LOWER AND UPPER BOUNDS FOR THE NORMALIZED DETERMINANT OF HADAMARD PRODUCT OF TWO POSITIVE OPERATORS IN HILBERT SPACES

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Abstract. For positive invertible operators A on a Hilbert space H and a fixed unit vector $x \in H$, define the normalized determinant by $\Delta_x(A) := \exp \langle \ln Ax, x\rangle$. In this paper we obtain some lower and upper bounds for the determinant $\Delta_x (A \circ B)$ of the Hadamard product of two operators under some natural assumptions such as $0 < m_1 \le A \le M_1$ and $0 < m_2 \le B \le M_2$, where m_i, M_i $(i = 1, 2)$ are constants.

1. Introduction

The famous *Young inequality* for scalars says that if $a, b > 0$ and $\nu \in [0, 1]$, then

$$a^{1-\nu} b^{\nu} \le (1-\nu)\, a + \nu b \tag{1.1}$$

with equality if and only if $a = b$. The inequality (1.1) is also called *ν-weighted arithmetic-geometric mean inequality*.

The following inequality provides a refinement and a multiplicative reverse for Young's inequality

$$S\left(\left(\frac{a}{b}\right)^r\right) a^{1-\nu} b^{\nu} \le (1-\nu)\, a + \nu b \le S\left(\frac{a}{b}\right) a^{1-\nu} b^{\nu}, \tag{1.2}$$

where $a, b > 0$, $\nu \in [0, 1]$, $r = \min\{1-\nu, \nu\}$.

The second inequality in (1.2) is due to Tominaga [16] while the first one is due to Furuichi [8].

We consider the *Kantorovich's constant* defined by

$$K(h) := \frac{(h+1)^2}{4h},\ h > 0. \tag{1.3}$$

The function K is decreasing on $(0, 1)$ and increasing on $[1, \infty)$, $K(h) \ge 1$ for any $h > 0$ and $K(h) = K\left(\frac{1}{h}\right)$ for any $h > 0$.

The following multiplicative refinement and reverse of Young inequality in terms of Kantorovich's constant holds

$$K^r\left(\frac{a}{b}\right) a^{1-\nu} b^{\nu} \le (1-\nu)\, a + \nu b \le K^R\left(\frac{a}{b}\right) a^{1-\nu} b^{\nu} \tag{1.4}$$

where $a, b > 0$, $\nu \in [0, 1]$, $r = \min\{1-\nu, \nu\}$ and $R = \max\{1-\nu, \nu\}$.

The first inequality in (1.4) was obtained by Zou et al. in [19] while the second by Liao et al. [14].

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In the recent paper [4] we obtained the following reverses of Young's inequality as well:

$$0 \leq (1-\nu)a + \nu b - a^{1-\nu}b^{\nu} \leq \nu(1-\nu)(a-b)(\ln a - \ln b) \tag{1.5}$$

and

$$1 \leq \frac{(1-\nu)a + \nu b}{a^{1-\nu}b^{\nu}} \leq \exp\left[4\nu(1-\nu)\left(K\left(\frac{a}{b}\right)-1\right)\right], \tag{1.6}$$

where $a,\ b > 0$, $\nu \in [0,1]$.

It has been shown in [4] that there is no ordering for some known upper bounds of the quantity $(1-\nu)a + \nu b - a^{1-\nu}b^{\nu}$ and the one provided by the inequality (1.5). The same conclusion is true for the upper bounds of the quantity $\frac{(1-\nu)a+\nu b}{a^{1-\nu}b^{\nu}}$ incorporated in the inequalities (1.2), (1.4) and (1.6).

Let $B(H)$ be the space of all bounded linear operators on a Hilbert space H, and I stands for the identity operator on H. An operator A in $B(H)$ is said to be positive (in symbol: $A \geq 0$) if $\langle Ax, x\rangle \geq 0$ for all $x \in H$. In particular, $A > 0$ means that A is positive and invertible. For a pair A, B of selfadjoint operators the order relation $A \geq B$ means as usual that $A - B$ is positive.

In 1998, Fujii et al. [6], [7], introduced the *normalized determinant* $\Delta_x(A)$ for positive invertible operators A on a Hilbert space H and a fixed unit vector $x \in H$, namely $\|x\| = 1$, defined by $\Delta_x(A) := \exp\langle \ln Ax, x\rangle$ and discussed it as a continuous geometric mean and observed some inequalities around the determinant from this point of view.

Some of the fundamental properties of normalized determinant are as follows, [6].

For each unit vector $x \in H$, see also [11], we have:

(i) *continuity*: the map $A \to \Delta_x(A)$ is norm continuous;
(ii) *bounds*: $\langle A^{-1}x, x\rangle^{-1} \leq \Delta_x(A) \leq \langle Ax, x\rangle$;
(iii) *continuous mean*: $\langle A^p x, x\rangle^{1/p} \downarrow \Delta_x(A)$ for $p \downarrow 0$ and $\langle A^p x, x\rangle^{1/p} \uparrow \Delta_x(A)$ for $p \uparrow 0$;
(iv) *power equality:* $\Delta_x(A^t) = \Delta_x(A)^t$ for all $t > 0$;
(v) *homogeneity*: $\Delta_x(tA) = t\Delta_x(A)$ and $\Delta_x(tI) = t$ for all $t > 0$;
(vi) *monotonicity*: $0 < A \leq B$ implies $\Delta_x(A) \leq \Delta_x(B)$;
(vii) *multiplicativity*: $\Delta_x(AB) = \Delta_x(A)\Delta_x(B)$ for commuting A and B;
(viii) *Ky Fan type inequality*: $\Delta_x((1-\alpha)A + \alpha B) \geq \Delta_x(A)^{1-\alpha}\Delta_x(B)^{\alpha}$ for $0 < \alpha < 1$.

We define the logarithmic mean of two positive numbers a, b by

$$L(a,b) := \begin{cases} \frac{b-a}{\ln b - \ln a} \text{ if } b \neq a, \\ \\ a \text{ if } b = a. \end{cases}$$

In [6] the authors obtained the following additive reverse inequality for the operator A which satisfy the condition $0 < mI \leq A \leq MI$, where m, M are positive numbers,

$$0 \leq \langle Ax, x\rangle - \Delta_x(A) \leq L(m,M)\left[\ln L(m,M) + \frac{M\ln m - m\ln M}{M-m} - 1\right] \tag{1.7}$$

for all $x \in H$, $\|x\| = 1$.

The famous *Young inequality* for scalars says that if $a, b > 0$ and $\nu \in [0,1]$, then

$$a^{1-\nu} b^{\nu} \leq (1-\nu) a + \nu b \tag{1.8}$$

with equality if and only if $a = b$. The inequality (1.8) is also called *ν-weighted arithmetic-geometric mean inequality.*

We recall that *Specht's ratio* is defined by [15]

$$S(h) := \begin{cases} \dfrac{h^{\frac{1}{h-1}}}{e \ln\left(h^{\frac{1}{h-1}}\right)} \text{ if } h \in (0,1) \cup (1,\infty) \\ \\ 1 \text{ if } h = 1. \end{cases} \tag{1.9}$$

It is well known that $\lim_{h \to 1} S(h) = 1$, $S(h) = S\left(\frac{1}{h}\right) > 1$ for $h > 0$, $h \neq 1$. The function is decreasing on $(0,1)$ and increasing on $(1,\infty)$.

In [7], the authors obtained the following multiplicative reverse inequality as well

$$1 \leq \frac{\langle Ax, x\rangle}{\Delta_x(A)} \leq S\left(\frac{M}{m}\right) \tag{1.10}$$

for $0 < mI \leq A \leq MI$ and $x \in H$, $\|x\| = 1$.

Since $0 < M^{-1}I \leq A^{-1} \leq m^{-1}I$, then by (1.10) for A^{-1} we get

$$1 \leq \frac{\langle A^{-1}x, x\rangle}{\Delta_x(A^{-1})} \leq S\left(\frac{m^{-1}}{M^{-1}}\right) = S\left(\left(\frac{m}{M}\right)^{-1}\right) = S\left(\frac{M}{m}\right),$$

which is equivalent to

$$1 \leq \frac{\Delta_x(A)}{\langle A^{-1}x, x\rangle^{-1}} \leq S\left(\frac{M}{m}\right) \tag{1.11}$$

for $x \in H$, $\|x\| = 1$.

In [7], the authors obtained the following multiplicative reverse inequality as well

$$1 \leq \frac{\langle Ax, x\rangle}{\Delta_x(A)} \leq S\left(\frac{M}{m}\right) \tag{1.12}$$

for $0 < mI \leq A \leq MI$ and $x \in H$, $\|x\| = 1$.

Since $0 < M^{-1}I \leq A^{-1} \leq m^{-1}I$, then by (1.10) for A^{-1} we get

$$1 \leq \frac{\langle A^{-1}x, x\rangle}{\Delta_x(A^{-1})} \leq S\left(\frac{m^{-1}}{M^{-1}}\right) = S\left(\left(\frac{m}{M}\right)^{-1}\right) = S\left(\frac{M}{m}\right),$$

which is equivalent to

$$1 \leq \frac{\Delta_x(A)}{\langle A^{-1}x, x\rangle^{-1}} \leq S\left(\frac{M}{m}\right) \tag{1.13}$$

for $x \in H$, $\|x\| = 1$.

Recall the *geometric operator mean* for the positive operators A, $B > 0$

$$A \#_t B := A^{1/2}(A^{-1/2}BA^{-1/2})^t A^{1/2}$$

where $t \in [0,1]$ and

$$A \# B := A^{1/2}(A^{-1/2}BA^{-1/2})^{1/2} A^{1/2}.$$

By the definitions of $\#$ and $\otimes$ we have

$$A \# B = B \# A \text{ and } (A \# B) \otimes (B \# A) = (A \otimes B) \# (B \otimes A).$$

In 2007, S. Wada [17] obtained the following *Callebaut type inequalities* for tensorial product

$$(A\#B)\otimes(A\#B)\leq\frac{1}{2}\left[(A\#_{\alpha}B)\otimes(A\#_{1-\alpha}B)+(A\#_{1-\alpha}B)\otimes(A\#_{\alpha}B)\right] \leq\frac{1}{2}(A\otimes B+B\otimes A) \tag{1.14}$$

for $A,\ B>0$ and $\alpha\in[0,1]$.

Recall that the *Hadamard product* of A and B in $B(H)$ is defined to be the operator $A\circ B\in B(H)$ satisfying

$$\langle(A\circ B)e_j,e_j\rangle=\langle Ae_j,e_j\rangle\langle Be_j,e_j\rangle$$

for all $j\in\mathbb{N}$, where $\{e_j\}_{j\in\mathbb{N}}$ is an *orthonormal basis* for the separable Hilbert space H.

It is known that, see [5], we have the representation

$$A\circ B=\mathcal{U}^{*}(A\otimes B)\mathcal{U} \tag{1.15}$$

where $\mathcal{U}:H\to H\otimes H$ is the isometry defined by $\mathcal{U}e_j=e_j\otimes e_j$ for all $j\in\mathbb{N}$.

If f *is super-multiplicative (sub-multiplicative)* on $[0,\infty)$, then also [10, p. 173]

$$f(A\circ B)\geq(\leq)f(A)\circ f(B)\ \text{ for all } A,\ B\geq 0. \tag{1.16}$$

We recall the following elementary inequalities for the Hadamard product

$$A^{1/2}\circ B^{1/2}\leq\left(\frac{A+B}{2}\right)\circ 1 \text{ for } A,\ B\geq 0$$

and *Fiedler inequality*

$$A\circ A^{-1}\geq 1 \text{ for } A>0. \tag{1.17}$$

As extension of Kadison's Schwarz inequality on the Hadamard product, Ando [1] showed that

$$A\circ B\leq\left(A^{2}\circ 1\right)^{1/2}\left(B^{2}\circ 1\right)^{1/2}\ \text{ for } A,\ B\geq 0$$

and Aujla and Vasudeva [3] gave an alternative upper bound

$$A\circ B\leq\left(A^{2}\circ B^{2}\right)^{1/2}\ \text{ for } A,\ B\geq 0.$$

It has been shown in [12] that $\left(A^{2}\circ 1\right)^{1/2}\left(B^{2}\circ 1\right)^{1/2}$ and $\left(A^{2}\circ B^{2}\right)^{1/2}$ are incomparable for 2-square positive definite matrices A and B.

Motivated by the above results, in this paper we obtain some lower and upper bounds for the determinant $\Delta_x(A\circ B)$ of the Hadamard product of two operators under some natural assumptions such as $0<m_1\leq A\leq M_1$ and $0<m_2\leq B\leq M_2$, where m_i, M_i $(i=1,2)$ are constants.

2. Multiplicative Inequalities

We have the following result:

Lemma 1. *Let $f:I\subset\mathbb{R}\to\mathbb{R}$ be a twice differentiable function on the interval $\mathring{I}$, the interior of I. If there exists the constants d, D such that*

$$d\leq f''(t)\leq D \text{ for any } t\in\mathring{I}, \tag{2.1}$$

then

$$\frac{1}{2}\nu(1-\nu)d(b-a)^2 \le (1-\nu)f(a)+\nu f(b)-f((1-\nu)a+\nu b) \le \frac{1}{2}\nu(1-\nu)D(b-a)^2 \tag{2.2}$$

for any $a,\ b\in \mathring{I}$ *and* $\nu\in[0,1]$.

In particular, we have

$$\frac{1}{8}(b-a)^2 d \le \frac{f(a)+f(b)}{2} - f\left(\frac{a+b}{2}\right) \le \frac{1}{8}(b-a)^2 D, \tag{2.3}$$

for any $a,\ b\in \mathring{I}$.

The constant $\frac{1}{8}$ *is best possible in both inequalities in (2.3).*

We have:

Corollary 1. *For any* $a,\ b>0$ *and* $\nu\in[0,1]$ *we have*

$$\frac{1}{2}\nu(1-\nu)(\ln a-\ln b)^2\min\{a,b\} \le (1-\nu)a+\nu b - a^{1-\nu}b^{\nu} \le \frac{1}{2}\nu(1-\nu)(\ln a-\ln b)^2\max\{a,b\} \tag{2.4}$$

and

$$\exp\left[\frac{1}{2}\nu(1-\nu)\frac{(b-a)^2}{\max^2\{a,b\}}\right] \le \frac{(1-\nu)a+\nu b}{a^{1-\nu}b^{\nu}} \le \exp\left[\frac{1}{2}\nu(1-\nu)\frac{(b-a)^2}{\min^2\{a,b\}}\right]. \tag{2.5}$$

We start to the following operator inequalities involving positive operators and positive linear maps:

Theorem 1. *Assume that the selfadjoint operator* P *satisfies the condition* $0<m\le P\le M$ *for some constants,* $m,\ M$ *and* Φ *a unital positive linear map from* $B(H)$ *into* $B(K)$. *Then*

$$\begin{aligned}(2.6)\qquad &\frac{1}{2M^2}(M-\Phi(P))(\Phi(P)-m)+\ln m\frac{M-\Phi(P)}{M-m}+\ln M\frac{\Phi(P)-m}{M-m}\\ &\le \ln\Phi(P)\\ &\le \frac{1}{2m^2}(M-\Phi(P))(\Phi(P)-m)+\ln m\frac{M-\Phi(P)}{M-m}+\ln M\frac{\Phi(P)-m}{M-m}\\ &\le \frac{1}{8}\left(\frac{M}{m}-1\right)^2+\ln m\frac{M-\Phi(P)}{M-m}+\ln M\frac{\Phi(P)-m}{M-m}\end{aligned}$$

and

$$\begin{aligned}
&\frac{1}{2M^2}\Phi\left[(M-P)(P-m)\right]+\ln m\frac{M-\Phi(P)}{M-m}+\ln M\frac{\Phi(P)-m}{M-m} \\
&\le \Phi(\ln P) \\
&\le \frac{1}{2m^2}\Phi\left[(M-P)(P-m)\right]+\ln m\frac{M-\Phi(P)}{M-m}+\ln M\frac{\Phi(P)-m}{M-m} \\
&\le \frac{1}{2m^2}(M-\Phi(P))(\Phi(P)-m)+\ln m\frac{M-\Phi(P)}{M-m}+\ln M\frac{\Phi(P)-m}{M-m} \\
&\le \frac{1}{8}\left(\frac{M}{m}-1\right)^2+\ln m\frac{M-\Phi(P)}{M-m}+\ln M\frac{\Phi(P)-m}{M-m}.
\end{aligned}\tag{2.7}$$

Proof. Assume that $0<a<b$. By taking the logarithm in (2.5) we get

$$\begin{aligned}
&\frac{1}{2}\nu(1-\nu)\left(1-\frac{a}{b}\right)^2 \\
&\le \ln((1-\nu)a+\nu b)-(1-\nu)\ln a-\nu\ln b \\
&\le \frac{1}{2}\nu(1-\nu)\left(\frac{b}{a}-1\right)^2
\end{aligned}\tag{2.8}$$

for all $\nu\in[0,1]$.

We take $\nu=\frac{t-a}{b-a}\in[0,1]$ for $t\in[a,b]$, then by (2.8) we get

$$\begin{aligned}
\frac{1}{2b^2}(b-t)(t-a)&\le \ln t-\frac{b-t}{b-a}\ln a-\frac{t-a}{b-a}\ln b \\
&\le \frac{1}{2a^2}(b-t)(t-a)
\end{aligned}\tag{2.9}$$

for all $t\in[a,b]$.

By utilizing the continuous functional calculus for selfadjoint operators T with spectra $\operatorname{Sp}(T)\subseteq[a,b]$, we obtain from (2.9) that

$$\begin{aligned}
&\frac{1}{2b^2}(b-T)(T-a)+\ln a\frac{b-T}{b-a}+\ln b\frac{T-a}{b-a} \\
&\le \ln T \\
&\le \frac{1}{2a^2}(b-T)(T-a)+\ln a\frac{b-T}{b-a}+\ln b\frac{T-a}{b-a}.
\end{aligned}\tag{2.10}$$

Now if $0<m\le P\le M$, then $0<m\le\Phi(P)\le M$ and by (2.10) we get for $T=\Phi(P)$, $a=m$ and $b=M$ the inequality

$$\begin{aligned}
&\frac{1}{2M^2}(M-\Phi(P))(\Phi(P)-m)+\ln m\frac{M-\Phi(P)}{M-m}+\ln M\frac{\Phi(P)-m}{M-m} \\
&\le \ln\Phi(P) \\
&\le \frac{1}{2m^2}(M-\Phi(P))(\Phi(P)-m)+\ln m\frac{M-\Phi(P)}{M-m}+\ln M\frac{\Phi(P)-m}{M-m} \\
&\le \frac{1}{8}\left(\frac{M}{m}-1\right)^2+\ln m\frac{M-\Phi(P)}{M-m}+\ln M\frac{\Phi(P)-m}{M-m},
\end{aligned}$$

since

$$(M-\Phi(P))(\Phi(P)-m)\le\frac{1}{4}(M-m)^2.$$

If we take $T = P$, $a = m$ and $b = M$ in (2.10), then we get

$$\begin{aligned}
&\frac{1}{2M^2}(M-P)(P-m) + \ln m \frac{M-P}{M-m} + \ln M \frac{P-m}{M-m} \\
&\leq \ln P \\
&\leq \frac{1}{2m^2}(M-P)(P-m) + \ln m \frac{M-P}{M-m} + \ln M \frac{P-m}{M-m}.
\end{aligned}$$

If we apply to this inequality the functional Φ we get

$$\begin{aligned}
&\frac{1}{2M^2}\Phi\left[(M-P)(P-m)\right] + \ln m \frac{M-\Phi(P)}{M-m} + \ln M \frac{\Phi(P)-m}{M-m} \\
&\leq \Phi(\ln P) \\
&\leq \frac{1}{2m^2}\Phi\left[(M-P)(P-m)\right] + \ln m \frac{M-\Phi(P)}{M-m} + \ln M \frac{\Phi(P)-m}{M-m}.
\end{aligned}$$

Since the function $g(t) = (M-t)(t-m)$, $t \in [m, M]$ is operator concave, then by Jensen's operator inequality we have

$$\Phi\left[(M-P)(P-m)\right] \leq (M-\Phi(P))(\Phi(P)-m).$$

The theorem is thus proved. □

Corollary 2. *With the assumptions of Theorem 1 we have the chain of inequalities*

$$\begin{aligned}
&\ln m \frac{M-\Phi(P)}{M-m} + \ln M \frac{\Phi(P)-m}{M-m} \\
&\leq \frac{1}{2M^2}\Phi\left[(M-P)(P-m)\right] + \ln m \frac{M-\Phi(P)}{M-m} + \ln M \frac{\Phi(P)-m}{M-m} \\
&\leq \Phi(\ln P) \leq \ln \Phi(P) \\
&\leq \frac{1}{2m^2}(M-\Phi(P))(\Phi(P)-m) + \ln m \frac{M-\Phi(P)}{M-m} + \ln M \frac{\Phi(P)-m}{M-m} \\
&\leq \frac{1}{8}\left(\frac{M}{m}-1\right)^2 + \ln m \frac{M-\Phi(P)}{M-m} + \ln M \frac{\Phi(P)-m}{M-m} \\
&\leq \frac{1}{8}\left(\frac{M}{m}-1\right)^2 + \Phi(\ln P).
\end{aligned} \tag{2.11}$$

The second inequality in (2.11) follows by Jensen's operator inequality for the operator concave function $\ln$. The rest is obvious.

We have the following inequalities for the determinant $\Delta_x(A \circ B)$ for $x \in H$, $\|x\| = 1$.

Theorem 2. *Assume that* $0 < m_1 \le A \le M_1$ *and* $0 < m_2 \le B \le M_2$, *then*

$$
\begin{aligned}
& (m_1 m_2)^{\frac{M_1 M_2 - \langle (A \circ B) x, x \rangle}{M_1 M_2 - m_1 m_2}} (M_1 M_2)^{\frac{\langle (A \circ B) x, x \rangle - m_1 m_2}{M_1 M_2 - m_1 m_2}} \\
& \le \exp\left(\frac{1}{2 (M_1 M_2)^2} \langle \mathcal{U}^* \left[(M_1 M_2 - P)(P - m_1 m_2) \right] \mathcal{U} x, x \rangle \right) \\
& \times (m_1 m_2)^{\frac{M_1 M_2 - \langle (A \circ B) x, x \rangle}{M_1 M_2 - m_1 m_2}} (M_1 M_2)^{\frac{\langle (A \circ B) x, x \rangle - m_1 m_2}{M_1 M_2 - m_1 m_2}} \\
& \le \exp \left[\langle \mathcal{U}^* (\ln (A \otimes B)) \mathcal{U} x, x \rangle \right] \\
& \le \Delta_x (A \circ B) \\
& \le (m_1 m_2)^{\frac{M_1 M_2 - \langle (A \circ B) x, x \rangle}{M_1 M_2 - m_1 m_2}} (M_1 M_2)^{\frac{\langle (A \circ B) x, x \rangle - m_1 m_2}{M_1 M_2 - m_1 m_2}} \\
& \times \exp\left(\frac{1}{2 (m_1 m_2)^2} \langle (M_1 M_2 - A \circ B)(A \circ B - m_1 m_2) x, x \rangle \right) \\
& \le \exp\left(\frac{1}{8} \left(\frac{M_1 M_2}{m_1 m_2} - 1 \right)^2 \right) \\
& \times (m_1 m_2)^{\frac{M_1 M_2 - \langle (A \circ B) x, x \rangle}{M_1 M_2 - m_1 m_2}} (M_1 M_2)^{\frac{\langle (A \circ B) x, x \rangle - m_1 m_2}{M_1 M_2 - m_1 m_2}} \\
& \le \exp\left(\frac{1}{8} \left(\frac{M_1 M_2}{m_1 m_2} - 1 \right)^2 \right) \exp \langle \mathcal{U}^* (\ln A \otimes B) \mathcal{U} x, x \rangle
\end{aligned} \tag{2.12}
$$

for $x \in H$, $\|x\| = 1$.

Proof. Since $0 < m_1 \le A \le M_1$ and $0 < m_2 \le B \le M_2$, then $0 < m_1 m_2 \le P = A \otimes B \le M_1 M_2$. From (2.11) for $m = m_1 m_2$, $M = M_1 M_2$, $\Phi(P) = \mathcal{U}^* (A \otimes B) \mathcal{U} = A \circ B$ we get

$$
\begin{aligned}
& \ln (m_1 m_2) \frac{M_1 M_2 - A \circ B}{M_1 M_2 - m_1 m_2} + \ln (M_1 M_2) \frac{A \circ B - m_1 m_2}{M_1 M_2 - m_1 m_2} \\
& \le \frac{1}{2 (M_1 M_2)^2} \mathcal{U}^* \left[(M_1 M_2 - P)(P - m_1 m_2) \right] \mathcal{U} \\
& + \ln (m_1 m_2) \frac{M_1 M_2 - A \circ B}{M_1 M_2 - m_1 m_2} + \ln (M_1 M_2) \frac{A \circ B - m_1 m_2}{M_1 M_2 - m_1 m_2} \\
& \le \mathcal{U}^* (\ln (A \otimes B)) \mathcal{U} \le \ln (A \circ B) \\
& \le \frac{1}{2 (m_1 m_2)^2} (M_1 M_2 - A \circ B)(A \circ B - m_1 m_2) \\
& + \ln (m_1 m_2) \frac{M_1 M_2 - A \circ B}{M_1 M_2 - m_1 m_2} + \ln (M_1 M_2) \frac{A \circ B - m_1 m_2}{M_1 M_2 - m_1 m_2} \\
& \le \frac{1}{8} \left(\frac{M_1 M_2}{m_1 m_2} - 1 \right)^2 \\
& + \ln (m_1 m_2) \frac{M_1 M_2 - A \circ B}{M_1 M_2 - m_1 m_2} + \ln (M_1 M_2) \frac{A \circ B - m_1 m_2}{M_1 M_2 - m_1 m_2} \\
& \le \frac{1}{8} \left(\frac{M_1 M_2}{m_1 m_2} - 1 \right)^2 + \mathcal{U}^* (\ln (A \otimes B)) \mathcal{U}.
\end{aligned} \tag{2.13}
$$

If we take the inner product for $x \in H$, $\|x\| = 1$, then we get

$$
\begin{aligned}
&\ln(m_1 m_2)\frac{M_1M_2 - \langle (A\circ B)x, x\rangle}{M_1M_2 - m_1m_2} + \ln(M_1M_2)\frac{\langle (A\circ B)x, x\rangle - m_1m_2}{M_1M_2 - m_1m_2} \\
&\le \frac{1}{2(M_1M_2)^2}\langle \mathcal{U}^*[(M_1M_2 - P)(P - m_1m_2)]\mathcal{U}x, x\rangle \\
&+ \ln(m_1 m_2)\frac{M_1M_2 - \langle (A\circ B)x, x\rangle}{M_1M_2 - m_1m_2} + \ln(M_1M_2)\frac{\langle (A\circ B)x, x\rangle - m_1m_2}{M_1M_2 - m_1m_2} \\
&\le \langle \mathcal{U}^*(\ln(A\otimes B))\mathcal{U}x, x\rangle \le \langle \ln(A\circ B)x, x\rangle \\
&\le \frac{1}{2(m_1m_2)^2}\langle (M_1M_2 - A\circ B)(A\circ B - m_1m_2)x, x\rangle \\
&+ \ln(m_1 m_2)\frac{M_1M_2 - \langle (A\circ B)x, x\rangle}{M_1M_2 - m_1m_2} + \ln(M_1M_2)\frac{\langle (A\circ B)x, x\rangle - m_1m_2}{M_1M_2 - m_1m_2} \\
&\le \frac{1}{8}\left(\frac{M_1M_2}{m_1m_2} - 1\right)^2 \\
&+ \ln(m_1 m_2)\frac{M_1M_2 - \langle (A\circ B)x, x\rangle}{M_1M_2 - m_1m_2} + \ln(M_1M_2)\frac{\langle (A\circ B)x, x\rangle - m_1m_2}{M_1M_2 - m_1m_2} \\
&\le \frac{1}{8}\left(\frac{M_1M_2}{m_1m_2} - 1\right)^2 + \langle \mathcal{U}^*(\ln(A\otimes B))\mathcal{U}x, x\rangle,
\end{aligned}
$$

which, by taking the exponential, is equivalent to (2.12). □

3. Connection to Oppenheim's Inequalities

In the finite dimensional case, if we consider the matrices $A = (a_{ij})$, $B = (b_{ij}) \in \mathbb{M}_n(\mathbb{C})$, then $A\circ B$ has an associated matrix $A\circ B = (a_{ij}b_{ij})$ in $\mathbb{M}_n(\mathbb{C})$.

Recall Hadamard determinant inequality [18, p. 218] for $A \ge 0$

$$\det A \le \det(A\circ 1) \quad (= \prod_{i=1}^n a_{ii})$$

and Oppenheim's inequality [18, p. 242] for A, $B \ge 0$

$$\det A \det B \le \det(A\circ B) \le \det(A\circ 1)\det(B\circ 1) \quad \left(= \prod_{i=1}^n a_{ii}b_{ii}\right).$$

In the recent paper [11] S. Hiramatsu and Y. Seo obtained the following interesting Oppenheim's type inequalities

$$\frac{1}{S(h_1)S(h_2)} \le \frac{\Delta_x(A\circ B)}{\Delta_x(A\circ 1)\Delta_x(1\circ B)} \le S(h_1h_2) \tag{3.1}$$

for $x \in H$, $\|x\| = 1$, provided that $0 < m_1 \le A \le M_1$ and $0 < m_2 \le B \le M_2$.

We have the following similar inequalities:

Proposition 1. *With the assumptions of Theorem 2 we have the determinant inequalities*

$$\frac{1}{D(h_1)D(h_2)} \le \frac{\Delta_x(A\circ B)}{\Delta_x(A\circ 1)\Delta_x(1\circ B)} \le D(h_1h_2), \tag{3.2}$$

where $h_1 = \frac{M_1}{m_1} > 1$, $h_2 = \frac{M_2}{m_2} > 1$ *and* $D(h) := \exp\left[\frac{1}{8}(h-1)^2\right]$.

Proof. By the properties of the tensorial product, we have that

$$A \otimes B = (A \otimes 1)(1 \otimes B)$$

where $A \otimes 1$ and $1 \otimes B$ are commutative operators.

Therefore

$$\ln(A \otimes B) = \ln[(A \otimes 1)(1 \otimes B)] = \ln(A \otimes 1) + \ln(1 \otimes B)$$

and

$$\begin{aligned}\mathcal{U}^*(\ln(A \otimes B))\mathcal{U} &= \mathcal{U}^*[\ln(A \otimes 1) + \ln(1 \otimes B)]\mathcal{U} \\ &= \mathcal{U}^*(\ln(A \otimes 1))\mathcal{U} + \mathcal{U}^*(\ln(1 \otimes B))\mathcal{U}.\end{aligned}$$

Using Jensen's operator inequality for the operator concave function ln, we also have

$$\mathcal{U}^*(\ln(A \otimes 1))\mathcal{U} \le \ln(\mathcal{U}^*(A \otimes 1)\mathcal{U}) = \ln(A \circ 1)$$

and

$$\mathcal{U}^*(\ln(1 \otimes B))\mathcal{U} \le \ln(\mathcal{U}^*((1 \otimes B))\mathcal{U}) = \ln(1 \circ B).$$

These imply for $x \in H$, $\|x\| = 1$ that

$$\begin{aligned}\exp\langle \mathcal{U}^*(\ln(A \otimes B))\mathcal{U}x, x\rangle &\le \exp[\langle \ln(A \circ 1)x, x\rangle + \langle \ln(1 \circ B)x, x\rangle] \\ &= \exp[\langle \ln(A \circ 1)x, x\rangle]\exp[\langle \ln(1 \circ B)x, x\rangle] \\ &= \Delta_x(A \circ 1)\Delta_x(1 \circ B)\end{aligned}$$

and by the second part of (2.12) for $m = m_1 m_2$ and $M = M_1 M_2$, we derive the second inequality in (3.2).

From (2.11) we have

$$\ln \Phi(P) \le \Phi(\ln P) + \frac{1}{8}\left(\frac{M}{m} - 1\right)^2$$

provided that $0 < m \le P \le M$.

Now, if we take in this inequality $0 < m_1 \le P = A \otimes 1 \le M_1$, then we get for $\Phi(P) = \mathcal{U}^*(A \otimes 1)\mathcal{U} = A \circ 1$ that

$$\ln(A \circ 1) \le \frac{1}{8}\left(\frac{M_1}{m_1} - 1\right)^2 + \mathcal{U}^*(\ln(A \otimes 1))\mathcal{U}$$

while for $0 < m_2 \le P = 1 \otimes B \le M_2$

$$\ln(1 \circ B) \le \frac{1}{8}\left(\frac{M_2}{m_2} - 1\right)^2 + \mathcal{U}^*(\ln(1 \otimes B))\mathcal{U},$$

which gives, by addition, that

$$\begin{aligned}&\ln(A \circ 1) + \ln(1 \circ B) \\ &- \ln\left[\exp\left[\frac{1}{8}\left(\frac{M_1}{m_1} - 1\right)^2 + \frac{1}{8}\left(\frac{M_2}{m_2} - 1\right)^2\right]\right] \\ &\le \mathcal{U}^*(\ln(A \otimes 1))\mathcal{U} + \mathcal{U}^*(\ln(1 \otimes B))\mathcal{U} = \mathcal{U}^*(\ln(A \otimes B))\mathcal{U}.\end{aligned}$$

By taking the inner product for $x \in H$, $\|x\| = 1$ we get that

$$\begin{aligned}
&\langle \ln (A \circ 1)\, x, x\rangle + \langle \ln (1 \circ B)\, x, x\rangle \\
&- \ln \left[\exp \left[\frac{1}{8} \left(\frac{M_1}{m_1} - 1 \right)^2 + \frac{1}{8} \left(\frac{M_2}{m_2} - 1 \right)^2 \right] \right] \\
&\leq \langle \mathcal{U}^* \left(\ln (A \otimes B) \right) \mathcal{U} x, x\rangle
\end{aligned}$$

and by taking the exponential, we derive

$$\frac{\exp \langle \ln (A \circ 1)\, x, x\rangle \exp \langle \ln (1 \circ B)\, x, x\rangle}{\exp \left[\frac{1}{8} \left(\frac{M_1}{m_1} - 1 \right)^2 + \frac{1}{8} \left(\frac{M_2}{m_2} - 1 \right)^2 \right]} \leq \exp \langle \mathcal{U}^* \left(\ln (A \otimes B) \right) \mathcal{U} x, x\rangle$$

for $x \in H$, $\|x\| = 1$ and by the third inequality in (2.12) we obtain the first part of (3.2). □

By conducting some numerical experiments we can state the following:

Conjecture 1. *With the above notations, we have for $h > 1$ that*

$$D(h) > S(h).$$

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