q-Deformed hyperbolic tangent relied Banach space valued multivariate multi layer neural network approximation

George A. Anastassiou
Department of Mathematical Sciences, University of Memphis,
Memphis, TN 38152, U.S.A.
ganastss@memphis.edu

Abstract

Here we study the multivariate quantitative approximation of Banach space valued continuous multivariate functions on a box or $\mathbb{R}^N$, $N \in \mathbb{N}$, by the multivariate normalized, quasi-interpolation, Kantorovich type and quadrature type neural network operators. We research also the case of approximation by iterated multilayer neural network operators of the last four types. These approximations are achieved by establishing multidimensional Jackson type inequalities involving the multivariate modulus of continuity of the engaged function or its high order Fréchet derivatives or partial derivatives. Our multivariate operators are defined by using a multidimensional density function induced by a q-deformed hyperbolic tangent sigmoid function. The approximations are pointwise and uniform. The related feed-forward neural network are with one or multi hidden layers.

2020 Mathematics Subject Classification: 41A17, 41A25, 41A30, 41A36.

Keywords and phrases: multi layer approximation, q-deformed hyperbolic tangent sigmoid function, multivariate neural network approximation, quasi-interpolation operator, Kantorovich type operator, quadrature type operator, multivariate modulus of continuity, abstract approximation, iterated approximation.

1 Introduction

The author in [2] and [3], see chapters 2-5, was the first to establish neural network approximations to continuous functions with rates by very specifically defined neural network operators of Cardaliaguet-Euvrard and "Squashing" types,

RGMIA Res. Rep. Coll. **25** (2022), Art. 133, 33 pp. Received 8/12/22

by employing the modulus of continuity of the engaged function or its high order derivative, and producing very tight Jackson type inequalities. He treats there both the univariate and multivariate cases. The defining these operators "bell-shaped" and "squashing" functions are assumed to be of compact support. Also in [3] he gives the Nth order asymptotic expansion for the error of weak approximation of these two operators to a special natural class of smooth functions, see chapters 4-5 there.

Motivations for this work are the article [15] of Z. Chen and F. Cao, also by [4]-[13], [16], [17].

Here we perform a q-deformed, $q > 1$, $q \neq 1$, hyperbolic tangent sigmoid function based neural network approximations to continuous functions over boxes or over the whole $\mathbb{R}^N$, $N \in \mathbb{N}$ and also iterated, multi layer approximations. All convergences here are with rates expressed via the multivariate modulus of continuity of the involved function or its high order Fréchet derivative or partial derivatives and given by very tight multidimensional Jackson type inequalities.

We come up with the "right" precisely defined multivariate normalized, quasi-interpolation neural network operators related to boxes or $\mathbb{R}^N$, as well as Kantorovich type and quadrature type related operators on $\mathbb{R}^N$. Our boxes are not necessarily symmetric to the origin. In preparation to prove our results we establish important properties of the basic multivariate density function induced by the q-deformed hyperbolic tangent sigmoid function.

Feed-forward neural networks (FNNs) with one hidden layer, the only type of networks we deal with in this article, are mathematically expressed as

$$N_n(x) = \sum_{j=0}^{n} c_j \sigma\left(\langle a_j \cdot x\rangle + b_j\right), \quad x \in \mathbb{R}^s, \quad s \in \mathbb{N},$$

where for $0 \leq j \leq n$, $b_j \in \mathbb{R}$ are the thresholds, $a_j \in \mathbb{R}^s$ are the connection weights, $c_j \in \mathbb{R}$ are the coefficients, $\langle a_j \cdot x\rangle$ is the inner product of a_j and x, and σ is the activation function of the network. In many fundamental network models, the activation function is the hyperbolic tangent sigmoid function. About neural networks read [19] - [21].

2 About q-deformed $\tanh_q x$

We found $\tanh_q x$ in [18].

We study $\tanh_q x$ and we have proved ([13]) that it is a sigmoid function and we will mention several of its properties related to the approximation by neural network operators. So the deformed $\tanh_q x$ is defined as follows:

$$h_q(x) := \tanh qx := \frac{e^x - qe^{-x}}{e^x + qe^{-x}}, \tag{1}$$

$x \in \mathbb{R}$, where $q \in (0, +\infty) - \{1\}$.

We have that

$$h_q(0) = \frac{1-q}{1+q} \neq 0, \quad q \neq 1. \tag{2}$$

We notice that

$$h_q(-x) = \frac{e^{-x} - qe^{x}}{e^{-x} + qe^{x}} = \frac{\frac{1}{q}e^{-x} - e^{x}}{\frac{1}{q}e^{-x} + e^{x}} = -\left(\frac{e^{x} - \frac{1}{q}e^{-x}}{e^{x} + \frac{1}{q}e^{-x}}\right) = -h_{\frac{1}{q}}(x). \tag{3}$$

That is

$$h_q(-x) = -h_{\frac{1}{q}}(x), \ \ \forall\, x \in \mathbb{R}, \tag{4}$$

and $h_{\frac{1}{q}}(x) = -h_q(-x)$, hence

$$h'_{\frac{1}{q}}(x) = h'_q(-x). \tag{5}$$

It is

$$h_q(x) = \frac{e^{2x} - q}{e^{2x} + q} = \frac{1 - \frac{q}{e^{2x}}}{1 + \frac{q}{e^{2x}}} \underset{(x \to +\infty)}{\to} 1,$$

i.e.

$$h_q(+\infty) = 1. \tag{6}$$

Furthermore

$$h_q(x) = \frac{e^{2x} - q}{e^{2x} + q} \underset{(x \to -\infty)}{\to} -\frac{q}{q} = -1,$$

i.e.

$$h_q(-\infty) = -1. \tag{7}$$

We find that

$$h'_q(x) = \frac{4qe^{2x}}{(e^{2x} + q)^2} > 0, \tag{8}$$

therefore h_q is striclty increasing.

Next we obtain ($x \in \mathbb{R}$)

$$h''_q(x) = 8qe^{2x}\left(\frac{q - e^{2x}}{(e^{2x} + q)^3}\right) \in C(\mathbb{R}). \tag{9}$$

We observe that

$$q - e^{2x} \gtrless 0 \Leftrightarrow q \gtrless e^{2x} \Leftrightarrow \ln q \gtrless 2x \Leftrightarrow x \lessgtr \frac{\ln q}{2}.$$

So, in case of $x < \frac{\ln q}{2}$, we have that h_q is strictly concave up, with $h''_q\left(\frac{\ln q}{2}\right) = 0$.

And in case of $x > \frac{\ln q}{2}$, we have that h_q is strictly concave down.

So h_q is a shifted sigmoid function with $h_q(0) = \frac{1-q}{1+q} \neq 0$, and $h_q(-x) = -h_{q^{-1}}(x)$ (a semi-odd function).

By $1 > -1$, $x+1 > x-1$, we consider the activation function

$$\psi_q(x) := \frac{1}{4}\left(h_q(x+1) - h_q(x-1)\right) > 0, \tag{10}$$

$\forall x \in \mathbb{R}$, $q > 0$, $q \neq 1$. Notice that $\psi_q(\pm\infty) = 0$, so the x-axis is horizontal asymptote.

We have that

$$\psi_q(-x) = \frac{1}{4}\left[h_q(-x+1) - h_q(-x-1)\right] =$$

$$\frac{1}{4}\left[\left(\frac{e^{-x+1} - qe^{x-1}}{e^{-x+1} + qe^{x-1}}\right) - \left(\frac{e^{-x-1} - qe^{x+1}}{e^{-x-1} + qe^{x+1}}\right)\right] =$$

$$\frac{1}{4}\left[\left(\frac{e^{x+1} - \frac{1}{q}e^{-x-1}}{e^{x+1} + \frac{1}{q}e^{-x-1}}\right) - \left(\frac{e^{x-1} - \frac{1}{q}e^{-x+1}}{q^{x-1} + \frac{1}{q}e^{-x+1}}\right)\right] =$$

$$\frac{1}{4}\left[h_{q^{-1}}(x+1) - h_{q^{-1}}(x-1)\right] = \psi_{\frac{1}{q}}(x), \quad \forall\, x \in \mathbb{R}. \tag{11}$$

Thus

$$\psi_q(-x) = \psi_{q^{-1}}(x), \ \ \forall\, x \in \mathbb{R}, \ \forall\, q > 0, \, q \neq 1, \ \text{a deformed symmetry.} \tag{12}$$

Next we have that

$$\psi_q'(x) = \frac{1}{4}\left[h_q'(x+1) - h_q'(x-1)\right], \quad \forall\, x \in \mathbb{R}. \tag{13}$$

Let $x < \frac{\ln q}{2} - 1$, then $x - 1 < x + 1 < \frac{\ln q}{2}$ and $h_q'(x+1) > h_q'(x-1)$ (by h_q being strictly concave up for $x < \frac{\ln q}{2}$), that is $\psi_q'(x) > 0$. Hence ψ_q is striclty increasing over $\left(-\infty, \frac{\ln q}{2} - 1\right)$.

Let now $x - 1 > \frac{\ln q}{2}$, then $x + 1 > x - 1 > \frac{\ln q}{2}$, and $h_q'(x+1) < h_q'(x-1)$, that is $\psi_q'(x) < 0$.

Therefore ψ_q is strictly decreasing over $\left(\frac{\ln q}{2} + 1, +\infty\right)$.

Next, let $\frac{\ln q}{2} - 1 \leq x \leq \frac{\ln q}{2} + 1$. We have that

$$\psi_q''(x) = \frac{1}{4}\left[h_q''(x+1) - h_q''(x-1)\right] =$$

$$2q\left[e^{2(x+1)}\left(\frac{q - e^{2(x+1)}}{\left(e^{2(x+1)} + q\right)^3}\right) - e^{2(x-1)}\left(\frac{q - e^{2(x-1)}}{\left(e^{2(x-1)} + q\right)^3}\right)\right]. \tag{14}$$

By $\frac{\ln q}{2} \leq x \Leftrightarrow \frac{\ln q}{2} \leq x + 1 \Leftrightarrow \ln q \leq 2(x+1) \Leftrightarrow q \leq e^{2(x+1)} \Leftrightarrow q - e^{2(x+1)} \leq 0$.

By $x \leq \frac{\ln q}{2} + 1 \Leftrightarrow x - 1 \leq \frac{\ln q}{2} \Leftrightarrow 2(x-1) \leq \ln q \Leftrightarrow e^{2(x-1)} \leq q \Leftrightarrow q - e^{2(x-1)} \geq 0$.

Clearly by (14) we get that $\psi_q''(x) \leq 0$, for $x \in \left[\frac{\ln q}{2} - 1, \frac{\ln q}{2} + 1\right]$.

More precisely ψ_q is concave down over $\left[\frac{\ln q}{2} - 1, \frac{\ln q}{2} + 1\right]$, and strictly concave down over $\left(\frac{\ln q}{2} - 1, \frac{\ln q}{2} + 1\right)$.

Consequently ψ_q has a bell-type shape over $\mathbb{R}$.

Of course it holds $\psi_q''\left(\frac{\ln q}{2}\right) < 0$.

At $x = \frac{\ln q}{2}$, we have

$$\psi_q'(x) = \frac{1}{4}\left[h_q'(x+1) - h_q'(x-1)\right] =$$

$$q\left[\frac{e^{2(x+1)}}{\left(e^{2(x+1)} + q\right)^2} - \frac{e^{2(x-1)}}{\left(e^{2(x-1)} + q\right)^2}\right], \tag{15}$$

and

$$\psi_q'\left(\frac{\ln q}{2}\right) = q\left[\frac{e^{2\left(\frac{\ln q}{2}+1\right)}}{\left(e^{2\left(\frac{\ln q}{2}+1\right)} + q\right)^2} - \frac{e^{2\left(\frac{\ln q}{2}-1\right)}}{\left(e^{2\left(\frac{\ln q}{2}-1\right)} + q\right)^2}\right] = \tag{16}$$

$$\frac{e^2\left(e^{-2}+1\right)^2 - e^{-2}\left(e^2+1\right)^2}{\left(e^2+1\right)^2\left(e^{-2}+1\right)^2} = 0.$$

Therefore at the only critical number $x = \frac{\ln q}{2}$ ψ_q achieves a maximum, which is

$$\psi(x) = \frac{1}{4}\left[h_q(x+1) - h_q(x-1)\right] =$$

$$\frac{1}{4}\left[\left(\frac{e^{x+1} - qe^{-x-1}}{e^{x+1} + qe^{-x-1}}\right) - \left(\frac{e^{x-1} - qe^{-x+1}}{q^{x-1} + qe^{-x+1}}\right)\right] = \tag{17}$$

$$\frac{1}{4}\left[\left(\frac{e - e^{-1}}{e + e^{-1}}\right) - \left(\frac{e^{-1} - e}{e^{-1} + e}\right)\right] = \frac{\left(e - e^{-1}\right)}{2\left(e + e^{-1}\right)}.$$

Conclusion: the maximum value of ψ_q is

$$\psi_q\left(\frac{\ln q}{2}\right) = \frac{\left(e - e^{-1}\right)}{2\left(e + e^{-1}\right)} = \frac{\tanh 1}{2}. \tag{18}$$

We need

Theorem 1 *([13]) We have that*

$$\sum_{i=-\infty}^{\infty} \psi_q(x-i) = 1, \ \ \forall\, x \in \mathbb{R}, \forall\, q > 0,\, q \neq 1. \tag{19}$$

It holds

Theorem 2 *([13]) It holds*

$$\int_{-\infty}^{\infty} \psi_q(x)\, dx = 1, \quad q > 0,\ q \neq 1. \tag{20}$$

So ψ_q is a density function on $\mathbb{R}$.

We need also the following result

Theorem 3 *([13]) Let $0 < \alpha < 1$, and $n \in \mathbb{N}$ with $n^{1-\alpha} > 2$; $q > 0$, $q \neq 1$. Then*

$$\sum_{\substack{k=-\infty \\ : |nx-k| \geq n^{1-\alpha}}}^{\infty} \psi_q(nx-k) < \max\left\{q, \frac{1}{q}\right\} e^4 e^{-2n^{(1-\alpha)}} = Qe^{-2n^{(1-\alpha)}};$$

$$Q := \max\left\{q, \frac{1}{q}\right\} e^4. \tag{21}$$

Let $\lceil\cdot\rceil$ the ceiling of the number, and $\lfloor\cdot\rfloor$ the integral part of the number. We need

Theorem 4 *([13]) Let $x \in [a,b] \subset \mathbb{R}$ and $n \in \mathbb{N}$ so that $\lceil na\rceil \leq \lfloor nb\rfloor$. For $q > 0$, $q \neq 1$, we consider the number $\lambda_q > z_0 > 0$ with $\psi(z_0) = \psi_q(0)$ and $\lambda_q > 1$. Then*

$$\frac{1}{\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} \psi_q(nx-k)} < max\left\{\frac{1}{\psi_q(\lambda_q)}, \frac{1}{\psi_{\frac{1}{q}}\left(\lambda_{\frac{1}{q}}\right)}\right\} =: \Phi(q). \tag{22}$$

We make

Remark 5 *([13])*

(i) We have that

$$\lim_{n\to+\infty} \sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} \psi_q(nx-k) \neq 1, \quad \textit{for at least some } x \in [a,b]. \tag{23}$$

(ii) Let $[a,b] \subset \mathbb{R}$. For large n we always have $\lceil na\rceil \leq \lfloor nb\rfloor$. Also $a \leq \frac{k}{n} \leq b$, iff $\lceil na\rceil \leq k \leq \lfloor nb\rfloor$.

In general it holds

$$\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} \psi_q(nx-k) \leq 1. \tag{24}$$

We make

Remark 6 *We introduce*

$$Z_q(x_1,...,x_N) := Z_q(x) := \prod_{i=1}^{N} \psi_q(x_i), \quad x = (x_1,...,x_N) \in \mathbb{R}^N,\ q>0,\ q\neq 1,\ N\in\mathbb{N}. \tag{25}$$

It has the properties:

(i) $Z_q(x) > 0,\ \forall\, x \in \mathbb{R}^N$,

(ii)

$$\sum_{k=-\infty}^{\infty} Z_q(x-k) := \sum_{k_1=-\infty}^{\infty}\sum_{k_2=-\infty}^{\infty}...\sum_{k_N=-\infty}^{\infty} Z_q(x_1-k_1,...,x_N-k_N) = 1, \tag{26}$$

where $k := (k_1,...,k_n) \in \mathbb{Z}^N$, $\forall\, x \in \mathbb{R}^N$,

hence

(iii)

$$\sum_{k=-\infty}^{\infty} Z_q(nx-k) = 1, \tag{27}$$

$\forall\, x \in \mathbb{R}^N$*;* $n \in \mathbb{N}$*,*

and

(iv)

$$\int_{\mathbb{R}^N} Z_q(x)\,dx = 1, \tag{28}$$

that is Z_q *is a multivariate density function.*

Here denote $\|x\|_\infty := \max\{|x_1|,...,|x_N|\}$, $x \in \mathbb{R}^N$*, also set* $\infty := (\infty,...,\infty)$*,* $-\infty := (-\infty,...,-\infty)$ *upon the multivariate context, and*

$$\begin{aligned} \lceil na \rceil &:= (\lceil na_1 \rceil,...,\lceil na_N \rceil), \\ \lfloor nb \rfloor &:= (\lfloor nb_1 \rfloor,...,\lfloor nb_N \rfloor), \end{aligned} \tag{29}$$

where $a := (a_1,...,a_N)$*,* $b := (b_1,...,b_N)$*.*

We obviously see that

$$\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} Z_q(nx-k) = \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor}\left(\prod_{i=1}^{N}\psi_q(nx_i-k_i)\right) =$$

$$\sum_{k_1=\lceil na_1 \rceil}^{\lfloor nb_1 \rfloor}...\sum_{k_N=\lceil na_N \rceil}^{\lfloor nb_N \rfloor}\left(\prod_{i=1}^{N}\psi_q(nx_i-k_i)\right) = \prod_{i=1}^{N}\left(\sum_{k_i=\lceil na_i \rceil}^{\lfloor nb_i \rfloor}\psi_q(nx_i-k_i)\right). \tag{30}$$

For $0<\beta<1$ and $n \in \mathbb{N}$, a fixed $x \in \mathbb{R}^N$, we have that

$$\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} Z_q (nx-k) =$$

$$\sum_{\begin{cases} k = \lceil na \rceil \\ \left\| \frac{k}{n} - x \right\|_\infty \leq \frac{1}{n^\beta} \end{cases}}^{\lfloor nb \rfloor} Z_q (nx-k) + \sum_{\begin{cases} k = \lceil na \rceil \\ \left\| \frac{k}{n} - x \right\|_\infty > \frac{1}{n^\beta} \end{cases}}^{\lfloor nb \rfloor} Z_q (nx-k) . \tag{31}$$

In the last two sums the counting is over disjoint vector sets of k's, because the condition $\left\| \frac{k}{n} - x \right\|_\infty > \frac{1}{n^\beta}$ implies that there exists at least one $\left| \frac{k_r}{n} - x_r \right| > \frac{1}{n^\beta}$, where $r \in \{1, ..., N\}$.

(v) By Theorem 3 and as in [10], pp. 379-380, we derive that

$$\sum_{\begin{cases} k = \lceil na \rceil \\ \left\| \frac{k}{n} - x \right\|_\infty > \frac{1}{n^\beta} \end{cases}}^{\lfloor nb \rfloor} Z_q (nx-k) < Q e^{-2n^{(1-\beta)}}, \;\; 0<\beta<1, \;\; m \in \mathbb{N}, \tag{32}$$

with $n \in \mathbb{N} : n^{1-\beta} > 2$, $x \in \prod_{i=1}^N [a_i, b_i]$.

(vi) By Theorem 4 we get that

$$0 < \frac{1}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} Z_q (nx-k)} < (\Phi(q))^N , \tag{33}$$

$\forall \; x \in \left(\prod_{i=1}^N [a_i, b_i] \right)$, $\;n \in \mathbb{N}$.

It is also clear that

(vii)

$$\sum_{\begin{cases} k = -\infty \\ \left\| \frac{k}{n} - x \right\|_\infty > \frac{1}{n^\beta} \end{cases}}^{\infty} Z_q (nx-k) < Q e^{-2n^{(1-\beta)}}, \tag{34}$$

$0<\beta<1$, $n \in \mathbb{N} : n^{1-\beta} > 2$, $x \in \mathbb{R}^N$, $m \in \mathbb{N}$.

Furthermore it holds

$$\lim_{n \to \infty} \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} Z_q (nx-k) \neq 1, \tag{35}$$

for at least some $x \in \left(\prod_{i=1}^N [a_i, b_i] \right)$.

Here $\left(X, \|\cdot\|_\gamma \right)$ is a Banach space.

Let $f \in C\left(\prod_{i=1}^{N}[a_i,b_i],X\right)$, $x=(x_1,...,x_N)\in\prod_{i=1}^{N}[a_i,b_i]$, $n\in\mathbb{N}$ *such that* $\lceil na_i\rceil \le \lfloor nb_i\rfloor$, $i=1,...,N$.

We introduce and define the following multivariate linear normalized neural network operator ($x:=(x_1,...,x_N)\in\left(\prod_{i=1}^{N}[a_i,b_i]\right)$):

$$A_n(f,x_1,...,x_N):=A_n(f,x):=\frac{\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} f\left(\frac{k}{n}\right)Z_q(nx-k)}{\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} Z_q(nx-k)}=$$

$$\frac{\sum_{k_1=\lceil na_1\rceil}^{\lfloor nb_1\rfloor}\sum_{k_2=\lceil na_2\rceil}^{\lfloor nb_2\rfloor}...\sum_{k_N=\lceil na_N\rceil}^{\lfloor nb_N\rfloor} f\left(\frac{k_1}{n},...,\frac{k_N}{n}\right)\left(\prod_{i=1}^{N}\psi_q(nx_i-k_i)\right)}{\prod_{i=1}^{N}\left(\sum_{k_i=\lceil na_i\rceil}^{\lfloor nb_i\rfloor}\psi_q(nx_i-k_i)\right)}. \quad (36)$$

For large enough $n\in\mathbb{N}$ *we always obtain* $\lceil na_i\rceil\le\lfloor nb_i\rfloor$, $i=1,...,N$. *Also* $a_i\le\frac{k_i}{n}\le b_i$, *iff* $\lceil na_i\rceil\le k_i\le\lfloor nb_i\rfloor$, $i=1,...,N$.

When $g\in C\left(\prod_{i=1}^{N}[a_i,b_i]\right)$ *we define the companion operator*

$$\widetilde{A}_n(g,x):=\frac{\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} g\left(\frac{k}{n}\right)Z_q(nx-k)}{\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} Z_q(nx-k)}. \quad (37)$$

Clearly $\widetilde{A}_n$ *is a positive linear operator. We have that*

$$\widetilde{A}_n(1,x)=1,\ \ \forall\ x\in\left(\prod_{i=1}^{N}[a_i,b_i]\right).$$

Notice that $A_n(f)\in C\left(\prod_{i=1}^{N}[a_i,b_i],X\right)$ *and* $\widetilde{A}_n(g)\in C\left(\prod_{i=1}^{N}[a_i,b_i]\right)$.

Furthermore it holds

$$\|A_n(f,x)\|_\gamma\le\frac{\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}\left\|f\left(\frac{k}{n}\right)\right\|_\gamma Z_q(nx-k)}{\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} Z_q(nx-k)}=\widetilde{A}_n\left(\|f\|_\gamma,x\right), \quad (38)$$

$\forall\ x\in\prod_{i=1}^{N}[a_i,b_i]$.

Clearly $\|f\|_\gamma\in C\left(\prod_{i=1}^{N}[a_i,b_i]\right)$.

So, we have that

$$\|A_n(f,x)\|_\gamma\le\widetilde{A}_n\left(\|f\|_\gamma,x\right), \quad (39)$$

$\forall\ x\in\prod_{i=1}^{N}[a_i,b_i]$, $\forall\ n\in\mathbb{N}$, $\forall\ f\in C\left(\prod_{i=1}^{N}[a_i,b_i],X\right)$.

Let $c\in X$ *and* $g\in C\left(\prod_{i=1}^{N}[a_i,b_i]\right)$, *then* $cg\in C\left(\prod_{i=1}^{N}[a_i,b_i],X\right)$.

Furthermore it holds

$$A_n(cg,x)=c\widetilde{A}_n(g,x),\ \ \forall\ x\in\prod_{i=1}^{N}[a_i,b_i]. \quad (40)$$

Since $\widetilde{A}_n(1) = 1$, *we get that*

$$A_n(c) = c, \ \ \forall \ c \in X. \tag{41}$$

We call $\widetilde{A}_n$ *the companion operator of* A_n.

For convenience we call

$$A_n^*(f,x) := \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} f\left(\frac{k}{n}\right) Z_q(nx-k) =$$

$$\sum_{k_1=\lceil na_1 \rceil}^{\lfloor nb_1 \rfloor} \sum_{k_2=\lceil na_2 \rceil}^{\lfloor nb_2 \rfloor} \dots \sum_{k_N=\lceil na_N \rceil}^{\lfloor nb_N \rfloor} f\left(\frac{k_1}{n}, \dots, \frac{k_N}{n}\right) \left(\prod_{i=1}^{N} \psi_q(nx_i - k_i)\right), \tag{42}$$

$\forall \ x \in \left(\prod_{i=1}^{N} [a_i, b_i]\right)$.

That is

$$A_n(f,x) := \frac{A_n^*(f,x)}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} Z_q(nx-k)}, \tag{43}$$

$\forall \ x \in \left(\prod_{i=1}^{N} [a_i, b_i]\right)$, $n \in \mathbb{N}$.

Hence

$$A_n(f,x) - f(x) = \frac{A_n^*(f,x) - f(x)\left(\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} Z_q(nx-k)\right)}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} Z_q(nx-k)}. \tag{44}$$

Consequently we derive

$$\|A_n(f,x) - f(x)\|_\gamma \overset{(33)}{\leq} (\Phi(q))^N \left\| A_n^*(f,x) - f(x) \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} Z_q(nx-k) \right\|_\gamma, \tag{45}$$

$\forall \ x \in \left(\prod_{i=1}^{N} [a_i, b_i]\right)$.

We will estimate the right hand side of (45).

For the last and others we need

Definition 7 *([11], p. 274) Let* M *be a convex and compact subset of* $\left(\mathbb{R}^N, \|\cdot\|_p\right)$, $p \in [1, \infty]$, *and* $\left(X, \|\cdot\|_\gamma\right)$ *be a Banach space. Let* $f \in C(M, X)$. *We define the first modulus of continuity of* f *as*

$$\omega_1(f,\delta) := \sup_{\substack{x,y \in M: \\ \|x-y\|_p \leq \delta}} \|f(x) - f(y)\|_\gamma, \ \ 0 < \delta \leq diam(M). \tag{46}$$

If $\delta > diam(M)$, *then*

$$\omega_1(f,\delta) = \omega_1(f, diam(M)). \tag{47}$$

Notice $\omega_1(f,\delta)$ is increasing in $\delta>0$. For $f\in C_B(M,X)$ (continuous and bounded functions) $\omega_1(f,\delta)$ is defined similarly.

Lemma 8 *([11], p. 274) We have $\omega_1(f,\delta)\to 0$ as $\delta\downarrow 0$, iff $f\in C(M,X)$, where M is a convex compact subset of $\left(\mathbb{R}^N,\|\cdot\|_p\right)$, $p\in[1,\infty]$.*

Clearly we have also: $f\in C_U\left(\mathbb{R}^N,X\right)$ (uniformly continuous functions), iff $\omega_1(f,\delta)\to 0$ as $\delta\downarrow 0$, where ω_1 is defined similarly to (46). The space $C_B\left(\mathbb{R}^N,X\right)$ denotes the continuous and bounded functions on $\mathbb{R}^N$.

Let now $f\in C^m\left(\prod_{i=1}^N[a_i,b_i]\right)$, $m,N\in\mathbb{N}$. Here f_α denotes a partial derivative of f, $\alpha:=(\alpha_1,...,\alpha_N)$, $\alpha_i\in\mathbb{Z}_+$, $i=1,...,N$, and $|\alpha|:=\sum_{i=1}^N\alpha_i=l$, where $l=0,1,...,m$. We write also $f_\alpha:=\frac{\partial^\alpha f}{\partial x^\alpha}$ and we say it is of order l.

We denote
$$\omega_{1,m}^{\max}(f_\alpha,h):=\max_{\alpha:|\alpha|=m}\omega_1(f_\alpha,h). \tag{48}$$
Call also
$$\|f_\alpha\|_{\infty,m}^{\max}:=\max_{|\alpha|=m}\left\{\|f_\alpha\|_\infty\right\}, \tag{49}$$
where $\|\cdot\|_\infty$ is the supremum norm.

When $f\in C_B\left(\mathbb{R}^N,X\right)$ we define,
$$B_n(f,x):=B_n(f,x_1,...,x_N):=\sum_{k=-\infty}^{\infty}f\left(\frac{k}{n}\right)Z_q(nx-k):=$$
$$\sum_{k_1=-\infty}^{\infty}\sum_{k_2=-\infty}^{\infty}...\sum_{k_N=-\infty}^{\infty}f\left(\frac{k_1}{n},\frac{k_2}{n},...,\frac{k_N}{n}\right)\left(\prod_{i=1}^N\psi_q(nx_i-k_i)\right), \tag{50}$$
$n\in\mathbb{N}$, $\forall\ x\in\mathbb{R}^N$, $N\in\mathbb{N}$, the multivariate quasi-interpolation neural network operator.

Also for $f\in C_B\left(\mathbb{R}^N,X\right)$ we define the multivariate Kantorovich type neural network operator
$$C_n(f,x):=C_n(f,x_1,...,x_N):=\sum_{k=-\infty}^{\infty}\left(n^N\int_{\frac{k}{n}}^{\frac{k+1}{n}}f(t)\,dt\right)Z_q(nx-k)=$$
$$\sum_{k_1=-\infty}^{\infty}\sum_{k_2=-\infty}^{\infty}...\sum_{k_N=-\infty}^{\infty}\left(n^N\int_{\frac{k_1}{n}}^{\frac{k_1+1}{n}}\int_{\frac{k_2}{n}}^{\frac{k_2+1}{n}}...\int_{\frac{k_N}{n}}^{\frac{k_N+1}{n}}f(t_1,...,t_N)\,dt_1...dt_N\right)$$
$$\cdot\left(\prod_{i=1}^N\psi_q(nx_i-k_i)\right), \tag{51}$$

$n \in \mathbb{N}$, $\forall\, x \in \mathbb{R}^N$.

Again for $f \in C_B\left(\mathbb{R}^N, X\right)$, $N \in \mathbb{N}$, we define the multivariate neural network operator of quadrature type $D_n(f,x)$, $n \in \mathbb{N}$, as follows.

Let $\theta = (\theta_1, ..., \theta_N) \in \mathbb{N}^N$, $r = (r_1, ..., r_N) \in \mathbb{Z}_+^N$, $w_r = w_{r_1,r_2,...r_N} \geq 0$, such that $\sum_{r=0}^{\theta} w_r = \sum_{r_1=0}^{\theta_1} \sum_{r_2=0}^{\theta_2} ... \sum_{r_N=0}^{\theta_N} w_{r_1,r_2,...r_N} = 1$; $k \in \mathbb{Z}^N$ and

$$\delta_{nk}(f) := \delta_{n,k_1,k_2,...,k_N}(f) := \sum_{r=0}^{\theta} w_r f\left(\frac{k}{n} + \frac{r}{n\theta}\right) =$$

$$\sum_{r_1=0}^{\theta_1} \sum_{r_2=0}^{\theta_2} ... \sum_{r_N=0}^{\theta_N} w_{r_1,r_2,...r_N} f\left(\frac{k_1}{n} + \frac{r_1}{n\theta_1}, \frac{k_2}{n} + \frac{r_2}{n\theta_2}, ..., \frac{k_N}{n} + \frac{r_N}{n\theta_N}\right), \tag{52}$$

where $\frac{r}{\theta} := \left(\frac{r_1}{\theta_1}, \frac{r_2}{\theta_2}, ..., \frac{r_N}{\theta_N}\right)$.

We set

$$D_n(f,x) := D_n(f, x_1, ..., x_N) := \sum_{k=-\infty}^{\infty} \delta_{nk}(f) Z_q(nx - k) = \tag{53}$$

$$\sum_{k_1=-\infty}^{\infty} \sum_{k_2=-\infty}^{\infty} ... \sum_{k_N=-\infty}^{\infty} \delta_{n,k_1,k_2,...,k_N}(f) \left(\prod_{i=1}^{N} \psi_q(nx_i - k_i)\right),$$

$\forall\, x \in \mathbb{R}^N$.

In this article we study the approximation properties of A_n, B_n, C_n, D_n neural network operators and as well of their iterates, that is acting with multilayer neural networks. Thus the quantitative pointwise and uniform convergence of these operators to the unit operator I.

3 Multivariate general Neural Network Approximations

Here we present several vectorial neural network approximations to Banach space valued functions given with rates.

We give

Theorem 9 *Let $f \in C\left(\prod_{i=1}^{N} [a_i, b_i], X\right)$, $0 < \beta < 1$, $x \in \left(\prod_{i=1}^{N} [a_i, b_i]\right)$, $N, n \in \mathbb{N}$ with $n^{1-\beta} > 2$. Then*

1)

$$\|A_n(f,x) - f(x)\|_\gamma \leq$$

$$(\Phi(q))^N \left[\omega_1\left(f, \frac{1}{n^\beta}\right) + 2Qe^{-2n^{(1-\beta)}} \left\| \|f\|_\gamma \right\|_\infty\right] =: \lambda_1(n), \tag{54}$$

and

2)

$$\left\| \left\| A_n(f) - f \right\|_\gamma \right\|_\infty \leq \lambda_1(n). \tag{55}$$

We notice that $\lim_{n\to\infty} A_n(f) \overset{\|\cdot\|_\gamma}{=} f$, *pointwise and uniformly.*

Above ω_1 *is with respect to* $p = \infty$.

Proof. We observe that

$$\Delta(x) := A_n^*(f, x) - f(x) \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} Z_q(nx - k) =$$

$$\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} f\left(\frac{k}{n}\right) Z_q(nx - k) - \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} f(x) Z_q(nx - k) =$$

$$\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \left(f\left(\frac{k}{n}\right) - f(x) \right) Z_q(nx - k). \tag{56}$$

Thus

$$\|\Delta(x)\|_\gamma \leq \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \left\| f\left(\frac{k}{n}\right) - f(x) \right\|_\gamma Z_q(nx - k) =$$

$$\sum_{\substack{k = \lceil na \rceil \\ \left\| \frac{k}{n} - x \right\|_\infty \leq \frac{1}{n^\beta}}}^{\lfloor nb \rfloor} \left\| f\left(\frac{k}{n}\right) - f(x) \right\|_\gamma Z_q(nx - k) +$$

$$\sum_{\substack{k = \lceil na \rceil \\ \left\| \frac{k}{n} - x \right\|_\infty > \frac{1}{n^\beta}}}^{\lfloor nb \rfloor} \left\| f\left(\frac{k}{n}\right) - f(x) \right\|_\gamma Z_q(nx - k) \overset{(27)}{\leq}$$

$$\omega_1\left(f, \frac{1}{n^\beta}\right) + 2 \left\| \|f\|_\gamma \right\|_\infty \sum_{\substack{k = \lceil na \rceil \\ \left\| \frac{k}{n} - x \right\|_\infty > \frac{1}{n^\beta}}}^{\lfloor nb \rfloor} Z_q(nx - k) \overset{(32)}{\leq}$$

$$\omega_1\left(f, \frac{1}{n^\beta}\right) + 2Qe^{-2n^{(1-\beta)}} \left\| \|f\|_\gamma \right\|_\infty. \tag{57}$$

So that

$$\|\Delta(x)\|_\gamma \leq \omega_1\left(f, \frac{1}{n^\beta}\right) + 2Qe^{-2n^{(1-\beta)}} \left\| \|f\|_\gamma \right\|_\infty. \tag{58}$$

Now using (45) we finish the proof. ■

We make

Remark 10 *([11], pp. 263-266) Let* $\left(\mathbb{R}^N, \|\cdot\|_p\right)$*,* $N \in \mathbb{N}$*; where* $\|\cdot\|_p$ *is the* L_p*-norm,* $1 \leq p \leq \infty$*.* $\mathbb{R}^N$ *is a Banach space, and* $\left(\mathbb{R}^N\right)^j$ *denotes the* j*-fold product space* $\mathbb{R}^N \times ... \times \mathbb{R}^N$ *endowed with the max-norm* $\|x\|_{(\mathbb{R}^N)^j} := \max_{1 \leq \lambda \leq j} \|x_\lambda\|_p$*, where* $x := (x_1, ..., x_j) \in \left(\mathbb{R}^N\right)^j$*.*

Let $\left(X, \|\cdot\|_\gamma\right)$ *be a general Banach space. Then the space* $L_j := L_j\left(\left(\mathbb{R}^N\right)^j; X\right)$ *of all* j*-multilinear continuous maps* $g : \left(\mathbb{R}^N\right)^j \to X$*,* $j = 1, ..., m$*, is a Banach space with norm*

$$\|g\| := \|g\|_{L_j} := \sup_{\left(\|x\|_{(\mathbb{R}^N)^j} = 1\right)} \|g(x)\|_\gamma = \sup \frac{\|g(x)\|_\gamma}{\|x_1\|_p \dots \|x_j\|_p}. \tag{59}$$

Let M *be a non-empty convex and compact subset of* $\mathbb{R}^k$ *and* $x_0 \in M$ *is fixed.*

Let O *be an open subset of* $\mathbb{R}^N : M \subset O$*. Let* $f : O \to X$ *be a continuous function, whose Fréchet derivatives (see [22])* $f^{(j)} : O \to L_j = L_j\left(\left(\mathbb{R}^N\right)^j; X\right)$ *exist and are continuous for* $1 \leq j \leq m$*,* $m \in \mathbb{N}$*.*

Call $(x - x_0)^j := (x - x_0, ..., x - x_0) \in \left(\mathbb{R}^N\right)^j$*,* $x \in M$*.*

We will work with $f|_M$*.*

Then, by Taylor's formula ([14]), ([22], p. 124), we get

$$f(x) = \sum_{j=0}^{m} \frac{f^{(j)}(x_0)(x - x_0)^j}{j!} + R_m(x, x_0), \quad \text{all } x \in M, \tag{60}$$

where the remainder is the Riemann integral

$$R_m(x, x_0) := \int_0^1 \frac{(1-u)^{m-1}}{(m-1)!} \left(f^{(m)}(x_0 + u(x - x_0)) - f^{(m)}(x_0)\right)(x - x_0)^m du, \tag{61}$$

here we set $f^{(0)}(x_0)(x - x_0)^0 = f(x_0)$*.*

We consider

$$w := \omega_1\left(f^{(m)}, h\right) := \sup_{\substack{x, y \in M: \\ \|x - y\|_p \leq h}} \left\|f^{(m)}(x) - f^{(m)}(y)\right\|, \tag{62}$$

$h > 0$*.*

We obtain

$$\left\|\left(f^{(m)}(x_0 + u(x - x_0)) - f^{(m)}(x_0)\right)(x - x_0)^m\right\|_\gamma \leq$$

$$\left\|f^{(m)}(x_0 + u(x - x_0)) - f^{(m)}(x_0)\right\| \cdot \|x - x_0\|_p^m \leq$$

$$w \|x - x_0\|_p^m \left\lceil \frac{u \|x - x_0\|_p}{h} \right\rceil, \tag{63}$$

by Lemma 7.1.1, [1], p. 208, where $\lceil\cdot\rceil$ *is the ceiling.*

Therefore for all $x \in M$ *(see [1], pp. 121-122):*

$$\left\|R_m\left(x, x_0\right)\right\|_\gamma \leq w\left\|x-x_0\right\|_p^m \int_0^1\left\lceil\frac{u\left\|x-x_0\right\|_p}{h}\right\rceil \frac{(1-u)^{m-1}}{(m-1)!} du$$

$$=w \Phi_m\left(\left\|x-x_0\right\|_p\right) \tag{64}$$

by a change of variable, where

$$\Phi_m(t):=\int_0^{|t|}\left\lceil\frac{s}{h}\right\rceil \frac{(|t|-s)^{m-1}}{(m-1)!} ds=\frac{1}{m!}\left(\sum_{j=0}^{\infty}(|t|-j h)_+^m\right), \quad \forall\, t \in \mathbb{R}, \tag{65}$$

is a (polynomial) spline function, see [1], p. 210-211.

Also from there we get

$$\Phi_m(t) \leq\left(\frac{|t|^{m+1}}{(m+1)!h}+\frac{|t|^m}{2 m!}+\frac{h|t|^{m-1}}{8(m-1)!}\right), \quad \forall\, t \in \mathbb{R}, \tag{66}$$

with equality true only at $t=0$*.*

Therefore it holds

$$\left\|R_m\left(x, x_0\right)\right\|_\gamma \leq w\left(\frac{\left\|x-x_0\right\|_p^{m+1}}{(m+1)!h}+\frac{\left\|x-x_0\right\|_p^m}{2 m!}+\frac{h\left\|x-x_0\right\|_p^{m-1}}{8(m-1)!}\right), \quad \forall\, x \in M. \tag{67}$$

We have found that

$$\left\|f(x)-\sum_{j=0}^m \frac{f^{(j)}\left(x_0\right)\left(x-x_0\right)^j}{j!}\right\|_\gamma \leq$$

$$\omega_1\left(f^{(m)}, h\right)\left(\frac{\left\|x-x_0\right\|_p^{m+1}}{(m+1)!h}+\frac{\left\|x-x_0\right\|_p^m}{2 m!}+\frac{h\left\|x-x_0\right\|_p^{m-1}}{8(m-1)!}\right)<\infty, \tag{68}$$

$\forall\, x, x_0 \in M$.

Here $0<\omega_1\left(f^{(m)}, h\right)<\infty$*, by* M *being compact and* $f^{(m)}$ *being continuous on* M*.*

One can rewrite (68) as follows:

$$\left\|f(\cdot)-\sum_{j=0}^m \frac{f^{(j)}\left(x_0\right)\left(\cdot-x_0\right)^j}{j!}\right\|_\gamma \leq$$

$$\omega_1\left(f^{(m)}, h\right)\left(\frac{\left\|\cdot-x_0\right\|_p^{m+1}}{(m+1)!h}+\frac{\left\|\cdot-x_0\right\|_p^m}{2 m!}+\frac{h\left\|\cdot-x_0\right\|_p^{m-1}}{8(m-1)!}\right), \ \forall\, x_0 \in M, \tag{69}$$

a pointwise functional inequality on M.

Here $(\cdot - x_0)^j$ *maps* M *into* $\left(\mathbb{R}^N\right)^j$ *and it is continuous, also* $f^{(j)}(x_0)$ *maps* $\left(\mathbb{R}^N\right)^j$ *into* X *and it is continuous. Hence their composition* $f^{(j)}(x_0)(\cdot - x_0)^j$ *is continuous from* M *into* X.

Clearly $f(\cdot) - \sum_{j=0}^{m} \frac{f^{(j)}(x_0)(\cdot - x_0)^j}{j!} \in C(M, X)$, *hence* $\left\| f(\cdot) - \sum_{j=0}^{m} \frac{f^{(j)}(x_0)(\cdot - x_0)^j}{j!} \right\|_\gamma \in C(M)$.

Let $\left\{\widetilde{L}_N\right\}_{N \in \mathbb{N}}$ *be a sequence of positive linear operators mapping* $C(M)$ *into* $C(M)$.

Therefore we obtain

$$\left(\widetilde{L}_N\left(\left\| f(\cdot) - \sum_{j=0}^{m} \frac{f^{(j)}(x_0)(\cdot - x_0)^j}{j!} \right\|_\gamma\right)\right)(x_0) \leq$$

$$\omega_1\left(f^{(m)}, h\right)\left[\frac{\left(\widetilde{L}_N\left(\|\cdot - x_0\|_p^{m+1}\right)\right)(x_0)}{(m+1)!h} + \frac{\left(\widetilde{L}_N\left(\|\cdot - x_0\|_p^{m}\right)\right)(x_0)}{2m!} + \frac{h\left(\widetilde{L}_N\left(\|\cdot - x_0\|_p^{m-1}\right)\right)(x_0)}{8(m-1)!}\right], \tag{70}$$

$\forall\ N \in \mathbb{N}$, $\forall\ x_0 \in M$.

Clearly (70) is valid when $M = \prod_{i=1}^{N} [a_i, b_i]$ and $\widetilde{L}_n = \widetilde{A}_n$, see (37).

All the above is preparation for the following theorem, where we assume Fréchet differentiability of functions.

This will be a direct application of Theorem 10.2, [11], pp. 268-270. The operators A_n, $\widetilde{A}_n$ fulfill its assumptions, see (36), (37), (39), (40) and (41).

We present the following high order approximation results.

Theorem 11 *Let* O *open subset of* $\left(\mathbb{R}^N, \|\cdot\|_p\right)$, $p \in [1, \infty]$, *such that* $\prod_{i=1}^{N} [a_i, b_i] \subset O \subseteq \mathbb{R}^N$, *and let* $\left(X, \|\cdot\|_\gamma\right)$ *be a general Banach space. Let* $m \in \mathbb{N}$ *and* $f \in C^m(O, X)$, *the space of* m*-times continuously Fréchet differentiable functions from* O *into* X. *We study the approximation of* $f|_{\prod_{i=1}^{N} [a_i, b_i]}$. *Let* $x_0 \in \left(\prod_{i=1}^{N} [a_i, b_i]\right)$ *and* $r > 0$. *Then*

1)

$$\left\| (A_n(f))(x_0) - \sum_{j=0}^{m} \frac{1}{j!}\left(A_n\left(f^{(j)}(x_0)(\cdot - x_0)^j\right)\right)(x_0) \right\|_\gamma \leq$$

$$\frac{\omega_1\left(f^{(m)}, r\left(\left(\widetilde{A}_n\left(\|\cdot - x_0\|_p^{m+1}\right)\right)(x_0)\right)^{\frac{1}{m+1}}\right)}{rm!}\left(\left(\widetilde{A}_n\left(\|\cdot - x_0\|_p^{m+1}\right)\right)(x_0)\right)^{\left(\frac{m}{m+1}\right)}$$

$$\left[\frac{1}{(m+1)} + \frac{r}{2} + \frac{mr^2}{8}\right], \tag{71}$$

2) additionally if $f^{(j)}(x_0) = 0$, $j = 1, ..., m$, we have

$$\|(A_n(f))(x_0) - f(x_0)\|_\gamma \leq$$

$$\frac{\omega_1\left(f^{(m)}, r\left(\left(\widetilde{A}_n\left(\|\cdot - x_0\|_p^{m+1}\right)\right)(x_0)\right)^{\frac{1}{m+1}}\right)}{rm!}\left(\left(\widetilde{A}_n\left(\|\cdot - x_0\|_p^{m+1}\right)\right)(x_0)\right)^{\left(\frac{m}{m+1}\right)} \tag{72}$$

$$\left[\frac{1}{(m+1)} + \frac{r}{2} + \frac{mr^2}{8}\right],$$

3)

$$\|(A_n(f))(x_0) - f(x_0)\|_\gamma \leq \sum_{j=1}^{m} \frac{1}{j!}\left\|\left(A_n\left(f^{(j)}(x_0)(\cdot - x_0)^j\right)\right)(x_0)\right\|_\gamma +$$

$$\frac{\omega_1\left(f^{(m)}, r\left(\left(\widetilde{A}_n\left(\|\cdot - x_0\|_p^{m+1}\right)\right)(x_0)\right)^{\frac{1}{m+1}}\right)}{rm!}\left(\left(\widetilde{A}_n\left(\|\cdot - x_0\|_p^{m+1}\right)\right)(x_0)\right)^{\left(\frac{m}{m+1}\right)} \tag{73}$$

$$\left[\frac{1}{(m+1)} + \frac{r}{2} + \frac{mr^2}{8}\right],$$

and

4)

$$\left\|\|A_n(f) - f\|_\gamma\right\|_{\infty, \prod_{i=1}^{N}[a_i, b_i]} \leq$$

$$\sum_{j=1}^{m} \frac{1}{j!}\left\|\left\|\left(A_n\left(f^{(j)}(x_0)(\cdot - x_0)^j\right)\right)(x_0)\right\|_\gamma\right\|_{\infty, x_0 \in \prod_{i=1}^{N}[a_i, b_i]} +$$

$$\frac{\omega_1\left(f^{(m)}, r\left\|\left(\widetilde{A}_n\left(\|\cdot - x_0\|_p^{m+1}\right)\right)(x_0)\right\|_{\infty, x_0 \in \prod_{i=1}^{N}[a_i, b_i]}^{\frac{1}{m+1}}\right)}{rm!}$$

$$\left\|\left(\widetilde{A}_n\left(\|\cdot - x_0\|_p^{m+1}\right)\right)(x_0)\right\|_{\infty, x_0 \in \prod_{i=1}^{N}[a_i, b_i]}^{\left(\frac{m}{m+1}\right)} \tag{74}$$

$$\left[\frac{1}{(m+1)} + \frac{r}{2} + \frac{mr^2}{8}\right].$$

We give

Corollary 12 *(to Theorem 11, case of $m=1$) Then*

1)

$$\left\|\left(A_{n}\left(f\right)\right)\left(x_{0}\right)-f\left(x_{0}\right)\right\|_{\gamma}\leq\left\|\left(A_{n}\left(f^{(1)}\left(x_{0}\right)\left(\cdot-x_{0}\right)\right)\right)\left(x_{0}\right)\right\|_{\gamma}+$$

$$\frac{1}{2r}\omega_{1}\left(f^{(1)},r\left(\left(\widetilde{A}_{n}\left(\left\|\cdot-x_{0}\right\|_{p}^{2}\right)\right)\left(x_{0}\right)\right)^{\frac{1}{2}}\right)\left(\left(\widetilde{A}_{n}\left(\left\|\cdot-x_{0}\right\|_{p}^{2}\right)\right)\left(x_{0}\right)\right)^{\frac{1}{2}} \tag{75}$$

$$\left[1+r+\frac{r^{2}}{4}\right],$$

and

2)

$$\left\|\left\|\left(A_{n}\left(f\right)\right)-f\right\|_{\gamma}\right\|_{\infty,\prod\limits_{i=1}^{N}\left[a_{i},b_{i}\right]}\leq$$

$$\left\|\left\|\left(A_{n}\left(f^{(1)}\left(x_{0}\right)\left(\cdot-x_{0}\right)\right)\right)\left(x_{0}\right)\right\|_{\gamma}\right\|_{\infty,x_{0}\in\prod\limits_{i=1}^{N}\left[a_{i},b_{i}\right]}+$$

$$\frac{1}{2r}\omega_{1}\left(f^{(1)},r\left\|\left(\widetilde{A}_{n}\left(\left\|\cdot-x_{0}\right\|_{p}^{2}\right)\right)\left(x_{0}\right)\right\|_{\infty,x_{0}\in\prod\limits_{i=1}^{N}\left[a_{i},b_{i}\right]}^{\frac{1}{2}}\right)$$

$$\left\|\left(\widetilde{A}_{n}\left(\left\|\cdot-x_{0}\right\|_{p}^{2}\right)\right)\left(x_{0}\right)\right\|_{\infty,x_{0}\in\prod\limits_{i=1}^{N}\left[a_{i},b_{i}\right]}^{\frac{1}{2}}\left[1+r+\frac{r^{2}}{4}\right], \tag{76}$$

$r>0$.

We make

Remark 13 *We estimate $0<\alpha<1$, $m,n\in\mathbb{N}:n^{1-\alpha}>2$,*

$$\widetilde{A}_{n}\left(\left\|\cdot-x_{0}\right\|_{\infty}^{m+1}\right)\left(x_{0}\right)=\frac{\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}\left\|\frac{k}{n}-x_{0}\right\|_{\infty}^{m+1}Z_{q}\left(nx_{0}-k\right)}{\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}Z_{q}\left(nx_{0}-k\right)}\overset{(33)}{<}$$

$$\left(\Phi\left(q\right)\right)^{N}\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}\left\|\frac{k}{n}-x_{0}\right\|_{\infty}^{m+1}Z_{q}\left(nx_{0}-k\right)= \tag{77}$$

$$\left(\Phi\left(q\right)\right)^{N}\left\{\sum_{\substack{k=\lceil na\rceil \\ \left\{:\left\|\frac{k}{n}-x_{0}\right\|_{\infty}\leq\frac{1}{n^{\alpha}}\right.}}^{\lfloor nb\rfloor}\left\|\frac{k}{n}-x_{0}\right\|_{\infty}^{m+1}Z_{q}\left(nx_{0}-k\right)+\right.$$

$$\left.\sum_{\left\{\begin{array}{c} k=\lceil na\rceil \\ :\left\|\frac{k}{n}-x_0\right\|_\infty>\frac{1}{n^\alpha}\end{array}\right.}^{\lfloor nb\rfloor} \left\|\frac{k}{n}-x_0\right\|_\infty^{m+1} Z_q\left(nx_0-k\right)\right\} \overset{(34)}{\leq}$$

$$\left(\Phi\left(q\right)\right)^N\left\{\frac{1}{n^{\alpha(m+1)}}+Qe^{-2n^{(1-\alpha)}}\left\|b-a\right\|_\infty^{m+1}\right\}, \tag{78}$$

(where $b-a=\left(b_1-a_1,...,b_N-a_N\right)$).

We have proved that ($\forall\ x_0\in\prod_{i=1}^N\left[a_i,b_i\right]$)

$$\widetilde{A}_n\left(\left\|\cdot-x_0\right\|_\infty^{m+1}\right)\left(x_0\right)<\left(\Phi\left(q\right)\right)^N\left\{\frac{1}{n^{\alpha(m+1)}}+Qe^{-2n^{(1-\alpha)}}\left\|b-a\right\|_\infty^{m+1}\right\}=:\varphi_1\left(n\right) \tag{79}$$

($0<\alpha<1$, $m,n\in\mathbb{N}:n^{1-\alpha}>2$).

And, consequently it holds

$$\left\|\widetilde{A}_n\left(\left\|\cdot-x_0\right\|_\infty^{m+1}\right)\left(x_0\right)\right\|_{\infty,x_0\in\prod_{i=1}^N\left[a_i,b_i\right]}<$$

$$\left(\Phi\left(q\right)\right)^N\left\{\frac{1}{n^{\alpha(m+1)}}+Qe^{-2n^{(1-\alpha)}}\left\|b-a\right\|_\infty^{m+1}\right\}=\varphi_1\left(n\right)\to 0,\quad \text{as } n\to+\infty. \tag{80}$$

So, we have that $\varphi_1\left(n\right)\to 0$, as $n\to+\infty$. Thus, when $p\in[1,\infty]$, from Theorem 11 we have the convergence to zero in the right hand sides of parts (1), (2).

Next we estimate $\left\|\left(\widetilde{A}_n\left(f^{(j)}\left(x_0\right)\left(\cdot-x_0\right)^j\right)\right)\left(x_0\right)\right\|_\gamma$.

We have that

$$\left(\widetilde{A}_n\left(f^{(j)}\left(x_0\right)\left(\cdot-x_0\right)^j\right)\right)\left(x_0\right)=\frac{\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} f^{(j)}\left(x_0\right)\left(\frac{k}{n}-x_0\right)^j Z_q\left(nx_0-k\right)}{\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} Z_q\left(nx_0-k\right)}. \tag{81}$$

When $p=\infty$, $j=1,...,m$, we obtain

$$\left\|f^{(j)}\left(x_0\right)\left(\frac{k}{n}-x_0\right)^j\right\|_\gamma\leq\left\|f^{(j)}\left(x_0\right)\right\|\left\|\frac{k}{n}-x_0\right\|_\infty^j. \tag{82}$$

We further have that

$$\left\|\left(\widetilde{A}_n\left(f^{(j)}\left(x_0\right)\left(\cdot-x_0\right)^j\right)\right)\left(x_0\right)\right\|_\gamma\overset{(33)}{<}$$

$$(\Phi(q))^N \left(\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \left\| f^{(j)}(x_0) \left(\frac{k}{n} - x_0 \right)^j \right\|_\gamma Z_q(nx_0 - k) \right) \leq$$

$$(\Phi(q))^N \left(\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \left\| f^{(j)}(x_0) \right\| \left\| \frac{k}{n} - x_0 \right\|_\infty^j Z_q(nx_0 - k) \right) = \tag{83}$$

$$(\Phi(q))^N \left\| f^{(j)}(x_0) \right\| \left(\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \left\| \frac{k}{n} - x_0 \right\|_\infty^j Z_q(nx_0 - k) \right) =$$

$$(\Phi(q))^N \left\| f^{(j)}(x_0) \right\| \left\{ \sum_{\begin{cases} k = \lceil na \rceil \\ : \left\| \frac{k}{n} - x_0 \right\|_\infty \leq \frac{1}{n^\alpha} \end{cases}}^{\lfloor nb \rfloor} \left\| \frac{k}{n} - x_0 \right\|_\infty^j Z_q(nx_0 - k) \right.$$

$$\left. + \sum_{\begin{cases} k = \lceil na \rceil \\ : \left\| \frac{k}{n} - x_0 \right\|_\infty > \frac{1}{n^\alpha} \end{cases}}^{\lfloor nb \rfloor} \left\| \frac{k}{n} - x_0 \right\|_\infty^j Z_q(nx_0 - k) \right\} \overset{(32)}{\leq} \tag{84}$$

$$(\Phi(q))^N \left\| f^{(j)}(x_0) \right\| \left\{ \frac{1}{n^{\alpha j}} + Qe^{-2n^{(1-\alpha)}} \|b - a\|_\infty^j \right\} \to 0, \text{ as } n \to \infty.$$

That is

$$\left\| \left(\widetilde{A}_n \left(f^{(j)}(x_0) (\cdot - x_0)^j \right) \right) (x_0) \right\|_\gamma \to 0, \text{ as } n \to \infty.$$

Therefore when $p = \infty$, *for* $j = 1, ..., m$, *we have proved:*

$$\left\| \left(\widetilde{A}_n \left(f^{(j)}(x_0) (\cdot - x_0)^j \right) \right) (x_0) \right\|_\gamma <$$

$$(\Phi(q))^N \left\| f^{(j)}(x_0) \right\| \left\{ \frac{1}{n^{\alpha j}} + Qe^{-2n^{(1-\alpha)}} \|b - a\|_\infty^j \right\} \leq \tag{85}$$

$$(\Phi(q))^N \left\| f^{(j)} \right\|_\infty \left\{ \frac{1}{n^{\alpha j}} + Qe^{-2n^{(1-\alpha)}} \|b - a\|_\infty^j \right\} =: \varphi_{2j}(n) < \infty,$$

and converges to zero, as $n \to \infty$.

We conclude:

In Theorem 11, the right hand sides of (73) and (74) converge to zero as $n \to \infty$, for any $p \in [1, \infty]$.

Also in Corollary 12, the right hand sides of (75) and (76) converge to zero as $n \to \infty$, for any $p \in [1, \infty]$.

Conclusion 14 *We have proved that the left hand sides of (71), (72), (73), (74) and (75), (76) converge to zero as $n \to \infty$, for $p \in [1, \infty]$. Consequently $A_n \to I$ (unit operator) pointwise and uniformly, as $n \to \infty$, where $p \in [1, \infty]$. In the presence of initial conditions we achieve a higher speed of convergence, see (72). Higher speed of convergence happens also to the left hand side of (71).*

We further give

Corollary 15 *(to Theorem 11) Let O open subset of $\left(\mathbb{R}^N, \|\cdot\|_\infty\right)$, such that $\prod_{i=1}^{N} [a_i, b_i] \subset O \subseteq \mathbb{R}^N$, and let $\left(X, \|\cdot\|_\gamma\right)$ be a general Banach space. Let $m \in \mathbb{N}$ and $f \in C^m(O, X)$, the space of m-times continuously Fréchet differentiable functions from O into X. We study the approximation of $f|_{\prod_{i=1}^{N} [a_i, b_i]}$. Let $x_0 \in \left(\prod_{i=1}^{N} [a_i, b_i]\right)$ and $r > 0$. Here $\varphi_1(n)$ as in (79) and $\varphi_{2j}(n)$ as in (85), where $n \in \mathbb{N} : n^{1-\alpha} > 2$, $0 < \alpha < 1$, $j = 1, ..., m$. Then*

1)

$$\left\| (A_n(f))(x_0) - \sum_{j=0}^{m} \frac{1}{j!} \left(A_n \left(f^{(j)}(x_0) (\cdot - x_0)^j \right) \right)(x_0) \right\|_\gamma \leq$$

$$\frac{\omega_1 \left(f^{(m)}, r (\varphi_1(n))^{\frac{1}{m+1}} \right)}{rm!} (\varphi_1(n))^{\left(\frac{m}{m+1}\right)} \left[\frac{1}{(m+1)} + \frac{r}{2} + \frac{mr^2}{8} \right], \tag{86}$$

2) additionally, if $f^{(j)}(x_0) = 0$, $j = 1, ..., m$, we have

$$\|(A_n(f))(x_0) - f(x_0)\|_\gamma \leq$$

$$\frac{\omega_1 \left(f^{(m)}, r (\varphi_1(n))^{\frac{1}{m+1}} \right)}{rm!} (\varphi_1(n))^{\left(\frac{m}{m+1}\right)} \left[\frac{1}{(m+1)} + \frac{r}{2} + \frac{mr^2}{8} \right], \tag{87}$$

3)

$$\left\| \|A_n(f) - f\|_\gamma \right\|_{\infty, \prod_{i=1}^{N} [a_i, b_i]} \leq \sum_{j=1}^{m} \frac{\varphi_{2j}(n)}{j!} +$$

$$\frac{\omega_1 \left(f^{(m)}, r (\varphi_1(n))^{\frac{1}{m+1}} \right)}{rm!} (\varphi_1(n))^{\left(\frac{m}{m+1}\right)} \tag{88}$$

$$\left[\frac{1}{(m+1)} + \frac{r}{2} + \frac{mr^2}{8} \right] =: \varphi_3(n) \to 0, \text{ as } n \to \infty.$$

In the next we discuss further the high order of approximation by using the smoothnedd of f, where $X = \mathbb{R}$.

We give

Theorem 16 *Let* $f \in C^m\left(\prod_{i=1}^{N}[a_i,b_i]\right)$, $0<\beta<1$, $n,m,N\in\mathbb{N}$, $n^{1-\beta}\geq 3$, $q>0$, $q\neq 1$, $x\in\left(\prod_{i=1}^{N}[a_i,b_i]\right)$. *Then*

i)

$$\left|\widetilde{A}_n(f,x)-f(x)-\sum_{j=1}^{m}\left(\sum_{|\alpha|=j}\left(\frac{f_\alpha(x)}{\prod_{=1}^{N}\alpha_i!}\right)\widetilde{A}_n\left(\prod_{i=1}^{N}(\cdot-x_i)^{\alpha_i},x\right)\right)\right|\leq \tag{89}$$

$$(\Phi(q))^N\left\{\frac{N^m}{m!n^{m\beta}}\omega_{1,m}^{\max}\left(f_\alpha,\frac{1}{n^\beta}\right)+\left(\frac{\|b-a\|_\infty^m\|f_\alpha\|_{\infty,m}^{\max}N^m}{m!}\right)2Qe^{-2n^{(1-\beta)}}\right\}.$$

ii)

$$\left|\widetilde{A}_n(f,x)-f(x)\right|\leq(\Phi(q))^N \tag{90}$$

$$\left\{\sum_{j=1}^{m}\left(\sum_{|\alpha|=j}\left(\frac{|f_\alpha(x)|}{\prod_{=1}^{N}\alpha_i!}\right)\left[\frac{1}{n^{\beta j}}+\left(\prod_{i=1}^{N}(b_i-a_i)^{\alpha_i}\right)Qe^{-2n^{(1-\beta)}}\right]\right)\right.$$

$$\left.+\frac{N^m}{m!n^{m\beta}}\omega_{1,m}^{\max}\left(f_\alpha,\frac{1}{n^\beta}\right)+\left(\frac{\|b-a\|_\infty^m\|f_\alpha\|_{\infty,m}^{\max}N^m}{m!}\right)2Qe^{-2n^{(1-\beta)}}\right\}.$$

iii)

$$\left\|\widetilde{A}_n(f)-f\right\|_\infty\leq(\Phi(q))^N \tag{91}$$

$$\left\{\sum_{j=1}^{m}\left(\sum_{|\alpha|=j}\left(\frac{\||f_\alpha|\|_\infty}{\prod_{=1}^{N}\alpha_i!}\right)\left[\frac{1}{n^{\beta j}}+\left(\prod_{i=1}^{N}(b_i-a_i)^{\alpha_i}\right)Qe^{-2n^{(1-\beta)}}\right]\right)\right.$$

$$\left.+\frac{N^m}{m!n^{m\beta}}\omega_{1,m}^{\max}\left(f_\alpha,\frac{1}{n^\beta}\right)+\left(\frac{\|b-a\|_\infty^m\|f_\alpha\|_{\infty,m}^{\max}N^m}{m!}\right)2Qe^{-2n^{(1-\beta)}}\right\}.$$

iv) Assume $f_\alpha(x_0)=0$, *for all* $\alpha:|\alpha|=1,...,m$; $x_0\in\left(\prod_{i=1}^{N}[a_i,b_i]\right)$. *Then*

$$\left|\widetilde{A}_n(f,x_0)-f(x_0)\right|\leq \tag{92}$$

$$(\Phi(q))^N\left\{\frac{N^m}{m!n^{m\beta}}\omega_{1,m}^{\max}\left(f_\alpha,\frac{1}{n^\beta}\right)+\left(\frac{\|b-a\|_\infty^m\|f_\alpha\|_{\infty,m}^{\max}N^m}{m!}\right)2Qe^{-2n^{(1-\beta)}}\right\},$$

notice in the last the extremely high rate of convergence at $n^{-\beta(m+1)}$.

Proof. As similar to [10], pp. 389-391, is omitted. ■

We continue with

Theorem 17 *Let $f \in C_B\left(\mathbb{R}^N, X\right)$, $0 < \beta < 1$, $x \in \mathbb{R}^N$, $q > 0$, $q \neq 1$, $m, N, n \in \mathbb{N}$ with $n^{1-\beta} > 2$, ω_1 is for $p = \infty$. Then*

1)

$$\left\| B_n(f,x) - f(x) \right\|_\gamma \leq \omega_1\left(f, \frac{1}{n^\beta}\right) + 2Qe^{-2n^{(1-\beta)}} \left\| \|f\|_\gamma \right\|_\infty =: \lambda_2(n), \tag{93}$$

2)

$$\left\| \left\| B_n(f) - f \right\|_\gamma \right\|_\infty \leq \lambda_2(n). \tag{94}$$

Given that $f \in \left(C_U\left(\mathbb{R}^N, X\right) \cap C_B\left(\mathbb{R}^N, X\right)\right)$, we obtain $\lim_{n\to\infty} B_n(f) = f$, uniformly.

Proof. We have that

$$B_n(f,x) - f(x) \overset{(27)}{=} \sum_{k=-\infty}^{\infty} f\left(\frac{k}{n}\right) Z_q(nx-k) - f(x) \sum_{k=-\infty}^{\infty} Z_q(nx-k) = \tag{95}$$

$$\sum_{k=-\infty}^{\infty} \left(f\left(\frac{k}{n}\right) - f(x) \right) Z_q(nx-k).$$

Hence

$$\left\| B_n(f,x) - f(x) \right\|_\gamma \leq \sum_{k=-\infty}^{\infty} \left\| f\left(\frac{k}{n}\right) - f(x) \right\|_\gamma Z_q(nx-k) =$$

$$\sum_{\begin{cases} k = -\infty \\ \left\| \frac{k}{n} - x \right\|_\infty \leq \frac{1}{n^\beta} \end{cases}}^{\infty} \left\| f\left(\frac{k}{n}\right) - f(x) \right\|_\gamma Z_q(nx-k) +$$

$$\sum_{\begin{cases} k = -\infty \\ \left\| \frac{k}{n} - x \right\|_\infty > \frac{1}{n^\beta} \end{cases}}^{\infty} \left\| f\left(\frac{k}{n}\right) - f(x) \right\|_\gamma Z_q(nx-k) \overset{(27)}{\leq}$$

$$\omega_1\left(f, \frac{1}{n^\beta}\right) + 2\left\| \|f\|_\gamma \right\|_\infty \sum_{\begin{cases} k = -\infty \\ \left\| \frac{k}{n} - x \right\|_\infty > \frac{1}{n^\beta} \end{cases}}^{\infty} Z_q(nx-k) \overset{(34)}{\leq}$$

$$\omega_1\left(f, \frac{1}{n^\beta}\right) + 2Qe^{-2n^{(1-\beta)}} \left\| \|f\|_\gamma \right\|_\infty, \tag{96}$$

proving the claim. ■

We give

Theorem 18 *Let $f \in C_B\left(\mathbb{R}^N, X\right)$, $0 < \beta < 1$, $x \in \mathbb{R}^N$, $q > 0$, $q \neq 1$, $m, N, n \in \mathbb{N}$ with $n^{1-\beta} > 2$, ω_1 is for $p = \infty$. Then*

1)

$$\left\| C_n(f,x) - f(x) \right\|_\gamma \leq \omega_1\left(f, \frac{1}{n} + \frac{1}{n^\beta}\right) + 2Qe^{-2n^{(1-\beta)}} \left\| \|f\|_\gamma \right\|_\infty =: \lambda_3(n), \tag{97}$$

2)

$$\left\| \left\| C_n(f) - f \right\|_\gamma \right\|_\infty \leq \lambda_3(n). \tag{98}$$

Given that $f \in \left(C_U\left(\mathbb{R}^N, X\right) \cap C_B\left(\mathbb{R}^N, X\right)\right)$, we obtain $\lim_{n\to\infty} C_n(f) = f$, uniformly.

Proof. We notice that

$$\int_{\frac{k}{n}}^{\frac{k+1}{n}} f(t)\,dt = \int_{\frac{k_1}{n}}^{\frac{k_1+1}{n}} \int_{\frac{k_2}{n}}^{\frac{k_2+1}{n}} \ldots \int_{\frac{k_N}{n}}^{\frac{k_N+1}{n}} f(t_1, t_2, \ldots, t_N)\, dt_1 dt_2 \ldots dt_N =$$

$$\int_0^{\frac{1}{n}} \int_0^{\frac{1}{n}} \ldots \int_0^{\frac{1}{n}} f\left(t_1 + \frac{k_1}{n}, t_2 + \frac{k_2}{n}, \ldots, t_N + \frac{k_N}{n}\right) dt_1 \ldots dt_N = \int_0^{\frac{1}{n}} f\left(t + \frac{k}{n}\right) dt. \tag{99}$$

Thus it holds (by (51))

$$C_n(f,x) = \sum_{k=-\infty}^{\infty} \left(n^N \int_0^{\frac{1}{n}} f\left(t + \frac{k}{n}\right) dt \right) Z_q(nx - k). \tag{100}$$

We observe that

$$\left\| C_n(f,x) - f(x) \right\|_\gamma =$$

$$\left\| \sum_{k=-\infty}^{\infty} \left(n^N \int_0^{\frac{1}{n}} f\left(t + \frac{k}{n}\right) dt \right) Z_q(nx-k) - \sum_{k=-\infty}^{\infty} f(x) Z_q(nx-k) \right\|_\gamma =$$

$$\left\| \sum_{k=-\infty}^{\infty} \left(\left(n^N \int_0^{\frac{1}{n}} f\left(t + \frac{k}{n}\right) dt \right) - f(x) \right) Z_q(nx-k) \right\|_\gamma =$$

$$\left\| \sum_{k=-\infty}^{\infty} \left(n^N \int_0^{\frac{1}{n}} \left(f\left(t + \frac{k}{n}\right) - f(x) \right) dt \right) Z_q(nx-k) \right\|_\gamma \leq \tag{101}$$

$$\sum_{k=-\infty}^{\infty} \left(n^N \int_0^{\frac{1}{n}} \left\| f\left(t + \frac{k}{n}\right) - f(x) \right\|_\gamma dt \right) Z_q(nx-k) =$$

$$\sum_{\substack{k=-\infty \\ \left\| \frac{k}{n} - x \right\|_\infty \leq \frac{1}{n^\beta}}}^{\infty} \left(n^N \int_0^{\frac{1}{n}} \left\| f\left(t + \frac{k}{n}\right) - f(x) \right\|_\gamma dt \right) Z_q(nx-k) +$$

$$\sum_{\begin{cases} k=-\infty \\ \left\| \frac{k}{n} - x \right\|_\infty > \frac{1}{n^\beta} \end{cases}}^{\infty} \left(n^N \int_0^{\frac{1}{n}} \left\| f\left(t + \frac{k}{n}\right) - f(x) \right\|_\gamma dt \right) Z_q(nx-k) \leq$$

$$\sum_{\begin{cases} k=-\infty \\ \left\| \frac{k}{n} - x \right\|_\infty \leq \frac{1}{n^\beta} \end{cases}}^{\infty} \left(n^N \int_0^{\frac{1}{n}} \omega_1\left(f, \|t\|_\infty + \left\| \frac{k}{n} - x \right\|_\infty \right) dt \right) Z_q(nx-k) +$$

$$2\left\| \|f\|_\gamma \right\|_\infty \left(\sum_{\begin{cases} k=-\infty \\ \left\| \frac{k}{n} - x \right\|_\infty > \frac{1}{n^\beta} \end{cases}}^{\infty} Z_q(|nx-k|) \right) \leq$$

$$\omega_1\left(f, \frac{1}{n} + \frac{1}{n^\beta}\right) + 2Qe^{-2n^{(1-\beta)}} \left\| \|f\|_\gamma \right\|_\infty, \tag{102}$$

proving the claim. ■

We also present

Theorem 19 *Let $f \in C_B(\mathbb{R}^N, X)$, $0 < \beta < 1$, $x \in \mathbb{R}^N$, $q > 0$, $q \neq 1$, $m, N, n \in \mathbb{N}$ with $n^{1-\beta} > 2$, ω_1 is for $p = \infty$. Then*

1)

$$\|D_n(f,x) - f(x)\|_\gamma \leq \omega_1\left(f, \frac{1}{n} + \frac{1}{n^\beta}\right) + 2Qe^{-2n^{(1-\beta)}} \left\| \|f\|_\gamma \right\|_\infty = \lambda_4(n), \tag{103}$$

2)

$$\left\| \|D_n(f) - f\|_\gamma \right\|_\infty \leq \lambda_4(n). \tag{104}$$

Given that $f \in \left(C_U(\mathbb{R}^N, X) \cap C_B(\mathbb{R}^N, X)\right)$, we obtain $\lim_{n\to\infty} D_n(f) = f$, uniformly.

Proof. Similar to the proof of Theorem 18, as such is omitted. ■

Definition 20 *Let $f \in C_B(\mathbb{R}^N, X)$, $N \in \mathbb{N}$, $q > 0$, $q \neq 1$, where $\left(X, \|\cdot\|_\gamma\right)$ is a Banach space. We define the general neural network operator*

$$F_n(f,x) := \sum_{k=-\infty}^{\infty} l_{nk}(f) Z_q(nx-k) =$$

$$\begin{cases} B_n(f,x), & \text{if } l_{nk}(f) = f\left(\frac{k}{n}\right), \\ C_n(f,x), & \text{if } l_{nk}(f) = n^N \int_{\frac{k}{n}}^{\frac{k+1}{n}} f(t)\,dt, \\ D_n(f,x), & \text{if } l_{nk}(f) = \delta_{nk}(f). \end{cases} \tag{105}$$

Clearly $l_{nk}(f)$ is an X-valued bounded linear functional such that $\|l_{nk}(f)\|_\gamma \leq \left\| \|f\|_\gamma \right\|_\infty$.

Hence $F_n(f)$ is a bounded linear operator with $\left\| \|F_n(f)\|_\gamma \right\|_\infty \leq \left\| \|f\|_\gamma \right\|_\infty$.

We need

Theorem 21 *Let $f \in C_B(\mathbb{R}^N, X)$, $N \geq 1$. Then $F_n(f) \in C_B(\mathbb{R}^N, X)$.*

Proof. Clearly $F_n(f)$ is a bounded function.

Next we prove the continuity of $F_n(f)$. Notice for $N = 1$, $Z_q = \psi_q$ by (10).

We will use the generalized Weierstrass M test: If a sequence of positive constants $M_1, M_2, M_3, ...$, can be found such that in some interval

(a) $\|u_n(x)\|_\gamma \leq M_n$, $n = 1, 2, 3, ...$

(b) $\sum M_n$ converges,

then $\sum u_n(x)$ is uniformly and absolutely convergent in the interval.

Also we will use:

If $\{u_n(x)\}$, $n = 1, 2, 3, ...$ are continuous in $[a, b]$ and if $\sum u_n(x)$ converges uniformly to the sum $S(x)$ in $[a, b]$, then $S(x)$ is continuous in $[a, b]$. I.e. a uniformly convergent series of continuous functions is a continuous function. First we prove claim for $N = 1$.

We will prove that $\sum_{k=-\infty}^{\infty} l_{nk}(f)\, \psi_q(nx - k)$ is continuous in $x \in \mathbb{R}$.

There always exists $\lambda \in \mathbb{N}$ such that $nx \in [-\lambda, \lambda]$. Call $\lambda^* := \lambda + \left\lceil \frac{\ln \frac{1}{q}}{2} \right\rceil$, $\lambda_* := -\lambda + \left\lfloor \frac{\ln \frac{1}{q}}{2} \right\rfloor$.

Since $nx \leq \lambda$, then $-nx \geq -\lambda$ and $k - nx \geq k - \lambda \geq \left\lceil \frac{\ln \frac{1}{q}}{2} \right\rceil$, when $k \geq \lambda^*$. Therefore

$$\sum_{k=\lambda^*}^{\infty} \psi_q(nx - k) = \sum_{k=\lambda^*}^{\infty} \psi_{q^{-1}}(k - nx) \leq \sum_{k=\lambda^*}^{\infty} \psi_{q^{-1}}(k - \lambda) = \sum_{k'=\left\lceil \frac{\ln \frac{1}{q}}{2} \right\rceil}^{\infty} \psi_{q^{-1}}(k') \leq 1.$$

So for $k \geq \lambda^*$ we get

$$\|l_{nk}(f)\|_\gamma\, \psi_q(nx - k) \leq \left\| \|f\|_\gamma \right\|_\infty \psi_{q^{-1}}(k - \lambda), \tag{106}$$

and

$$\left\| \|f\|_\gamma \right\|_\infty \sum_{k=\lambda^*}^{\infty} \psi_{q^{-1}}(k - \lambda) \leq \left\| \|f\|_\gamma \right\|_\infty. \tag{107}$$

Hence by the generalized Weierstrass M test we obtain that $\sum_{k=\lambda^*}^{\infty} l_{nk}(f)\, \psi_q(nx - k)$ is uniformly and absolutely convergent on $\left[-\frac{\lambda}{n}, \frac{\lambda}{n}\right]$.

Since $l_{nk}(f)\, \psi_q(nx - k)$ is continuous in x, then $\sum_{k=\lambda^*}^{\infty} l_{nk}(f)\, \psi_q(nx - k)$ is continuous on $\left[-\frac{\lambda}{n}, \frac{\lambda}{n}\right]$.

Because $nx \geq -\lambda$, then $-nx \leq \lambda$, and $k - nx \leq k + \lambda \leq \left\lfloor \frac{\ln \frac{1}{q}}{2} \right\rfloor$, when $k \leq \lambda_*$. Therefore

$$\sum_{k=-\infty}^{\lambda_*} \psi_q (nx - k) = \sum_{k=-\infty}^{\lambda_*} \psi_{q^{-1}} (k - nx) \leq \sum_{k=-\infty}^{\lambda_*} \psi_{q^{-1}} (k + \lambda) = \sum_{k'=-\infty}^{\left\lfloor \frac{\ln \frac{1}{q}}{2} \right\rfloor} \psi_{q^{-1}} (k') \leq 1.$$

So for $k \leq \lambda_*$ we get

$$\|l_{nk} (f)\|_\gamma \psi_q (nx - k) \leq \left\| \|f\|_\gamma \right\|_\infty \psi_{q^{-1}} (k + \lambda), \tag{108}$$

and

$$\left\| \|f\|_\gamma \right\|_\infty \sum_{k=-\infty}^{\lambda_*} \psi_{q^{-1}} (k + \lambda) \leq \left\| \|f\|_\gamma \right\|_\infty. \tag{109}$$

Hence by Weierstrass M test we obtain that $\sum_{k=-\infty}^{\lambda_*} l_{nk} (f) \psi_{q^{-1}} (nx - k)$ is uniformly and absolutely convergent on $\left[-\frac{\lambda}{n}, \frac{\lambda}{n}\right]$.

Since $l_{nk} (f) \psi_q (nx - k)$ is continuous in x, then $\sum_{k=-\infty}^{\lambda_*} l_{nk} (f) \psi_q (nx - k)$ is continuous on $\left[-\frac{\lambda}{n}, \frac{\lambda}{n}\right]$.

So we proved that $\sum_{k=\lambda^*}^{\infty} l_{nk} (f) \psi_q (nx - k)$ and $\sum_{k=-\infty}^{\lambda_*} l_{nk} (f) \psi_q (nx - k)$ are continuous on $\mathbb{R}$. Since $\sum_{k=\lambda_*+1}^{\lambda^*-1} l_{nk} (f) \psi_q (nx - k)$ is a finite sum of continuous functions on $\mathbb{R}$, it is also a continuous function on $\mathbb{R}$.

Writing

$$\sum_{k=-\infty}^{\infty} l_{nk} (f) \psi_q (nx - k) = \sum_{k=-\infty}^{\lambda_*} l_{nk} (f) \psi_q (nx - k) +$$

$$\sum_{k=\lambda_*+1}^{\lambda^*-1} l_{nk} (f) \psi_q (nx - k) + \sum_{k=\lambda^*}^{\infty} l_{nk} (f) \psi_q (nx - k) \tag{110}$$

we have it as a continuous function on $\mathbb{R}$. Therefore $F_n (f)$, when $N = 1$, is a continuous function on $\mathbb{R}$.

When $N = 2$ we have

$$F_n (f, x_1, x_2) = \sum_{k_1=-\infty}^{\infty} \sum_{k_2=-\infty}^{\infty} l_{nk} (f) \psi_q (nx_1 - k_1) \psi_q (nx_2 - k_2) =$$

$$\sum_{k_1=-\infty}^{\infty} \psi_q (nx_1 - k_1) \left(\sum_{k_2=-\infty}^{\infty} l_{nk} (f) \psi_q (nx_2 - k_2) \right)$$

(there always exist $\lambda_1, \lambda_2 \in \mathbb{N}$ such that $nx_1 \in [-\lambda_1, \lambda_1]$ and $nx_2 \in [-\lambda_2, \lambda_2]$, also call $\lambda_1^* := \lambda_1 + \left\lceil \frac{\ln \frac{1}{q}}{2} \right\rceil$, $\lambda_{1*} := -\lambda_1 + \left\lfloor \frac{\ln \frac{1}{q}}{2} \right\rfloor$, $\lambda_2^* := \lambda_2 + \left\lceil \frac{\ln \frac{1}{q}}{2} \right\rceil$, and

$\lambda_{2*} := -\lambda_2 + \left\lfloor \frac{\ln \frac{1}{q}}{2} \right\rfloor)$

$$= \sum_{k_1=-\infty}^{\infty} \psi_q(nx_1 - k_1) \left[\sum_{k_2=-\infty}^{\lambda_{2*}} l_{nk}(f) \psi_q(nx_2 - k_2) + \right.$$

$$\left. \sum_{k_2=\lambda_{2*}+1}^{\lambda_2^*-1} l_{nk}(f) \psi_q(nx_2 - k_2) + \sum_{k_2=\lambda_2^*}^{\infty} l_{nk}(f) \psi_q(nx_2 - k_2) \right] =$$

$$= \sum_{k_1=-\infty}^{\infty} \sum_{k_2=-\infty}^{\lambda_{2*}} l_{nk}(f) \psi_q(nx_1 - k_1) \psi_q(nx_2 - k_2) + \tag{111}$$

$$\sum_{k_1=-\infty}^{\infty} \sum_{k_2=\lambda_{2*}+1}^{\lambda_2^*-1} l_{nk}(f) \psi_q(nx_1 - k_1) \psi_q(nx_2 - k_2) +$$

$$\sum_{k_1=-\infty}^{\infty} \sum_{k_2=\lambda_2^*}^{\infty} l_{nk}(f) \psi_q(nx_1 - k_1) \psi_q(nx_2 - k_2) =: (*).$$

(For convenience call

$$F_q(k_1, k_2, x_1, x_2) := l_{nk}(f) \psi_q(nx_1 - k_1) \psi_q(nx_2 - k_2). \,)$$

Thus

$$(*) = \sum_{k_1=-\infty}^{\lambda_{1*}} \sum_{k_2=-\infty}^{\lambda_{2*}} F_q(k_1, k_2, x_1, x_2) + \sum_{k_1=\lambda_{1*}+1}^{\lambda_1^*-1} \sum_{k_2=-\infty}^{\lambda_{2*}} F_q(k_1, k_2, x_1, x_2) +$$

$$\sum_{k_1=\lambda_1^*}^{\infty} \sum_{k_2=-\infty}^{\lambda_{2*}} F_q(k_1, k_2, x_1, x_2) + \sum_{k_1=-\infty}^{\lambda_{1*}} \sum_{k_2=\lambda_{2*}+1}^{\lambda_2^*-1} F_q(k_1, k_2, x_1, x_2) +$$

$$\sum_{k_1=\lambda_{1*}+1}^{\lambda_1^*-1} \sum_{k_2=\lambda_{2*}+1}^{\lambda_2^*-1} F_q(k_1, k_2, x_1, x_2) + \sum_{k_1=\lambda_1^*}^{\infty} \sum_{k_2=\lambda_{2*}+1}^{\lambda_2^*-1} F_q(k_1, k_2, x_1, x_2) +$$

$$\sum_{k_1=-\infty}^{\lambda_{1*}} \sum_{k_2=\lambda_2^*}^{\infty} F_q(k_1, k_2, x_1, x_2) + \sum_{k_1=\lambda_{1*}+1}^{\lambda_1^*-1} \sum_{k_2=\lambda_2^*}^{\infty} F_q(k_1, k_2, x_1, x_2) + \tag{112}$$

$$\sum_{k_1=\lambda_1^*}^{\infty} \sum_{k_2=\lambda_2^*}^{\infty} F_q(k_1, k_2, x_1, x_2).$$

Notice that the finite sum of continuous functions $F_q(k_1, k_2, x_1, x_2)$: $\sum_{k_1=\lambda_{1*}+1}^{\lambda_1^*-1} \sum_{k_2=\lambda_{2*}+1}^{\lambda_2^*-1} F_q(k_1, k_2, x_1, x_2)$ is a continuous function.

The rest of the summands of $F_n(f, x_1, x_2)$ are treated all the same way and similarly to the case of $N = 1$. The method is demonstrated as follows.

We will prove that $\sum_{k_1=\lambda_1^*}^{\infty}\sum_{k_2=-\infty}^{\lambda_{2*}} l_{nk}(f)\,\psi_q(nx_1-k_1)\,\psi_q(nx_2-k_2)$ is continuous in $(x_1,x_2)\in\mathbb{R}^2$.

The continuous function

$$\|l_{nk}(f)\|_\gamma\,\psi_q(nx_1-k_1)\,\psi_q(nx_2-k_2)\le\left\|\|f\|_\gamma\right\|_\infty\psi_{q^{-1}}(k_1-\lambda_1)\,\psi_{q^{-1}}(k_2+\lambda_2),$$

and

$$\left\|\|f\|_\gamma\right\|_\infty\sum_{k_1=\lambda_1^*}^{\infty}\sum_{k_2=-\infty}^{\lambda_{2*}}\psi_{q^{-1}}(k_1-\lambda_1)\,\psi_{q^{-1}}(k_2+\lambda_2)=$$

$$\left\|\|f\|_\gamma\right\|_\infty\left(\sum_{k_1=\lambda_1^*}^{\infty}\psi_{q^{-1}}(k_1-\lambda_1)\right)\left(\sum_{k_2=-\infty}^{\lambda_{2*}}\psi_{q^{-1}}(k_2+\lambda_2)\right)\le$$

$$\left\|\|f\|_\gamma\right\|_\infty\left(\sum_{k_1'=\left\lceil\frac{\ln\frac{1}{q}}{2}\right\rceil}^{\infty}\psi_{q^{-1}}(k_1')\right)\left(\sum_{k_2'=-\infty}^{\left\lfloor\frac{\ln\frac{1}{q}}{2}\right\rfloor}\psi_{q^{-1}}(k_2')\right)\le\left\|\|f\|_\gamma\right\|_\infty. \tag{113}$$

So by the Weierstrass M test we get that $\sum_{k_1=\lambda_1^*}^{\infty}\sum_{k_2=-\infty}^{\lambda_{2*}} l_{nk}(f)\,\psi_q(nx_1-k_1)\,\psi_q(nx_2-k_2)$ is uniformly and absolutely convergent. Therefore it is continuous on $\mathbb{R}^2$.

Next we prove continuity on $\mathbb{R}^2$ of $\sum_{k_1=\lambda_{1*}+1}^{\lambda_1^*-1}\sum_{k_2=-\infty}^{\lambda_{2*}} l_{nk}(f)\,\psi_q(nx_1-k_1)\,\psi_q(nx_2-k_2)$.

Notice here that

$$\|l_{nk}(f)\|_\gamma\,\psi_q(nx_1-k_1)\,\psi_q(nx_2-k_2)\le\left\|\|f\|_\gamma\right\|_\infty\psi_q(nx_1-k_1)\,\psi_{q^{-1}}(k_2+\lambda_2) \tag{114}$$

$$\le\left\|\|f\|_\gamma\right\|_\infty\psi_q\left(\frac{\ln q}{2}\right)\psi_{q^{-1}}(k_2+\lambda_2)=\frac{\tanh 1}{2}\left\|\|f\|_\gamma\right\|_\infty\psi_{q^{-1}}(k_2+\lambda_2),$$

and

$$\frac{\tanh 1}{2}\left\|\|f\|_\gamma\right\|_\infty\left(\sum_{k_1=\lambda_{1*}+1}^{\lambda_1^*-1}1\right)\left(\sum_{k_2=-\infty}^{\lambda_{2*}}\psi_{q^{-1}}(k_2+\lambda_2)\right)= \tag{115}$$

$$\frac{\tanh 1}{2}\left\|\|f\|_\gamma\right\|_\infty\left(2\lambda_1+\left\lceil\frac{\ln\frac{1}{q}}{2}\right\rceil-\left\lfloor\frac{\ln\frac{1}{q}}{2}\right\rfloor-1\right)\left(\sum_{k_2'=-\infty}^{\left\lfloor\frac{\ln\frac{1}{q}}{2}\right\rfloor}\psi(k_2')\right)\le$$

$$\frac{\tanh 1}{2}\left(2\lambda_1+\left\lceil\frac{\ln\frac{1}{q}}{2}\right\rceil-\left\lfloor\frac{\ln\frac{1}{q}}{2}\right\rfloor-1\right)\left\|\|f\|_\gamma\right\|_\infty.$$

So the double series under consideration is uniformly convergent and continuous. Clearly $F_n(f,x_1,x_2)$ is proved to be continuous on $\mathbb{R}^2$.

Similarly reasoning one can prove easily now, but with more tedious work, that $F_n(f, x_1, ..., x_N)$ is continuous on $\mathbb{R}^N$, for any $N \geq 1$. We choose to omit this similar extra work. ■

Remark 22 *By (36) it is obvious that* $\left\| \|A_n(f)\|_\gamma \right\|_\infty \leq \left\| \|f\|_\gamma \right\|_\infty < \infty$, *and* $A_n(f) \in C\left(\prod_{i=1}^N [a_i, b_i], X\right)$, *given that* $f \in C\left(\prod_{i=1}^N [a_i, b_i], X\right)$.

Call L_n *any of the operators* A_n, B_n, C_n, D_n.

Clearly then

$$\left\| \|L_n^2(f)\|_\gamma \right\|_\infty = \left\| \|L_n(L_n(f))\|_\gamma \right\|_\infty \leq \left\| \|L_n(f)\|_\gamma \right\|_\infty \leq \left\| \|f\|_\gamma \right\|_\infty, \tag{116}$$

etc.

Therefore we get

$$\left\| \|L_n^k(f)\|_\gamma \right\|_\infty \leq \left\| \|f\|_\gamma \right\|_\infty, \quad \forall\, k \in \mathbb{N}, \tag{117}$$

the contraction property.

Also we see that

$$\left\| \|L_n^k(f)\|_\gamma \right\|_\infty \leq \left\| \|L_n^{k-1}(f)\|_\gamma \right\|_\infty \leq ... \leq \left\| \|L_n(f)\|_\gamma \right\|_\infty \leq \left\| \|f\|_\gamma \right\|_\infty. \tag{118}$$

Here L_n^k *are bounded linear operators.*

Notation 23 *Here* $q > 0$, $q \neq 1$, $N \in \mathbb{N}$, $0 < \beta < 1$. *Denote by*

$$c_N := \begin{cases} (\Phi(q))^N, & \text{if } L_n = A_n, \\ 1, & \text{if } L_n = B_n, C_n, D_n, \end{cases} \tag{119}$$

$$\varphi(n) := \begin{cases} \frac{1}{n^\beta}, & \text{if } L_n = A_n,\ B_n, \\ \frac{1}{n} + \frac{1}{n^\beta}, & \text{if } L_n = C_n, D_n, \end{cases} \tag{120}$$

$$\Omega := \begin{cases} C\left(\prod_{i=1}^N [a_i, b_i], X\right), & \text{if } L_n = A_n, \\ C_B\left(\mathbb{R}^N, X\right), & \text{if } L_n = B_n, C_n, D_n, \end{cases} \tag{121}$$

and

$$Y := \begin{cases} \prod_{i=1}^N [a_i, b_i], & \text{if } L_n = A_n, \\ \mathbb{R}^N, & \text{if } L_n = B_n, C_n, D_n. \end{cases} \tag{122}$$

We give the condensed

Theorem 24 *Let* $f \in \Omega$, $0 < \beta < 1$, $x \in Y$; $q > 0$, $q \neq 1$, n, $N \in \mathbb{N}$ *with* $n^{1-\beta} > 2$. *Then*

(i)

$$\|L_n(f,x)-f(x)\|_\gamma \le c_N\left[\omega_1(f,\varphi(n))+2Qe^{-2n^{(1-\beta)}}\left\|\|f\|_\gamma\right\|_\infty\right] =: \tau(n), \tag{123}$$

where ω_1 is for $p=\infty$,

and

(ii)

$$\left\|\|L_n(f)-f\|_\gamma\right\|_\infty \le \tau(n)\to 0, \text{ as } n\to\infty. \tag{124}$$

For f uniformly continuous and in Ω we obtain

$$\lim_{n\to\infty} L_n(f)=f,$$

pointwise and uniformly.

Proof. By Theorems 9, 17, 18, 19. ■

Next we talk about iterated multilayer neural network approximation (see also [9]).

We give

Theorem 25 *All here as in Theorem 24 and $r\in\mathbb{N}$, $\tau(n)$ as in (123). Then*

$$\left\|\|L_n^r f-f\|_\gamma\right\|_\infty \le r\tau(n). \tag{125}$$

So that the speed of convergence to the unit operator of L_n^r is not worse than of L_n.

Proof. As similar to [12], pp. 172-173, is omitted. ■

We also present

Theorem 26 *Let $f\in\Omega$; $q>0$, $q\neq 1$, $N, m_1,m_2,...,m_r\in\mathbb{N}: m_1\le m_2\le ...\le m_r$, $0<\beta<1$; $m_i^{1-\beta}>2$, $i=1,...,r$, $x\in Y$, and let $(L_{m_1},...,L_{m_r})$ as $(A_{m_1},...,A_{m_r})$ or $(B_{m_1},...,B_{m_r})$ or $(C_{m_1},...,C_{m_r})$ or $(D_{m_1},...,D_{m_r})$, $p=\infty$. Then*

$$\begin{gathered}\left\|L_{m_r}\left(L_{m_{r-1}}\left(...L_{m_2}\left(L_{m_1}f\right)\right)\right)(x)-f(x)\right\|_\gamma \le \\ \left\|\left\|L_{m_r}\left(L_{m_{r-1}}\left(...L_{m_2}\left(L_{m_1}f\right)\right)\right)-f\right\|_\gamma\right\|_\infty \le \\ \sum_{i=1}^{r}\left\|\|L_{m_i}f-f\|_\gamma\right\|_\infty \le \\ c_N\sum_{i=1}^{r}\left[\omega_1(f,\varphi(m_i))+2Qe^{-2n^{(1-\beta)}}\left\|\|f\|_\gamma\right\|_\infty\right] \le \\ rc_N\left[\omega_1(f,\varphi(m_1))+2Qe^{-2n^{(1-\beta)}}\left\|\|f\|_\gamma\right\|_\infty\right].\end{gathered} \tag{126}$$

Clearly, we notice that the speed of convergence to the unit operator of the multiply iterated operator is not worse than the speed of L_{m_1}.

Proof. As similar to [12], pp. 173-175, is omitted. ■

We also give

Theorem 27 *Let all as in Corollary 15, and $r \in \mathbb{N}$. Here $\varphi_3(n)$ is as in (88). Then*

$$\left\| \left\| A_n^r f - f \right\|_\gamma \right\|_\infty \leq r \left\| \left\| A_n f - f \right\|_\gamma \right\|_\infty \leq r \varphi_3(n) . \tag{127}$$

Proof. As similar to [12], p. 175, is omitted. ■

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