a+\frac{t-a}{b-a}\ln b \\ & \leq \ln t\leq \ln\left[a^{\frac{b-t}{b-a}}b^{\frac{t-a}{b-a}}+(b-t)(t-a)\left(\frac{\ln a-\ln b}{a-b}\right)\right] \\ & \leq \ln\left[a^{\frac{b-t}{b-a}}b^{\frac{t-a}{b-a}}+\frac{1}{4}(a-b)(\ln a-\ln b)\right]\end{aligned}$$

for all $t\in[a,b]$.

By utilizing the continuous functional calculus for selfadjoint operators T with spectra $\operatorname{Sp}(T)\subseteq[a,b]$, we obtain from (3.4) that

$$\begin{aligned}(3.5)\qquad & \ln a\frac{b-T}{b-a}+\ln b\frac{T-a}{b-a} \\ & \leq \ln T\leq \ln\left[a^{\frac{b-T}{b-a}}b^{\frac{T-a}{b-a}}+(b-T)(T-a)\left(\frac{\ln a-\ln b}{a-b}\right)\right] \\ & \leq \ln\left[a^{\frac{b-T}{b-a}}b^{\frac{T-a}{b-a}}+\frac{1}{4}(a-b)(\ln a-\ln b)\right].\end{aligned}$$

Now if $0<m\leq P\leq M$, then $0<m\leq\Phi(P)\leq M$ and by the second part of (3.5) we get for $T=\Phi(P)$, $a=m$ and $b=M$ the inequality

$$\begin{aligned}(3.6)\qquad & \ln\Phi(P) \\ & \leq \ln\left[m^{\frac{M-\Phi(P)}{M-m}}M^{\frac{\Phi(P)-m}{M-m}}+(M-\Phi(P))(\Phi(P)-m)\left(\frac{\ln M-\ln m}{M-m}\right)\right] \\ & \leq \ln\left[m^{\frac{M-\Phi(P)}{M-m}}M^{\frac{\Phi(P)-m}{M-m}}+\frac{1}{4}(M-m)(\ln M-\ln m)\right].\end{aligned}$$

From the first part of (3.5) we have

$$\ln m\frac{M-P}{M-m}+\ln M\frac{P-m}{M-m}\leq \ln P.$$

If we take the positive linear map Φ we obtain

$$\ln m\frac{M-\Phi(P)}{M-m}+\ln M\frac{\Phi(P)-m}{M-m}\leq\Phi(\ln P),$$

which is equivalent to

$$(3.7)\qquad \ln\left(m^{\frac{M-\Phi(P)}{M-m}}M^{\frac{\Phi(P)-m}{M-m}}\right)\leq\Phi(\ln P).$$

Now, by operator Jensen's inequality we have $\Phi(\ln P)\leq\ln\Phi(P)$. By collecting all these inequalities we obtain the desired result (3.1). □

Corollary 4. *With the assumptions of Theorem 1 we have*

$$(3.8)\qquad \Phi(\ln P)\leq\ln\Phi(P)\leq\Phi(\ln P)+\frac{1}{4}\left(\frac{M}{m}-1\right)\ln\left(\frac{M}{m}\right).$$

Proof. By the concavity of the function ln we have x, $y>0$ that

$$\ln x-\ln y\leq\frac{x}{y}-1.$$

This implies that

$$\ln(t+k)\leq\ln t+kt^{-1}$$

for all t, $k > 0$.

By the functional calculus we get in the operator order

$$\ln(T+k) \le \ln T + kT^{-1}$$

for all operators $T > 0$ and $k > 0$.

Therefore

$$\begin{aligned}
(3.9)\qquad & \ln\left[m^{\frac{M-\Phi(P)}{M-m}}M^{\frac{\Phi(P)-m}{M-m}} + \frac{1}{4}(M-m)(\ln M - \ln m)\right] \\
&\le \ln\left(m^{\frac{M-\Phi(P)}{M-m}}M^{\frac{\Phi(P)-m}{M-m}}\right) \\
&+ \frac{1}{4}(M-m)(\ln M - \ln m)\left(m^{\frac{M-\Phi(P)}{M-m}}M^{\frac{\Phi(P)-m}{M-m}}\right)^{-1} \\
&\le \ln\left(m^{\frac{M-\Phi(P)}{M-m}}M^{\frac{\Phi(P)-m}{M-m}}\right) + \frac{1}{4m}(M-m)(\ln M - \ln m) \\
&= \ln\left(m^{\frac{M-\Phi(P)}{M-m}}M^{\frac{\Phi(P)-m}{M-m}}\right) + \frac{1}{4}\left(\frac{M}{m}-1\right)\ln\left(\frac{M}{m}\right)
\end{aligned}$$

since, obviously,

$$m^{\frac{M-\Phi(P)}{M-m}}M^{\frac{\Phi(P)-m}{M-m}} \ge m.$$

Finally, since, by the first part of (3.1)

$$\ln\left(m^{\frac{M-\Phi(P)}{M-m}}M^{\frac{\Phi(P)-m}{M-m}}\right) \le \Phi(\ln P),$$

hence by (3.1) and (3.9) we derive (3.8). □

We also have the following inequalities for the determinants:

Theorem 4. *Assume that* $0 < m_1 \le A \le M_1$ *and* $0 < m_2 \le B \le M_2$, *then for* $m = m_1 m_2$, $M = M_1 M_2$,

$$\begin{aligned}
(3.10)\qquad & \Delta_x\left(m^{\frac{M-A\circ B}{M-m}}M^{\frac{A\circ B-m}{M-m}}\right) \\
&\le \exp\left[\langle \mathcal{U}^*(\ln(A\otimes B))\mathcal{U}x, x\rangle\right] \\
&\le \Delta_x(A\circ B) \\
&\le \Delta_x\left[m^{\frac{M-A\circ B}{M-m}}M^{\frac{A\circ B-m}{M-m}} + (M-A\circ B)(A\circ B-m)\left(\frac{\ln M-\ln m}{M-m}\right)\right] \\
&\le \ln\left(\frac{M}{m}\right)^{\frac{1}{4}(M-m)}\Delta_x\left(m^{\frac{M-A\circ B}{M-m}}M^{\frac{A\circ B-m}{M-m}}\right)
\end{aligned}$$

for $x \in H$, $\|x\| = 1$.

Proof. Since $0 < m_1 \leq A \leq M_1$ and $0 < m_2 \leq B \leq M_2$, then $0 < m_1 m_2 = m \leq P = A \otimes B \leq M_1 M_2 = M$. From (3.1) for $\Phi(P) = \mathcal{U}^*(A \otimes B)\mathcal{U} = A \circ B$ we get

$$\begin{aligned}(3.11)\qquad & \ln\left(m^{\frac{M-A\circ B}{M-m}} M^{\frac{A\circ B-m}{M-m}}\right) \\ & \leq \mathcal{U}^*\Phi\left(\ln\left(A\otimes B\right)\right)\mathcal{U} \leq \ln\left(A\circ B\right) \\ & \leq \ln\left[m^{\frac{M-A\circ B}{M-m}} M^{\frac{A\circ B-m}{M-m}} \right.\\ & \left. + \left(M - A\circ B\right)\left(A\circ B - m\right)\left(\frac{\ln M - \ln m}{M-m}\right)\right] \\ & \leq \ln\left[m^{\frac{M-A\circ B}{M-m}} M^{\frac{A\circ B-m}{M-m}} + \frac{1}{4}\left(M-m\right)\left(\ln M - \ln m\right)\right].\end{aligned}$$

If we take the inner product for $x \in H$, $\|x\| = 1$, then we get

$$\begin{aligned} & \left\langle \ln\left(m^{\frac{M-A\circ B}{M-m}} M^{\frac{A\circ B-m}{M-m}}\right) x, x\right\rangle \\ & \leq \left\langle \mathcal{U}^*\Phi\left(\ln\left(A\otimes B\right)\right)\mathcal{U}x, x\right\rangle \leq \left\langle \ln\left(A\circ B\right)x, x\right\rangle \\ & \leq \left\langle \ln\left[m^{\frac{M-A\circ B}{M-m}} M^{\frac{A\circ B-m}{M-m}}\right.\right. \\ & \left.\left. + \left(M - A\circ B\right)\left(A\circ B - m\right)\left(\frac{\ln M - \ln m}{M-m}\right)\right] x, x\right\rangle \\ & \leq \left\langle \ln\left(m^{\frac{M-A\circ B}{M-m}} M^{\frac{A\circ B-m}{M-m}} + \frac{1}{4}\left(M-m\right)\left(\ln M - \ln m\right)\right) x, x\right\rangle,\end{aligned}$$

which, by taking the exponential, is equivalent to

$$\begin{aligned}(3.12)\quad & \Delta_x\left(m^{\frac{M-A\circ B}{M-m}} M^{\frac{A\circ B-m}{M-m}}\right) \\ & \leq \exp\left[\left\langle \mathcal{U}^*\left(\ln\left(A\otimes B\right)\right)\mathcal{U}x, x\right\rangle\right] \\ & \leq \Delta_x\left(A\circ B\right) \\ & \leq \Delta_x\left(m^{\frac{M-A\circ B}{M-m}} M^{\frac{A\circ B-m}{M-m}} + \left(M - A\circ B\right)\left(A\circ B - m\right)\left(\frac{\ln M - \ln m}{M-m}\right)\right) \\ & \leq \Delta_x\left(m^{\frac{M-A\circ B}{M-m}} M^{\frac{A\circ B-m}{M-m}} + \frac{1}{4}\left(M-m\right)\left(\ln M - \ln m\right)\right).\end{aligned}$$

Since for any positive operator T and positive constant k we have

$$\Delta_x\left(T + k\right) = k\Delta_x\left(T\right)$$

hence

$$\begin{aligned} & \Delta_x\left(m^{\frac{M-A\circ B}{M-m}} M^{\frac{A\circ B-m}{M-m}} + \frac{1}{4}\left(M-m\right)\left(\ln M - \ln m\right)\right) \\ & = \ln\left(\frac{M}{m}\right)^{\frac{1}{4}(M-m)} \Delta_x\left(m^{\frac{M-A\circ B}{M-m}} M^{\frac{A\circ B-m}{M-m}}\right)\end{aligned}$$

and the inequality (3.10) is proved. $\square$

From Theorem 4 we also have

Corollary 5. *With the assumptions of Theorem 4 we have*

$$\begin{aligned}(3.13)\qquad &\exp\left[\langle\mathcal{U}^*\left(\ln\left(A\otimes B\right)\right)\mathcal{U}x,x\rangle\right]\\ &\leq \Delta_x\left(A\circ B\right)\\ &\leq \ln\left(\frac{M_1M_2}{m_1m_2}\right)^{\frac{1}{4}\left(M_1M_2-m_1m_2\right)}\exp\left[\langle\mathcal{U}^*\left(\ln\left(A\otimes B\right)\right)\mathcal{U}x,x\rangle\right]\end{aligned}$$

for $x\in H$, $\|x\|=1$.

4. Connection to Oppenheim's Inequalities

In the finite dimensional case, if we consider the matrices $A=(a_{ij})$, $B=(b_{ij})\in\mathbb{M}_n(\mathbb{C})$, then $A\circ B$ has an associated matrix $A\circ B=(a_{ij}b_{ij})$ in $\mathbb{M}_n(\mathbb{C})$.

Recall Hadamard determinant inequality [18, p. 218] for $A\geq 0$

$$\det A\leq\det\left(A\circ 1\right)\ \left(=\prod_{i=1}^{n}a_{ii}\right)$$

and Oppenheim's inequality [18, p. 242] for A, $B\geq 0$

$$\det A\det B\leq\det\left(A\circ B\right)\leq\det\left(A\circ 1\right)\det\left(B\circ 1\right)\ \left(=\prod_{i=1}^{n}a_{ii}b_{ii}\right).$$

In the recent paper [11] S. Hiramatsu and Y. Seo obtained the following interesting Oppenheim's type inequalities

$$\frac{1}{S\left(h_1\right)S\left(h_2\right)}\leq\frac{\Delta_x\left(A\circ B\right)}{\Delta_x\left(A\circ 1\right)\Delta_x\left(1\circ B\right)}\leq S\left(h_1h_2\right)\tag{4.1}$$

for $x\in H$, $\|x\|=1$, provided that $0<m_1\leq A\leq M_1$ and $0<m_2\leq B\leq M_2$.

We have the following similar inequalities:

Proposition 1. *With the assumptions of Theorem 2 we have the determinant inequalities*

$$\frac{1}{E\left(h_1\right)E\left(h_2\right)}\leq\frac{\Delta_x\left(A\circ B\right)}{\Delta_x\left(A\circ 1\right)\Delta_x\left(1\circ B\right)}\leq E\left(h_1h_2\right),\tag{4.2}$$

where $h_1=\frac{M_1}{m_1}>1$, $h_2=\frac{M_2}{m_2}>1$ *and* $E(h):=\exp\left[\frac{1}{4h}(h-1)^2\right]$.

Proof. By the properties of the tensorial product, we have that

$$A\otimes B=\left(A\otimes 1\right)\left(1\otimes B\right)$$

where $A\otimes 1$ and $1\otimes B$ are commutative operators.

Therefore

$$\ln\left(A\otimes B\right)=\ln\left[\left(A\otimes 1\right)\left(1\otimes B\right)\right]=\ln\left(A\otimes 1\right)+\ln\left(1\otimes B\right)$$

and

$$\begin{aligned}\mathcal{U}^*\left(\ln\left(A\otimes B\right)\right)\mathcal{U}&=\mathcal{U}^*\left[\ln\left(A\otimes 1\right)+\ln\left(1\otimes B\right)\right]\mathcal{U}\\ &=\mathcal{U}^*\left(\ln\left(A\otimes 1\right)\right)\mathcal{U}+\mathcal{U}^*\left(\ln\left(1\otimes B\right)\right)\mathcal{U}.\end{aligned}$$

Using Jensen's operator inequality for the operator concave function ln, we also have

$$\mathcal{U}^*\left(\ln\left(A\otimes 1\right)\right)\mathcal{U}\leq\ln\left(\mathcal{U}^*\left(A\otimes 1\right)\mathcal{U}\right)=\ln\left(A\circ 1\right)$$

and

$$\mathcal{U}^*(\ln(1\otimes B))\mathcal{U}\le \ln(\mathcal{U}^*((1\otimes B))\mathcal{U})=\ln(1\circ B).$$

These imply for $x\in H$, $\|x\|=1$ that

$$\begin{aligned}\exp\langle\mathcal{U}^*(\ln(A\otimes B))\mathcal{U}x,x\rangle &\le \exp[\langle\ln(A\circ 1)x,x\rangle+\langle\ln(1\circ B)x,x\rangle]\\ &=\exp[\langle\ln(A\circ 1)x,x\rangle]\exp[\langle\ln(1\circ B)x,x\rangle]\\ &=\Delta_x(A\circ 1)\Delta_x(1\circ B)\end{aligned}$$

and by the second part of (2.13) for $m=m_1m_2$, $M=M_1M_2$, we derive the second inequality in (4.2).

From (2.12) we have

$$\ln\Phi(P)\le\Phi(\ln P)+\frac{m}{4M}\left(\frac{M}{m}-1\right)^2$$

provided that $0<m\le P\le M$.

Now, if we take in this inequality $0<m_1\le P=A\otimes 1\le M_1$, then we get for $\Phi(P)=\mathcal{U}^*(A\otimes 1)\mathcal{U}=A\circ 1$ that

$$\ln(A\circ 1)\le\frac{m_1}{4M_1}\left(\frac{M_1}{m_1}-1\right)^2+\mathcal{U}^*(\ln(A\otimes 1))\mathcal{U}$$

while for $0<m_2\le P=1\otimes B\le M_2$

$$\ln(1\circ B)\le\frac{m_2}{4M_2}\left(\frac{M_2}{m_2}-1\right)^2+\mathcal{U}^*(\ln(1\otimes B))\mathcal{U},$$

which gives, by addition, that

$$\begin{aligned}&\ln(A\circ 1)+\ln(1\circ B)\\ &-\ln\left[\exp\left[\frac{m_1}{4M_1}\left(\frac{M_1}{m_1}-1\right)^2+\frac{m_2}{4M_2}\left(\frac{M_2}{m_2}-1\right)^2\right]\right]\\ &\le\mathcal{U}^*(\ln(A\otimes 1))\mathcal{U}+\mathcal{U}^*(\ln(1\otimes B))\mathcal{U}=\mathcal{U}^*(\ln(A\otimes B))\mathcal{U}.\end{aligned}$$

By taking the inner product for $x\in H$, $\|x\|=1$ we get that

$$\begin{aligned}&\langle\ln(A\circ 1)x,x\rangle+\langle\ln(1\circ B)x,x\rangle\\ &-\ln\left[\exp\left[\frac{m_1}{4M_1}\left(\frac{M_1}{m_1}-1\right)^2+\frac{m_2}{4M_2}\left(\frac{M_2}{m_2}-1\right)^2\right]\right]\\ &\le\langle\mathcal{U}^*(\ln(A\otimes B))\mathcal{U}x,x\rangle\end{aligned}$$

and by taking the exponential, we derive

$$\frac{\exp\langle\ln(A\circ 1)x,x\rangle\exp\langle\ln(1\circ B)x,x\rangle}{\exp\left[\frac{m_1}{4M_1}\left(\frac{M_1}{m_1}-1\right)^2+\frac{m_2}{4M_2}\left(\frac{M_2}{m_2}-1\right)^2\right]}\le\exp\langle\mathcal{U}^*(\ln(A\otimes B))\mathcal{U}x,x\rangle$$

for $x\in H$, $\|x\|=1$ and by the third inequality in (2.13) we obtain the first part of (4.2). $\square$

We also can state:

Proposition 2. *With the assumptions of Theorem 2 we have the determinant inequalities*

$$\frac{1}{F(h_1)F(h_2)} \le \frac{\Delta_x(A\circ B)}{\Delta_x(A\circ 1)\Delta_x(1\circ B)} \le F(h_1h_2), \tag{4.3}$$

where $h_1 = \frac{M_1}{m_1} > 1$, $h_2 = \frac{M_2}{m_2} > 1$ *and* $F(h) := h^{\frac{1}{4}(h-1)}$ *for* $h > 0$.

The proof follows from the inequality (3.8) by using a similar argument as in the proof of Proposition 1 and we omit the details.

By conducting some numerical experiments we can state the following:

Conjecture 1. *With the above notations, we have for* $h > 1$ *that*

$$F(h) > E(h) > S(h).$$

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