SOME BOUNDS FOR TRACE CLASS P-DETERMINANT OF HADAMARD PRODUCT OF TWO POSITIVE OPERATORS IN HILBERT SPACES VIA TOMINAGA'S RESULTS

SILVESTRU SEVER DRAGOMIR[1,2]

Abstract. Let H be a complex Hilbert space. For a given operator $P \geq 0$ with $P \in \mathcal{B}_1(H)$, the trace class associated to $\mathcal{B}(H)$ and $\operatorname{tr}(P) = 1$, we define the *P-determinant* of the positive invertible operator A by

$$\Delta_P(A) := \exp \operatorname{tr}(P \ln A).$$

In this paper we show, among others that, if $0 < m_1 \leq A \leq M_1$ and $0 < m_2 \leq B \leq M_2$, then for $P \geq 0$ with $P \in \mathcal{B}_1(H)$ and $\operatorname{tr}(P) = 1$,

$$\frac{1}{S(h_1) S(h_2)} \leq \frac{\Delta_P(A \circ B)}{\Delta_P(A \circ 1) \Delta_P(1 \circ B)} \leq S(h_1 h_2),$$

where $h_1 = \frac{M_1}{m_1} > 1$, $h_2 = \frac{M_2}{m_2} > 1$ and $S(\cdot)$ is *Specht's ratio.*

1. Introduction

In 1952, in the paper [6], B. Fuglede and R. V. Kadison introduced the determinant of a (invertible) operator and established its fundamental properties. The notion generalizes the usual determinant and can be considered for any operator in a finite von Neumann algebra (M, τ) with a faithful normal trace.

Let $T \in M$ be normal and $|T| := (T^*T)^{1/2}$ its modulus. By the spectral theorem one can represent T as an integral

$$T = \int_{\operatorname{Sp}(T)} \lambda dE(\lambda),$$

where $E(\lambda)$ is a projection valued measure and $\operatorname{Sp}(T)$ is the spectrum of T. The measure $\mu_T := \tau \circ E$ becomes a probability measure on the complex plane and has the support in the spectrum $\operatorname{Sp}(T)$.

For any $T \in M$ the Fuglede-Kadison determinant (*FK-determinant*) is defined by

$$\Delta_{FK}(T) := \exp\left(\int_0^\infty \ln t d\mu_{|T|}\right).$$

If T is invertible, then

$$\Delta_{FK}(T) := \exp(\tau(\ln(|T|))),$$

where $\ln(|T|)$ is defined by the use of functional calculus.

Let $B(H)$ be the space of all bounded linear operators on a Hilbert space H, and I stands for the identity operator on H. An operator A in $B(H)$ is said to be positive (in symbol: $A \geq 0$) if $\langle Ax, x\rangle \geq 0$ for all $x \in H$. In particular, $A > 0$

1991 *Mathematics Subject Classification.* 47A63, 26D15, 46C05.

Key words and phrases. Positive operators, Trace class operators, Determinants, Inequalities.

means that A is positive and invertible. For a pair A, B of selfadjoint operators the order relation $A \geq B$ means as usual that $A - B$ is positive.

In 1998, Fujii et al. [7], [8], introduced the *normalized determinant* $\Delta_x(A)$ for positive invertible operators A on a Hilbert space H and a fixed unit vector $x \in H$, namely $\|x\| = 1$, defined by

$$\Delta_x(A) := \exp \langle \ln Ax, x \rangle$$

and discussed it as a continuous geometric mean and observed some inequalities around the determinant from this point of view. For some recent results, see [10].

We need now some preparations for trace of operators in Hilbert spaces.

Let $(H, \langle \cdot, \cdot \rangle)$ be a complex Hilbert space and $\{e_i\}_{i \in I}$ an *orthonormal basis* of H. We say that $A \in \mathcal{B}(H)$ is a *Hilbert-Schmidt operator* if

$$\sum_{i \in I} \|Ae_i\|^2 < \infty. \tag{1.1}$$

It is well know that, if $\{e_i\}_{i \in I}$ and $\{f_j\}_{j \in J}$ are orthonormal bases for H and $A \in \mathcal{B}(H)$ then

$$\sum_{i \in I} \|Ae_i\|^2 = \sum_{j \in I} \|Af_j\|^2 = \sum_{j \in I} \|A^* f_j\|^2 \tag{1.2}$$

showing that the definition (1.1) is independent of the orthonormal basis and A is a Hilbert-Schmidt operator iff A^* is a Hilbert-Schmidt operator.

Let $\mathcal{B}_2(H)$ the set of *Hilbert-Schmidt operators* in $\mathcal{B}(H)$. For $A \in \mathcal{B}_2(H)$ we define

$$\|A\|_2 := \left(\sum_{i \in I} \|Ae_i\|^2 \right)^{1/2} \tag{1.3}$$

for $\{e_i\}_{i \in I}$ an orthonormal basis of H.

Using the triangle inequality in $l^2(I)$, one checks that $\mathcal{B}_2(H)$ is a *vector space* and that $\|\cdot\|_2$ is a norm on $\mathcal{B}_2(H)$, which is usually called in the literature as the *Hilbert-Schmidt norm.*

Denote *the modulus* of an operator $A \in \mathcal{B}(H)$ by $|A| := (A^*A)^{1/2}$.

Because $\||A| x\| = \|Ax\|$ for all $x \in H$, A is Hilbert-Schmidt iff $|A|$ is Hilbert-Schmidt and $\|A\|_2 = \||A|\|_2$. From (1.2) we have that if $A \in \mathcal{B}_2(H)$, then $A^* \in \mathcal{B}_2(H)$ and $\|A\|_2 = \|A^*\|_2$.

The following theorem collects some of the most important properties of Hilbert-Schmidt operators:

Theorem 1. *We have:*

(i) $(\mathcal{B}_2(H), \|\cdot\|_2)$ *is a Hilbert space with inner product*

$$\langle A, B \rangle_2 := \sum_{i \in I} \langle Ae_i, Be_i \rangle = \sum_{i \in I} \langle B^* Ae_i, e_i \rangle \tag{1.4}$$

and the definition does not depend on the choice of the orthonormal basis $\{e_i\}_{i \in I}$;

(ii) We have the inequalities

$$\|A\| \leq \|A\|_2 \tag{1.5}$$

for any $A \in \mathcal{B}_2(H)$ *and, if* $A \in \mathcal{B}_2(H)$ *and* $T \in \mathcal{B}(H)$, *then* AT, $TA \in \mathcal{B}_2(H)$ *with*

$$\|AT\|_2, \ \|TA\|_2 \leq \|T\| \|A\|_2 \tag{1.6}$$

(iii) $\mathcal{B}_2(H)$ *is an operator ideal in* $\mathcal{B}(H)$, *i.e.*

$$\mathcal{B}(H)\mathcal{B}_2(H)\mathcal{B}(H) \subseteq \mathcal{B}_2(H).$$

If $\{e_i\}_{i\in I}$ an orthonormal basis of H, we say that $A \in \mathcal{B}(H)$ is *trace class* if

$$\|A\|_1 := \sum_{i\in I} \langle |A| e_i, e_i\rangle < \infty. \tag{1.7}$$

The definition of $\|A\|_1$ does not depend on the choice of the orthonormal basis $\{e_i\}_{i\in I}$. We denote by $\mathcal{B}_1(H)$ the set of trace class operators in $\mathcal{B}(H)$.

The following proposition holds:

Proposition 1. *If* $A \in \mathcal{B}(H)$, *then the following are equivalent:*
(i) $A \in \mathcal{B}_1(H)$;
(ii) $|A|^{1/2} \in \mathcal{B}_2(H)$.

The following properties are also well known:

Theorem 2. *With the above notations:*
(i) We have

$$\|A\|_1 = \|A^*\|_1 \text{ and } \|A\|_2 \le \|A\|_1 \tag{1.8}$$

for any $A \in \mathcal{B}_1(H)$;
(ii) $\mathcal{B}_1(H)$ *is an operator ideal in* $\mathcal{B}(H)$, *i.e.*

$$\mathcal{B}(H)\mathcal{B}_1(H)\mathcal{B}(H) \subseteq \mathcal{B}_1(H);$$

(iii) We have

$$\mathcal{B}_2(H)\mathcal{B}_2(H) = \mathcal{B}_1(H);$$

(iv) We have

$$\|A\|_1 = \sup\{\langle A, B\rangle_2 \mid B \in \mathcal{B}_2(H), \ \|B\|_2 \le 1\};$$

(v) $(\mathcal{B}_1(H), \|\cdot\|_1)$ *is a Banach space.*

We define the *trace* of a trace class operator $A \in \mathcal{B}_1(H)$ to be

$$\operatorname{tr}(A) := \sum_{i\in I} \langle Ae_i, e_i\rangle, \tag{1.9}$$

where $\{e_i\}_{i\in I}$ an orthonormal basis of H. Note that this coincides with the usual definition of the trace if H is finite-dimensional. We observe that the series (1.9) converges absolutely and it is independent from the choice of basis.

The following result collects some properties of the trace:

Theorem 3. *We have:*
(i) If $A \in \mathcal{B}_1(H)$ *then* $A^* \in \mathcal{B}_1(H)$ *and*

$$\operatorname{tr}(A^*) = \overline{\operatorname{tr}(A)}; \tag{1.10}$$

(ii) If $A \in \mathcal{B}_1(H)$ *and* $T \in \mathcal{B}(H)$, *then* AT, $TA \in \mathcal{B}_1(H)$,

$$\operatorname{tr}(AT) = \operatorname{tr}(TA) \text{ and } |\operatorname{tr}(AT)| \le \|A\|_1 \|T\|; \tag{1.11}$$

(iii) $\operatorname{tr}(\cdot)$ *is a bounded linear functional on* $\mathcal{B}_1(H)$ *with* $\|\operatorname{tr}\| = 1$;
(iv) If A, $B \in \mathcal{B}_2(H)$ *then* AB, $BA \in \mathcal{B}_1(H)$ *and* $\operatorname{tr}(AB) = \operatorname{tr}(BA)$.

Now, if we assume that $P \geq 0$ and $P \in \mathcal{B}_1(H)$, then for all $T \in \mathcal{B}(H)$, PT, $TP \in \mathcal{B}_1(H)$ and $\operatorname{tr}(PT) = \operatorname{tr}(TP)$. Also, since $P^{1/2} \in \mathcal{B}_2(H)$, $TP^{1/2} \in \mathcal{B}_2(H)$, hence $P^{1/2}TP^{1/2}$ and $TP^{1/2}P^{1/2} = TP \in \mathcal{B}_1(H)$ with $\operatorname{tr}\left(P^{1/2}TP^{1/2}\right) = \operatorname{tr}(TP)$. Therefore, if $P \geq 0$ and $P \in \mathcal{B}_1(H)$,

$$\operatorname{tr}(PT) = \operatorname{tr}(TP) = \operatorname{tr}\left(P^{1/2}TP^{1/2}\right)$$

for all $T \in \mathcal{B}(H)$.

If $T \geq 0$, then $P^{1/2}TP^{1/2} \geq 0$, which implies that $\operatorname{tr}(PT) \geq 0$ that shows that the functional $\mathcal{B}(H) \ni T \longmapsto \operatorname{tr}(PT)$ is linear and isotonic functional. Also, by (1.11), if $T_n \to T$ for $n \to \infty$ in $\mathcal{B}(H)$ then $\lim_{n\to\infty} \operatorname{tr}(PT_n) = \operatorname{tr}(PT)$, namely $\mathcal{B}(H) \ni T \longmapsto \operatorname{tr}(PT)$ is also continuous in the norm topology.

For a survey on recent trace inequalities see [4] and the references therein.

Now, for a given $P \geq 0$ with $P \in \mathcal{B}_1(H)$ and $\operatorname{tr}(P) = 1$, we define the *P-determinant* of the positive invertible operator A by [5]

$$\Delta_P(A) := \exp\operatorname{tr}(P\ln A) = \exp\operatorname{tr}((\ln A)P) = \exp\operatorname{tr}\left(P^{1/2}(\ln A)P^{1/2}\right). \tag{1.12}$$

Assume that $P \geq 0$ with $P \in \mathcal{B}_1(H)$ and $\operatorname{tr}(P) = 1$. We observe that we have the following elementary properties

(i) *continuity*: the map $A \to \Delta_P(A)$ is norm continuous;
(ii) *power equality:* $\Delta_P(A^t) = \Delta_P(A)^t$ for all $t > 0$;
(iii) *homogeneity*: $\Delta_P(tA) = t\Delta_P(A)$ and $\Delta_P(tI) = t$ for all $t > 0$;
(iv) *monotonicity*: $0 < A \leq B$ implies $\Delta_P(A) \leq \Delta_P(B)$.

We have the following result [5]:

Theorem 4. *Let $P \geq 0$ with $P \in \mathcal{B}_1(H)$ and $\operatorname{tr}(P) = 1$, then for all A, $B > 0$ and $t \in [0,1]$,*

$$\Delta_P((1-t)A + tB) \geq [\Delta_P(A)]^{1-t}[\Delta_P(B)]^t. \tag{1.13}$$

Also, we have that

$$1 \leq \frac{\operatorname{tr}(PA)}{\Delta_P(A)} \leq \exp\left[\operatorname{tr}(PA)\operatorname{tr}\left(PA^{-1}\right) - 1\right]$$

and

$$1 \leq \frac{\Delta_P(A)}{\left[\operatorname{tr}\left(PA^{-1}\right)\right]^{-1}} \leq \exp\left[\operatorname{tr}\left(PA^{-1}\right)\operatorname{tr}(PA) - 1\right],$$

for $A > 0$ and $P \geq 0$ with $P \in \mathcal{B}_1(H)$ and $\operatorname{tr}(P) = 1$.

We recall that *Specht's ratio* is defined by [11]

$$S(h) := \begin{cases} \dfrac{h^{\frac{1}{h-1}}}{e\ln\left(h^{\frac{1}{h-1}}\right)} \text{ if } h \in (0,1)\cup(1,\infty) \\ \\ 1 \text{ if } h = 1. \end{cases} \tag{1.14}$$

It is well known that $\lim_{h\to 1} S(h) = 1$, $S(h) = S\left(\frac{1}{h}\right) > 1$ for $h > 0$, $h \neq 1$. The function is decreasing on $(0,1)$ and increasing on $(1,\infty)$.

The following inequality provides a refinement and a multiplicative reverse for Young's inequality

$$a^{1-\nu}b^{\nu} \leq S\left(\left(\frac{a}{b}\right)^{r}\right)a^{1-\nu}b^{\nu} \leq (1-\nu)\,a+\nu b \leq S\left(\frac{a}{b}\right)a^{1-\nu}b^{\nu}, \tag{1.15}$$

where a, $b>0$, $\nu \in [0,1]$, $r=\min\{1-\nu,\nu\}$.

The second inequality in (1.15) is due to Tominaga [12] while the first one is due to Furuichi [9].

In [12] Tominaga also obtained the following additive reverse inequality

$$0 \leq (1-\nu)\,a+\nu b-a^{1-\nu}b^{\nu} \leq L\,(a,b)\ln S\left(\frac{a}{b}\right) \tag{1.16}$$

where the *logarithmic mean* of two positive numbers a, b is defined by

$$L\,(a,b) := \begin{cases} \frac{b-a}{\ln b-\ln a} \text{ if } b \neq a, \\ \\ a \text{ if } b=a. \end{cases}$$

Motivated by the above results, in this paper we show, among others that, if $0<m_1 \leq A \leq M_1$ and $0<m_2 \leq B \leq M_2$, then for $P \geq 0$ with $P \in \mathcal{B}_1(H)$ and $\operatorname{tr}(P)=1$,

$$\frac{1}{S\,(h_1)\,S\,(h_2)} \leq \frac{\Delta_P\,(A \circ B)}{\Delta_P\,(A \circ 1)\,\Delta_P\,(1 \circ B)} \leq S\,(h_1 h_2)\,,$$

where $h_1=\frac{M_1}{m_1}>1$, $h_2=\frac{M_2}{m_2}>1$ and $S\,(\cdot)$ is *Specht's ratio.*

2. Multiplicative Inequalities

We start to the following operator inequalities involving positive operators and positive linear maps:

Lemma 1. *Assume that the selfadjoint operator V satisfies the condition $0<m \leq V \leq M$ for some constants, m, M and Φ a unital positive linear map from $B\,(H)$ into $B\,(K)$. Then*

$$\begin{aligned} &\ln m\frac{M-\Phi\,(V)}{M-m}+\ln M\frac{\Phi\,(V)-m}{M-m} \\ &\leq S\left(\left(\frac{M}{m}\right)^{\frac{1}{2}-\frac{1}{M-m}\left|\Phi(V)-\frac{m+M}{2}\right|}\right)+\ln m\frac{M-\Phi\,(V)}{M-m}+\ln M\frac{\Phi\,(V)-m}{M-m} \\ &\leq \ln\Phi\,(V) \leq \ln m\frac{M-\Phi\,(V)}{M-m}+\ln M\frac{\Phi\,(V)-m}{M-m}+\ln S\left(\frac{M}{m}\right) \end{aligned} \tag{2.1}$$

and

$$\begin{aligned} &\ln m\frac{M-\Phi\,(V)}{M-m}+\ln M\frac{\Phi\,(V)-m}{M-m} \\ &\leq \Phi S\left(\left(\frac{M}{m}\right)^{\frac{1}{2}-\frac{1}{M-m}\left|V-\frac{m+M}{2}\right|}\right)+\ln m\frac{M-\Phi\,(V)}{M-m}+\ln M\frac{\Phi\,(V)-m}{M-m} \\ &\leq \Phi\,(\ln V) \\ &\leq \ln m\frac{M-\Phi\,(V)}{M-m}+\ln M\frac{\Phi\,(V)-m}{M-m}+\ln S\left(\frac{M}{m}\right). \end{aligned} \tag{2.2}$$

Proof. From (1.15) we get, by taking the logarithm, that

$$\begin{aligned}(2.3)\qquad &(1-\nu)\ln a+\nu\ln b\\ &\le S\left(\left(\frac{a}{b}\right)^{r}\right)+(1-\nu)\ln a+\nu\ln b\le \ln\left[(1-\nu)a+\nu b\right]\\ &\le \ln S\left(\frac{a}{b}\right)+(1-\nu)\ln a+\nu\ln b,\end{aligned}$$

where a, $b>0$, $\nu\in[0,1]$, $r=\min\{1-\nu,\nu\}$.

Assume that $0<a<b$. We take $\nu=\frac{t-a}{b-a}\in[0,1]$ for $t\in[a,b]$ and observe that

$$(1-\nu)a+\nu b=\frac{b-t}{b-a}a+\frac{t-a}{b-a}b=t$$

and

$$\begin{aligned}\min\{\nu,1-\nu\} &= \min\left\{\frac{t-a}{b-a},\frac{b-t}{b-a}\right\}\\ &= \frac{1}{2}-\frac{1}{b-a}\left|t-\frac{a+b}{2}\right|.\end{aligned}$$

From (2.3) we get

$$\begin{aligned}(2.4)\qquad &\frac{b-t}{b-a}\ln a+\frac{t-a}{b-a}\ln b\\ &\le S\left(\left(\frac{a}{b}\right)^{\frac{1}{2}-\frac{1}{b-a}\left|t-\frac{a+b}{2}\right|}\right)+\frac{b-t}{b-a}\ln a+\frac{t-a}{b-a}\ln b\\ &\le \ln t\le \frac{b-t}{b-a}\ln a+\frac{t-a}{b-a}\ln b+\ln S\left(\frac{a}{b}\right)\end{aligned}$$

for all $t\in[a,b]$.

By utilizing the continuous functional calculus for selfadjoint operators T with spectra $\operatorname{Sp}(T)\subseteq[a,b]$, we obtain from (2.4) that

$$\begin{aligned}(2.5)\qquad &\ln a\frac{b-T}{b-a}+\ln b\frac{T-a}{b-a}\\ &\le S\left(\left(\frac{a}{b}\right)^{\frac{1}{2}-\frac{1}{b-a}\left|T-\frac{a+b}{2}\right|}\right)+\ln a\frac{b-T}{b-a}+\ln b\frac{T-a}{b-a}\\ &\le \ln T\le \ln a\frac{b-T}{b-a}+\ln b\frac{T-a}{b-a}+\ln S\left(\frac{a}{b}\right).\end{aligned}$$

Now if $0<m\le V\le M$, then $0<m\le\Phi(V)\le M$ and by (2.5) we get for $T=\Phi(V)$, $a=m$ and $b=M$ that

$$\begin{aligned}&\ln m\frac{M-\Phi(V)}{M-m}+\ln M\frac{\Phi(V)-m}{M-m}\\ &\le S\left(\left(\frac{m}{M}\right)^{\frac{1}{2}-\frac{1}{M-m}\left|\Phi(V)-\frac{m+M}{2}\right|}\right)+\ln m\frac{M-\Phi(V)}{M-m}+\ln M\frac{\Phi(V)-m}{M-m}\\ &\le \ln\Phi(V)\le \ln m\frac{M-\Phi(V)}{M-m}+\ln M\frac{\Phi(V)-m}{M-m}+\ln S\left(\frac{m}{M}\right)\end{aligned}$$

and the inequality (2.1) is proved.

If we take $T = V$, $a = m$ and $b = M$ in (2.5) and then apply Φ we obtain

$$\begin{aligned}
&\Phi\left(\ln m\frac{M-V}{M-m}+\ln M\frac{V-m}{M-m}\right)\\
&\le \Phi\left[S\left(\left(\frac{M}{m}\right)^{\frac{1}{2}-\frac{1}{M-m}\left|V-\frac{m+M}{2}\right|}\right)+\ln m\frac{M-V}{M-m}+\ln M\frac{V-m}{M-m}\right]\\
&\le \Phi\ln V\le \Phi\left[\ln m\frac{M-V}{M-m}+\ln M\frac{V-m}{M-m}+\ln S\left(\frac{M}{m}\right)\right],
\end{aligned}$$

which is equivalent to (2.2). □

Corollary 1. *With the assumptions of Lemma 1 we have the chain of inequalities*

$$\begin{aligned}
&\ln m\frac{M-\Phi(V)}{M-m}+\ln M\frac{\Phi(V)-m}{M-m}\\
&\le \Phi S\left(\left(\frac{M}{m}\right)^{\frac{1}{2}-\frac{1}{M-m}\left|V-\frac{m+M}{2}\right|}\right)+\ln m\frac{M-\Phi(V)}{M-m}+\ln M\frac{\Phi(V)-m}{M-m}\\
&\le \Phi(\ln V)\le \ln\Phi(V)\\
&\le \ln m\frac{M-\Phi(V)}{M-m}+\ln M\frac{\Phi(V)-m}{M-m}+\ln S\left(\frac{M}{m}\right)\\
&\le \Phi(\ln V)+\ln S\left(\frac{M}{m}\right).
\end{aligned} \tag{2.6}$$

Proof. Second inequality follows by Jensen's operator inequality for the operator concave function ln, while the last inequality follows by the fact that

$$\ln m\frac{M-\Phi(V)}{M-m}+\ln M\frac{\Phi(V)-m}{M-m}\le \Phi(\ln V)$$

from the first part of (2.6). □

We have the following inequalities for the determinant $\Delta_P(A\circ B)$ for $P\ge 0$ with $P\in\mathcal{B}_1(H)$ and $\operatorname{tr}(P)=1$.

Theorem 5. *Assume that* $0<m_1\le A\le M_1$ *and* $0<m_2\le B\le M_2$, *then for* $P\ge 0$ *with* $P\in\mathcal{B}_1(H)$ *and* $\operatorname{tr}(P)=1$,

$$\begin{aligned}
&m^{\frac{M-\operatorname{tr}[P(A\circ B)]}{M-m}}M^{\frac{\operatorname{tr}[P(A\circ B)]-m}{M-m}}\\
&\le m^{\frac{M-\operatorname{tr}[P(A\circ B)]}{M-m}}M^{\frac{\operatorname{tr}[P(A\circ B)]-m}{M-m}}\\
&\times\exp\operatorname{tr}\left[P\mathcal{U}^*S\left(\left(\frac{M}{m}\right)^{\frac{1}{2}-\frac{1}{M-m}\left|A\otimes B-\frac{m+M}{2}\right|}\right)\mathcal{U}\right]\\
&\le \exp\operatorname{tr}\left[P\mathcal{U}^*(\ln(A\otimes B))\mathcal{U}\right]\le\Delta_P(A\circ B)\\
&\le S\left(\frac{M}{m}\right)m^{\frac{M-\operatorname{tr}[P(A\circ B)]}{M-m}}M^{\frac{\operatorname{tr}[P(A\circ B)]-m}{M-m}}\\
&\le S\left(\frac{M}{m}\right)\exp\operatorname{tr}\left[P\mathcal{U}^*(\ln(A\otimes B))\mathcal{U}\right],
\end{aligned} \tag{2.7}$$

where $m=m_1m_2$, $M=M_1M_2$.

Proof. Since $0 < m_1 \leq A \leq M_1$ and $0 < m_2 \leq B \leq M_2$, then $0 < m_1 m_2 = m \leq V = A \otimes B \leq M = M_1 M_2$. From (2.6) for $\Phi(V) = \mathcal{U}^*(A \otimes B)\mathcal{U} = A \circ B$ we get

$$
\begin{aligned}
&\ln m \frac{M - A \circ B}{M - m} + \ln M \frac{A \circ B - m}{M - m} \\
&\leq \mathcal{U}^* S\left(\left(\frac{M}{m}\right)^{\frac{1}{2} - \frac{1}{M-m}\left|A \otimes B - \frac{m+M}{2}\right|}\right)\mathcal{U} \\
&+ \ln m \frac{M - A \circ B}{M - m} + \ln M \frac{A \circ B - m}{M - m} \\
&\leq \mathcal{U}^* (\ln(A \otimes B))\mathcal{U} \leq \ln(A \circ B) \\
&\leq \ln m \frac{M - A \circ B}{M - m} + \ln M \frac{A \circ B - m}{M - m} + \ln S\left(\frac{M}{m}\right) \\
&\leq \mathcal{U}^* (\ln(A \otimes B))\mathcal{U} + \ln S\left(\frac{M}{m}\right).
\end{aligned}
\tag{2.8}
$$

If we multiply the inequality (2.8) to the left and to the right with $P^{1/2}$, then we get

$$
\begin{aligned}
&\ln m \frac{MP - P^{1/2}(A \circ B)P^{1/2}}{M - m} + \ln M \frac{P^{1/2}(A \circ B)P^{1/2} - mP}{M - m} \\
&\leq P^{1/2}\mathcal{U}^* S\left(\left(\frac{M}{m}\right)^{\frac{1}{2} - \frac{1}{M-m}\left|A \otimes B - \frac{m+M}{2}\right|}\right)\mathcal{U}P^{1/2} \\
&+ \ln m \frac{MP - P^{1/2}(A \circ B)P^{1/2}}{M - m} + \ln M \frac{P^{1/2}(A \circ B)P^{1/2} - mP}{M - m} \\
&\leq P^{1/2}\mathcal{U}^* (\ln(A \otimes B))\mathcal{U}P^{1/2} \leq P^{1/2} \ln(A \circ B) P^{1/2} \\
&\leq \ln m \frac{MP - P^{1/2}(A \circ B)P^{1/2}}{M - m} + \ln M \frac{P^{1/2}(A \circ B)P^{1/2} - mP}{M - m} \\
&+ \ln S\left(\frac{M}{m}\right) P \\
&\leq P^{1/2}\mathcal{U}^* \Phi(\ln(A \otimes B))\mathcal{U}P^{1/2} + \ln S\left(\frac{M}{m}\right) P.
\end{aligned}
\tag{2.9}
$$

If we take the trace in (2.9) and use its properties and the fact that $\operatorname{tr}(P) = 1$, then we get

$$
\begin{aligned}
&\frac{M - \operatorname{tr}[P(A \circ B)]}{M - m} \ln m + \ln M \frac{\operatorname{tr}[P(A \circ B)] - m}{M - m} \\
&\leq \operatorname{tr}\left[P\mathcal{U}^* S\left(\left(\frac{M}{m}\right)^{\frac{1}{2} - \frac{1}{M-m}\left|A \otimes B - \frac{m+M}{2}\right|}\right)\mathcal{U}\right] \\
&+ \ln m \frac{M - \operatorname{tr}[P(A \circ B)]}{M - m} + \ln M \frac{\operatorname{tr}[P(A \circ B)] - m}{M - m}
\end{aligned}
\tag{2.10}
$$

$$\leq \operatorname{tr}\left[P\mathcal{U}^{*}\left(\ln\left(A\otimes B\right)\right)\mathcal{U}\right] \leq \operatorname{tr}\left[P\ln\left(A\circ B\right)\right]$$
$$\leq \ln m \frac{M-\operatorname{tr}\left[P\left(A\circ B\right)\right]}{M-m} + \ln M \frac{\operatorname{tr}\left[P\left(A\circ B\right)\right]-m}{M-m} + \ln S\left(\frac{M}{m}\right)$$
$$\leq \operatorname{tr}\left[P\mathcal{U}^{*}\left(\ln\left(A\otimes B\right)\right)\mathcal{U}\right] + \ln S\left(\frac{M}{m}\right) P.$$

Now, if we take the exponential in (2.10), then we get the desired result (2.7). □

3. Additive Inequalities

We also have the following inequalities:

Theorem 6. *Assume that the selfadjoint operator V satisfies the condition $0 < m \leq V \leq M$ for some constants, m, M and Φ a unital positive linear map from $B(H)$ into $B(K)$. Then*

$$\begin{aligned} &\ln\left(m^{\frac{M-\Phi(V)}{M-m}} M^{\frac{\Phi(V)-m}{M-m}}\right) \\ &\leq \Phi\left(\ln V\right) \\ &\leq \ln\Phi\left(V\right) \leq \ln\left[m^{\frac{M-\Phi(V)}{M-m}} M^{\frac{\Phi(V)-m}{M-m}} + L\left(m,M\right)\ln S\left(\frac{M}{m}\right)\right]. \end{aligned} \tag{3.1}$$

Proof. From (1.16) we have

$$a^{1-\nu}b^{\nu} \leq (1-\nu)\,a + \nu b \leq a^{1-\nu}b^{\nu} + L\left(a,b\right)\ln S\left(\frac{a}{b}\right) \tag{3.2}$$

for $a, b > 0$ and $\nu \in [0,1]$.

Assume that $0 < a < b$. We take $\nu = \frac{t-a}{b-a} \in [0,1]$ for $t \in [a,b]$, then by (3.2) we get

$$a^{\frac{b-t}{b-a}} b^{\frac{t-a}{b-a}} \leq t \leq a^{\frac{b-t}{b-a}} b^{\frac{t-a}{b-a}} + L\left(a,b\right)\ln S\left(\frac{a}{b}\right)$$

and by taking the logarithm, we get

$$\frac{b-t}{b-a}\ln a + \frac{t-a}{b-a}\ln b \leq \ln t \leq \ln\left[a^{\frac{b-t}{b-a}} b^{\frac{t-a}{b-a}} + L\left(a,b\right)\ln S\left(\frac{a}{b}\right)\right] \tag{3.3}$$

for $t \in [a,b] \subset (0,\infty)$.

By utilizing the continuous functional calculus for selfadjoint operators T with spectra $\operatorname{Sp}(T) \subseteq [a,b]$, we obtain from (3.3) that

$$\frac{b-T}{b-a}\ln a + \frac{T-a}{b-a}\ln b \leq \ln T \leq \ln\left[a^{\frac{b-T}{b-a}} b^{\frac{T-a}{b-a}} + L\left(a,b\right)\ln S\left(\frac{a}{b}\right)\right]. \tag{3.4}$$

Now if $0 < m \leq V \leq M$, then $0 < m \leq \Phi(V) \leq M$ and by the second part of (3.4) we get for $T = \Phi(V)$, $a = m$ and $b = M$ the inequality

$$\ln\Phi\left(V\right) \leq \ln\left[m^{\frac{M-\Phi(V)}{M-m}} M^{\frac{\Phi(V)-m}{M-m}} + L\left(m,M\right)\ln S\left(\frac{M}{m}\right)\right].$$

From the first part of (3.4) we have

$$\ln m\frac{M-V}{M-m} + \ln M\frac{V-m}{M-m} \leq \ln V.$$

If we take the positive linear map Φ we obtain

$$\ln m\frac{M-\Phi(V)}{M-m} + \ln M\frac{\Phi(V)-m}{M-m} \leq \Phi\left(\ln V\right),$$

which is equivalent to

$$\ln\left(m^{\frac{M-\Phi(V)}{M-m}}M^{\frac{\Phi(V)-m}{M-m}}\right)\leq\Phi\left(\ln V\right).$$

Now, by operator Jensen's inequality we have $\Phi\left(\ln V\right)\leq\ln\Phi\left(V\right)$. By collecting all these inequalities we obtain the desired result (3.1). $\square$

Corollary 2. *With the assumptions of Theorem 6,*

$$\begin{aligned}
&\ln\left(m^{\frac{M-\Phi(V)}{M-m}}M^{\frac{\Phi(V)-m}{M-m}}\right) \\
&\leq\Phi\left(\ln V\right)\leq\ln\Phi\left(V\right) \\
&\leq\ln\left(m^{\frac{M-\Phi(V)}{M-m}}M^{\frac{\Phi(V)-m}{M-m}}\right) \\
&+L\left(m,M\right)\ln S\left(\frac{M}{m}\right)\left[\frac{M-\Phi\left(V\right)}{M-m}m^{-1}+\frac{\Phi\left(V\right)-m}{M-m}M^{-1}\right] \\
&\leq\Phi\left(\ln V\right)+L\left(m,M\right)\ln S\left(\frac{M}{m}\right)\left[\frac{M-\Phi\left(V\right)}{M-m}m^{-1}+\frac{\Phi\left(V\right)-m}{M-m}M^{-1}\right] \\
&\leq\Phi\left(\ln V\right)+L\left(\frac{M}{m},1\right)\ln S\left(\frac{M}{m}\right).
\end{aligned}\tag{3.5}$$

Proof. By the concavity of the function ln we have x, $y>0$ that

$$\ln x-\ln y\leq\frac{x}{y}-1.$$

This implies that

$$\ln\left(t+k\right)\leq\ln t+kt^{-1}$$

for all t, $k>0$.

By the functional calculus we get in the operator order

$$\ln\left(T+k\right)\leq\ln T+kT^{-1}$$

for all operators $T>0$ and $k>0$.

$$\begin{aligned}
&\ln\left[m^{\frac{M-\Phi(V)}{M-m}}M^{\frac{\Phi(V)-m}{M-m}}+L\left(m,M\right)\ln S\left(\frac{M}{m}\right)\right] \\
&\leq\ln\left(m^{\frac{M-\Phi(V)}{M-m}}M^{\frac{\Phi(V)-m}{M-m}}\right)+L\left(m,M\right)\ln S\left(\frac{M}{m}\right)\left(m^{\frac{M-\Phi(V)}{M-m}}M^{\frac{\Phi(V)-m}{M-m}}\right)^{-1}.
\end{aligned}$$

By the geometric mean-harmonic mean inequality we have

$$m^{\frac{M-\Phi(V)}{M-m}}M^{\frac{\Phi(V)-m}{M-m}}\geq\left(\frac{M-\Phi\left(V\right)}{M-m}m^{-1}+\frac{\Phi\left(V\right)-m}{M-m}M^{-1}\right)^{-1}\geq m,$$

which implies that

$$\left(m^{\frac{M-\Phi(V)}{M-m}}M^{\frac{\Phi(V)-m}{M-m}}\right)^{-1}\leq\frac{M-\Phi\left(V\right)}{M-m}m^{-1}+\frac{\Phi\left(V\right)-m}{M-m}M^{-1}\leq m^{-1}.$$

Therefore

$$
\begin{aligned}
&\ln\left[m^{\frac{M-\Phi(V)}{M-m}}M^{\frac{\Phi(V)-m}{M-m}}+L(m,M)\ln S\left(\frac{M}{m}\right)\right]\\
&\le \ln\left(m^{\frac{M-\Phi(V)}{M-m}}M^{\frac{\Phi(V)-m}{M-m}}\right)+L(m,M)\ln S\left(\frac{M}{m}\right)\left(m^{\frac{M-\Phi(V)}{M-m}}M^{\frac{\Phi(V)-m}{M-m}}\right)^{-1}\\
&\le \ln\left(m^{\frac{M-\Phi(V)}{M-m}}M^{\frac{\Phi(V)-m}{M-m}}\right)\\
&+L(m,M)\ln S\left(\frac{M}{m}\right)\left[\frac{M-\Phi(V)}{M-m}m^{-1}+\frac{\Phi(V)-m}{M-m}M^{-1}\right]\\
&\le \Phi(\ln V)+L(m,M)\ln S\left(\frac{M}{m}\right)\left[\frac{M-\Phi(V)}{M-m}m^{-1}+\frac{\Phi(V)-m}{M-m}M^{-1}\right]
\end{aligned}
$$

and by Theorem 6 we derive the third and fourth inequalities in (3.5). The last part is obvious. $\square$

Theorem 7. *Assume that $0<m_1\le A\le M_1$ and $0<m_2\le B\le M_2$, then for $P\ge 0$ with $P\in\mathcal{B}_1(H)$ and $\operatorname{tr}(P)=1$,*

$$
\begin{aligned}
&\Delta_P\left(m^{\frac{M-A\circ B}{M-m}}M^{\frac{A\circ B-m}{M-m}}\right)\\
&\le \exp\operatorname{tr}\left[P\mathcal{U}^*\left(\ln(A\otimes B)\right)\mathcal{U}\right]\le\Delta_P(A\circ B)\\
&\le \Delta_P\left(m^{\frac{M-A\circ B}{M-m}}(M)^{\frac{A\circ B-m}{M-m}}\right)\\
&\times S\left(\frac{M}{m}\right)^{L(m,M)\left[\frac{M-\operatorname{tr}[P(A\circ B)]}{M-m}m^{-1}+\frac{\operatorname{tr}[P(A\circ B)]-m}{M-m}M^{-1}\right]}\\
&\le S\left(\frac{M}{m}\right)^{L(m,M)\left[\frac{M-\operatorname{tr}[P(A\circ B)]}{M-m}m^{-1}+\frac{\operatorname{tr}[P(A\circ B)]-m}{M-m}M^{-1}\right]}\\
&\times\exp\operatorname{tr}\left[P\mathcal{U}^*\left(\ln(A\otimes B)\right)\mathcal{U}\right]\\
&\le S\left(\frac{M}{m}\right)^{L\left(\frac{M}{m},1\right)}\exp\operatorname{tr}\left[P\mathcal{U}^*\left(\ln(A\otimes B)\right)\mathcal{U}\right],
\end{aligned}
\tag{3.6}
$$

where $m=m_1m_2$ and $M=M_1M_2$.

Proof. Since $0<m_1\le A\le M_1$ and $0<m_2\le B\le M_2$, then $0<m=m_1m_2\le V=A\otimes B\le M=M_1M_2$. From (3.5) for $m=m$, $M=M$, $\Phi(V)=\mathcal{U}^*(A\otimes B)\mathcal{U}=A\circ B$ we get

$$
\begin{aligned}
&\ln\left((m)^{\frac{M-A\circ B}{M-m}}(M)^{\frac{A\circ B-m}{M-m}}\right)\\
&\le\mathcal{U}^*\left(\ln(A\otimes B)\right)\mathcal{U}\le\ln(A\circ B)\\
&\le\ln\left((m)^{\frac{M-A\circ B}{M-m}}(M)^{\frac{A\circ B-m}{M-m}}\right)+L(m,M)\ln S\left(\frac{M}{m}\right)\\
&\times\left[\frac{M-A\circ B}{M-m}(m)^{-1}+\frac{A\circ B-m}{M-m}(M)^{-1}\right]
\end{aligned}
\tag{3.7}
$$

$$
\begin{aligned}
&\leq \mathcal{U}^{*}\left(\ln\left(A\otimes B\right)\right)\mathcal{U}+L\left(m,M\right)\ln S\left(\frac{M}{m}\right)\\
&\times\left[\frac{M-A\circ B}{M-m}\left(m\right)^{-1}+\frac{A\circ B-m}{M-m}\left(M\right)^{-1}\right]\\
&\leq \mathcal{U}^{*}\left(\ln\left(A\otimes B\right)\right)\mathcal{U}+L\left(\frac{M}{m},1\right)\ln S\left(\frac{M}{m}\right).
\end{aligned}
$$

If we multiply the inequality (3.7) to the left and to the right with $P^{1/2}$, take the trace, then we get

$$
\begin{aligned}
&\operatorname{tr}\left[P\ln\left(\left(m\right)^{\frac{M-A\circ B}{M-m}}\left(M\right)^{\frac{A\circ B-m}{M-m}}\right)\right]\\
&\leq \operatorname{tr}\left[P\mathcal{U}^{*}\left(\ln\left(A\otimes B\right)\right)\mathcal{U}\right]\leq \operatorname{tr}\left[P\ln\left(A\circ B\right)\right]\\
&\leq \operatorname{tr}\left[P\ln\left(\left(m\right)^{\frac{M-A\circ B}{M-m}}\left(M\right)^{\frac{A\circ B-m}{M-m}}\right)\right]\\
&+L\left(m,M\right)\ln S\left(\frac{M}{m}\right)\\
&\times\left[\frac{M-\operatorname{tr}\left[P\left(A\circ B\right)\right]}{M-m}m^{-1}+\frac{\operatorname{tr}\left[P\left(A\circ B\right)\right]-m}{M-m}M^{-1}\right]\\
&\leq \operatorname{tr}\left[P\mathcal{U}^{*}\left(\ln\left(A\otimes B\right)\right)\mathcal{U}\right]+L\left(m,M\right)\ln S\left(\frac{M}{m}\right)\\
&\times\left[\frac{M-\operatorname{tr}\left[P\left(A\circ B\right)\right]}{M-m}m^{-1}+\frac{\operatorname{tr}\left[P\left(A\circ B\right)\right]-m}{M-m}M^{-1}\right]\\
&\leq \operatorname{tr}\left[P\mathcal{U}^{*}\left(\ln\left(A\otimes B\right)\right)\mathcal{U}\right]+L\left(\frac{M}{m},1\right)\ln S\left(\frac{M}{m}\right).
\end{aligned}
$$

If we take the exponential, then we get

$$
\begin{aligned}
&\exp\operatorname{tr}\left[P\ln\left(\left(m\right)^{\frac{M-A\circ B}{M-m}}\left(M\right)^{\frac{A\circ B-m}{M-m}}\right)\right]\\
&\leq \exp\operatorname{tr}\left[P\mathcal{U}^{*}\left(\ln\left(A\otimes B\right)\right)\mathcal{U}\right]\leq \exp\operatorname{tr}\left[P\ln\left(A\circ B\right)\right]\\
&\leq \exp\operatorname{tr}\left[P\ln\left(\left(m\right)^{\frac{M-A\circ B}{M-m}}\left(M\right)^{\frac{A\circ B-m}{M-m}}\right)\right]\\
&\times S\left(\frac{M}{m}\right)^{L(m,M)\left[\frac{M-\operatorname{tr}[P(A\circ B)]}{M-m}m^{-1}+\frac{\operatorname{tr}[P(A\circ B)]-m}{M-m}M^{-1}\right]}\\
&\leq \exp\operatorname{tr}\left[P\mathcal{U}^{*}\left(\ln\left(A\otimes B\right)\right)\mathcal{U}\right]\\
&\times S\left(\frac{M}{m}\right)^{L(m,M)\left[\frac{M-\operatorname{tr}[P(A\circ B)]}{M-m}m^{-1}+\frac{\operatorname{tr}[P(A\circ B)]-m}{M-m}M^{-1}\right]}\\
&\leq S\left(\frac{M}{m}\right)^{L\left(\frac{M}{m},1\right)}\exp\operatorname{tr}\left[P\mathcal{U}^{*}\left(\ln\left(A\otimes B\right)\right)\mathcal{U}\right]
\end{aligned}
$$

and the inequality (3.6) is proved. □

4. Connection to Oppenheim's Inequalities

In the finite dimensional case, if we consider the matrices $A=(a_{ij})$, $B=(b_{ij})\in\mathbb{M}_n(\mathbb{C})$, then $A\circ B$ has an associated matrix $A\circ B=(a_{ij}b_{ij})$ in $\mathbb{M}_n(\mathbb{C})$.

Recall Hadamard determinant inequality [13, p. 218] for $A \geq 0$

$$\det A \leq \det(A \circ 1) \ \ (= \prod_{i=1}^{n} a_{ii})$$

and Oppenheim's inequality [13, p. 242] for A, $B \geq 0$

$$\det A \det B \leq \det(A \circ B) \leq \det(A \circ 1) \det(B \circ 1) \ \ \left(= \prod_{i=1}^{n} a_{ii} b_{ii}\right).$$

We recall that for an operator $Q > 0$ we consider the normalized determinant defined by [7]

$$\Delta_x(Q) := \exp \langle \ln Qx, x \rangle$$

for $x \in H$, $\|x\| = 1$.

In the recent paper [10], S. Hiramatsu and Y. Seo obtained the following interesting Oppenheim's type inequalities

$$\frac{1}{S(h_1) S(h_2)} \leq \frac{\Delta_x(A \circ B)}{\Delta_x(A \circ 1) \Delta_x(1 \circ B)} \leq S(h_1 h_2) \tag{4.1}$$

for $x \in H$, $\|x\| = 1$, provided that $0 < m_1 \leq A \leq M_1$ and $0 < m_2 \leq B \leq M_2$.

We have the following similar inequalities for the trace determinant:

Theorem 8. *Assume that $0 < m_1 \leq A \leq M_1$ and $0 < m_2 \leq B \leq M_2$, then for $P \geq 0$ with $P \in \mathcal{B}_1(H)$ and $\operatorname{tr}(P) = 1$,*

$$\frac{1}{S(h_1) S(h_2)} \leq \frac{\Delta_P(A \circ B)}{\Delta_P(A \circ 1) \Delta_P(1 \circ B)} \leq S(h_1 h_2), \tag{4.2}$$

where $h_1 = \frac{M_1}{m_1} > 1$, $h_2 = \frac{M_2}{m_2} > 1$.

Proof. By the properties of the tensorial product, we have that

$$A \otimes B = (A \otimes 1)(1 \otimes B)$$

where $A \otimes 1$ and $1 \otimes B$ are commutative operators.

Therefore

$$\ln(A \otimes B) = \ln[(A \otimes 1)(1 \otimes B)] = \ln(A \otimes 1) + \ln(1 \otimes B)$$

and

$$\begin{aligned} \mathcal{U}^*(\ln(A \otimes B))\mathcal{U} &= \mathcal{U}^*[\ln(A \otimes 1) + \ln(1 \otimes B)]\mathcal{U} \\ &= \mathcal{U}^*(\ln(A \otimes 1))\mathcal{U} + \mathcal{U}^*(\ln(1 \otimes B))\mathcal{U}. \end{aligned}$$

Using Jensen's operator inequality for the operator concave function ln, we also have

$$\mathcal{U}^*(\ln(A \otimes 1))\mathcal{U} \leq \ln(\mathcal{U}^*(A \otimes 1)\mathcal{U}) = \ln(A \circ 1)$$

and

$$\mathcal{U}^*(\ln(1 \otimes B))\mathcal{U} \leq \ln(\mathcal{U}^*((1 \otimes B))\mathcal{U}) = \ln(1 \circ B).$$

These imply for $P \geq 0$ with $P \in \mathcal{B}_1(H)$ and $\operatorname{tr}(P) = 1$ that

$$P^{1/2}\mathcal{U}^*(\ln(A \otimes B))\mathcal{U}P^{1/2} \leq P^{1/2} \ln(A \circ 1) P^{1/2} + P^{1/2} \ln(1 \circ B) P^{1/2}$$

and by taking the trace

$$\operatorname{tr}[P\mathcal{U}^*(\ln(A \otimes B))\mathcal{U}] \leq \operatorname{tr}[P \ln(A \circ 1)] + \operatorname{tr}[P \ln(1 \circ B)].$$

If we take the exponential we then obtain

$$\begin{aligned}(4.3)\qquad \exp\operatorname{tr}\left[P\mathcal{U}^*\left(\ln\left(A\otimes B\right)\right)\mathcal{U}\right] &\le \exp\left(\operatorname{tr}\left[P\ln\left(A\circ 1\right)\right]+\operatorname{tr}\left[P\ln\left(1\circ B\right)\right]\right)\\ &=\exp\left(\operatorname{tr}\left[P\ln\left(A\circ 1\right)\right]\right)\exp\operatorname{tr}\left[P\ln\left(1\circ B\right)\right]\\ &=\Delta_P\left(A\circ 1\right)\Delta_P\left(1\circ B\right).\end{aligned}$$

Since by (2.7) we have for $M=M_1M_2$ and $m=m_1m_2$, that

$$\Delta_P\left(A\circ B\right)\le S\left(\frac{M}{m}\right)\exp\operatorname{tr}\left[P\mathcal{U}^*\left(\ln\left(A\otimes B\right)\right)\mathcal{U}\right], \tag{4.4}$$

then by (4.3) and (4.4) we get

$$\Delta_P\left(A\circ B\right)\le S\left(\frac{M}{m}\right)\Delta_P\left(A\circ 1\right)\Delta_P\left(1\circ B\right),$$

which proves the second part of (4.2).

From (2.6) we have, see also [10]

$$\ln\Phi\left(V\right)\le\Phi\left(\ln V\right)+\ln S\left(\frac{M}{m}\right)$$

provided that $0<m\le V\le M$.

Now, if we take in this inequality $0<m_1\le V=A\otimes 1\le M_1$, then we get for $\Phi\left(V\right)=\mathcal{U}^*\left(A\otimes 1\right)\mathcal{U}=A\circ 1$ that

$$\ln\left(A\circ 1\right)\le\ln S\left(\frac{M_1}{m_1}\right)+\mathcal{U}^*\left(\ln\left(A\otimes 1\right)\right)\mathcal{U}$$

while for $0<m_2\le V=1\otimes B\le M_2$

$$\ln\left(1\circ B\right)\le\ln S\left(\frac{M_2}{m_2}\right)+\mathcal{U}^*\left(\ln\left(1\otimes B\right)\right)\mathcal{U},$$

which gives, by addition, that

$$\begin{aligned}&\ln\left(A\circ 1\right)+\ln\left(1\circ B\right)-\ln S\left(\frac{M_1}{m_1}\right)-\ln S\left(\frac{M_2}{m_2}\right)\\ &\le\mathcal{U}^*\left(\ln\left(A\otimes 1\right)\right)\mathcal{U}+\mathcal{U}^*\left(\ln\left(1\otimes B\right)\right)\mathcal{U}=\mathcal{U}^*\left(\ln\left(A\otimes B\right)\right)\mathcal{U}.\end{aligned}$$

These imply for $P\ge 0$ with $P\in\mathcal{B}_1\left(H\right)$ and $\operatorname{tr}\left(P\right)=1$ that

$$\begin{aligned}&P^{1/2}\ln\left(A\circ 1\right)P^{1/2}+P^{1/2}\ln\left(1\circ B\right)P^{1/2}-\ln S\left(\frac{M_1}{m_1}\right)P-\ln S\left(\frac{M_2}{m_2}\right)P\\ &\le P^{1/2}\mathcal{U}^*\left(\ln\left(A\otimes B\right)\right)\mathcal{U}P^{1/2}\end{aligned}$$

and by taking the trace we get

$$\begin{aligned}&\operatorname{tr}\left[P\ln\left(A\circ 1\right)\right]+\operatorname{tr}\left[P\ln\left(1\circ B\right)\right]-\ln S\left(\frac{M_1}{m_1}\right)-\ln S\left(\frac{M_2}{m_2}\right)\\ &\le\operatorname{tr}\left[P\mathcal{U}^*\left(\ln\left(A\otimes B\right)\right)\mathcal{U}\right].\end{aligned}$$

Finally, by taking the exponential we derive

$$\frac{\exp\operatorname{tr}\left[P\ln\left(A\circ 1\right)\right]\exp\operatorname{tr}\left[P\ln\left(1\circ B\right)\right]}{S\left(\frac{M_1}{m_1}\right)S\left(\frac{M_2}{m_2}\right)}\le\exp\operatorname{tr}\left[P\mathcal{U}^*\left(\ln\left(A\otimes B\right)\right)\mathcal{U}\right]. \tag{4.5}$$

Since by (2.7)

$$\exp\operatorname{tr}\left[P\mathcal{U}^*\left(\ln\left(A\otimes B\right)\right)\mathcal{U}\right]\le\Delta_P\left(A\circ B\right), \tag{4.6}$$

hence by (4.5) and (4.6) we derive the first inequality in (4.2). □

Remark 1. *In* [10] *the authors showed that*

$$S(h_1)\,S(h_2) \leq S(h_1 h_2)$$

for all $h_1, h_2 > 1$. *Therefore, by (4.2) we get the more symetrical result*

$$S^{-1}(h_1 h_2) \leq \frac{\Delta_P(A \circ B)}{\Delta_P(A \circ 1)\,\Delta_P(1 \circ B)} \leq S(h_1 h_2)\,, \tag{4.7}$$

where $h_1 = \frac{M_1}{m_1} > 1$, $h_2 = \frac{M_2}{m_2} > 1$.

References

[1] S. S. Dragomir, Hermite–Hadamard's type inequalities for operator convex functions, *Applied Mathematics and Computation*, **218** (2011), Issue 3, pp. 766–772.

[2] S. S. Dragomir, Some Grüss type inequalities for trace of operators in Hilbert spaces, *Oper. Matrices*, **10** (2016), no. 4, 923–943. Preprint *RGMIA Res. Rep. Coll.* **17** (2014), Art. 114. [https://rgmia.org/papers/v17/v17a114.pdf].

[3] S. S. Dragomir, Jensen's type trace inequalities for convex functions of selfadjoint operators in Hilbert spaces, *Facta Univ. Ser. Math. Inform.*, **31** (2016), no. 5, 981–998. Preprint *RGMIA Res. Rep. Coll.*, **17** (2014), Art. 116. [https://rgmia.org/papers/v17/v17a116.pdf].

[4] S. S. Dragomir, Trace inequalities for operators in Hilbert spaces: a survey of recent results, *Aust. J. Math. Anal. Appl.* Vol. **19** (2022), No. 1, Art. 1, 202 pp. [Online https://ajmaa.org/searchroot/files/pdf/v19n1/v19i1p1.pdf].

[5] S. S. Dragomir, Some properties of trace class V-determinant of positive operators in Hilbert spaces, Preprint RGMIA Res. Rep. Coll. 25

[6] B. Fuglede and R. V. Kadison, Determinant theory in finite factors, *Ann. of Math.* (2) **55** (1952), 520–530.

[7] J. I. Fujii and Y. Seo, Determinant for positive operators, *Sci. Math.,* **1** (1998), 153–156.

[8] J. I. Fujii, S. Izumino and Y. Seo, Determinant for positive operators and Specht's Theorem, *Sci. Math.*, **1** (1998), 307–310.

[9] S. Furuichi, Refined Young inequalities with Specht's ratio, *J. Egyptian Math. Soc.* **20** (2012), 46–49.

[10] S. Hiramatsu and Y. Seo, Determinant for positive operators and Oppenheim's inequality, *J. Math. Inequal.*, Volume 15 (2021), Number 4, 1637–1645.

[11] W. Specht, Zer Theorie der elementaren Mittel, *Math. Z.* **74** (1960), pp. 91-98.

[12] M. Tominaga, Specht's ratio in the Young inequality, *Sci. Math. Japon.*, **55** (2002), 583-588.

[13] F. Zhang, *Matrix Theory - Basic Results and Techniques, Second edition*, Universitext, Springer, 2011.

[1]Mathematics, College of Engineering & Science, Victoria University, PO Box 14428, Melbourne City, MC 8001, Australia.

E-mail address: sever.dragomir@vu.edu.au

URL: http://rgmia.org/dragomir

[2]DST-NRF Centre of Excellence in the Mathematical, and Statistical Sciences, School of Computer Science, & Applied Mathematics, University of the Witwatersrand,, Private Bag 3, Johannesburg 2050, South Africa