# Approximation of Time Separating Stochastic Processes by Neural Networks

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**Abstract**

Here we study the univariate quantitative approximation of time separating stochastic process over a compact interval or all the real line by quasi-interpolation neural network operators. We perform also the related fractional approximation. These approximations are derived by establishing Jackson type inequalities involving the modulus of continuity of the engaged stochastic function or its high order derivative or fractional derivatives. Our operators are defined by using a density function induced by a general sigmoid function. The approximations are pointwise and with respect to the uniform norm. The feed-forward neural networks are with one hidden layer. We finish with a lot of interesting applications.

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## 1 Introduction

The first author in [1] and [2], see Chapters 2-5, was the first to establish neural network approximation to continuous functions with rates by very specifically defined neural network operators of Cardaliaguet-Euvrard and "Squashing" types, by employing the modulus of continuity of the engaged function or its high order derivative, and producing very tight Jackson type inequalities. He treats there both the univariate and multivariate cases. The defining these operators "bell-shaped" and "squashing" functions are assumed to be of compact suport.



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Also in [2] he gives the $N$th order asymptotic expansion for the error of weak approximation of these two operators to a special natural class of smooth functions, see Chapters 4-5 there. The first author inspired by [15], continued his studies on neural networks approximation by introducing and using the proper quasi interpolation operators of sigmoidal and hyperbolic tangent type which resulted into [3]-[7], by treating both the univariate and multivariate cases. He did also the corresponding fractional case [8]. In this article we are also inspired by the related works [16], [17]. The authors here use general sigmoid function based neural network quantitative approximations to continuous functions over compact intervals of the real line or over the whole $\mathbb{R}$ with values to $\mathbb{R}$. All convergences here are with rates expressed via the modulus of continuity of the involved function or its high order derivative, or fractional derivatives and given by very tight Jackson type inequalities. More precisely, here we perform quantitative approximations of time separating stochastic processes by neural networks. We give plenty of varied and interesting applications. Specific motivations came by:

1. Stationary Gaussian processes with an explicit representation such as

$$X_t = \cos(\alpha t)\,\xi_1 + \sin(\alpha t)\,\xi_2, \alpha \in \mathbb{R},$$

   where $\xi_1, \xi_2$ are independent random variables with the standard normal distribution, see [19],

2. by the "Fourier model" of a stationary process, see [20].

Feed-forward neural networks (FNNs) with one hidden layer, the only type of networks we deal with in this article, are mathematically expressed as

$$N_n(x) = \sum_{j=0}^{n} c_j \sigma\left(\langle a_j \cdot x \rangle + b_j\right), \quad x \in \mathbb{R}^s, s \in \mathbb{N},$$

where for $0 \leq j \leq n$, $b_j \in \mathbb{R}$ are the thresholds, $a_j \in \mathbb{R}^s$ are the connection weights, $c_j \in X$ are the coefficients, $\langle a_j \cdot x \rangle$ is the inner product of $a_j$ and $x$, and $\sigma$ is the activation function of the network. In many fundamental neural network models, the activation function is derived from various specific sigmoid functions. Here we work for a general sigmoid function. About neural networks in general read [18], [21],[23]. See also [9] for a complete study of real valued approximation by neural network operators.

## 2 Background

Here we follow [14].

### 2.1 Basics on Neural Network

Let $h : \mathbb{R} \to [-1, 1]$ be a general sigmoid function, such that it is strictly increasing, $h(0) = 0, h(-x) = -h(x)$ for every $x \in \mathbb{R}$, $h(+\infty) = 1, h(-\infty) = -1$. Also $h$ is strictly convex over $(-\infty, 0]$ and strictly concave over $[0, +\infty)$, with $h^{(2)} \in C(\mathbb{R})$.
Some examples of related sigmoid functions follow: $\frac{1}{1+e^{-x}}$; $\tanh x$; $\frac{2}{\pi}\arctan\left(\frac{\pi}{2}x\right)$; $\frac{x}{\sqrt[2m]{1+x^{2m}}}, m \in$

$\mathbb{N}$; $\frac{2}{\pi}gd(x)$; $\frac{x}{\left(1+|x|^{\lambda}\right)^{\frac{1}{\lambda}}}$, $\lambda$ is odd ; $erf\left(\frac{\sqrt{\pi}}{2}x\right)$ ; $\frac{1}{1+e^{-\mu x}}$; $\tanh(\mu x)$, $\mu > 0$ for all $x \in \mathbb{R}$,
We consider the activation function

$$\psi(x) := \frac{1}{4}\left(h(x+1) - h(x-1)\right), x \in \mathbb{R}, , \tag{1}$$

As in [13], p.285, we get that

$$\psi(-x) = \psi(x), \text{ for every } x \in \mathbb{R}.$$

Thus $\psi$ is an even function.
Since $x+1 > x-1$, then $h(x+1) > h(x-1)$, and $\psi(x) > 0$, for all $x \in \mathbb{R}$.
We see that

$$\psi(0) = \frac{h(1)}{2}. \tag{2}$$

Let $x > 1$, we have that

$$\psi'(x) = \frac{1}{4}\left(h'(x+1) - h'(x-1)\right) < 0, \tag{3}$$

by $h'$ be strictly decreasing on $[0, +\infty)$.
Let now $0 < x < 1$, then $1 - x > 0$ and $0 < 1 - x < 1 + x$. It holds $h'(x-1) = h'(1-x) > h'(x+1)$, so that again $\psi'(x) < 0$. Consequently $\psi$ is strictly decreasing on $(0, +\infty)$. Clearly $\psi$ is strictly increasing on $(-\infty, 0)$ and $\psi'(0) = 0$. See that

$$\begin{gathered} \lim_{x \to +\infty} \psi(x) = \frac{1}{4}\left(h(+\infty) - h(+\infty)\right) = 0, \\ \text{and} \\ \lim_{x \to -\infty} \psi(x) = \frac{1}{4}\left(h(-\infty) - h(-\infty)\right) = 0. \end{gathered} \tag{4}$$

That is the $x$-axis is the horizontal asymptote on $\psi$ .
Conclusion, $\psi$ is a bell symmetric function with maximum $\psi(0) = \frac{h(1)}{2}$ .
We need

**Theorem 1.** *([14]) We have that*

$$\sum_{i=-\infty}^{+\infty} \psi(x-i) = 1, \textit{ for every } x \in \mathbb{R}. \tag{5}$$

**Theorem 2.** *([14]) It holds*

$$\int_{-\infty}^{+\infty} \psi(x)dx = 1. \tag{6}$$

Thus, $\psi(x)$ is a density function on $\mathbb{R}$.
We give

**Theorem 3.** *([14]) Let $0 < \alpha < 1$, and $n \in \mathbb{N}$ with $n^{1-\alpha} > 2$. It holds*

$$\sum_{\begin{cases} k = -\infty \\ : |nx-k| \geq n^{1-\alpha} \end{cases}}^{\infty} \psi(nx-k) < \frac{\left(1 - h\left(n^{1-\alpha} - 2\right)\right)}{2}. \tag{7}$$

*Notice that*

$$\lim_{n \to +\infty} \frac{\left(1 - h\left(n^{1-\alpha} - 2\right)\right)}{2} = 0.$$

Denote by $\lfloor \cdot \rfloor$ the integral part of the number and by $\lceil \cdot \rceil$ the ceiling of the number. We further give

**Theorem 4.** *([14])Let $x \in [a,b] \subset \mathbb{R}$ and $n \in \mathbb{N}$ so that $\lceil na \rceil \leq \lfloor nb \rfloor$. It holds*

$$\frac{1}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \psi(nx-k)} < \frac{1}{\psi(1)}, \quad \forall\, x \in [a,b]. \tag{8}$$

**Remark 5.** *We have that*

$$\lim_{n \to \infty} \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \psi(nx-k) \neq 1, \tag{9}$$

*for at least some $x \in [a,b]$. See [13], p. 290, same reasoning.*

**Note 6.** *For large enough $n$ we always obtain $\lceil na \rceil \leq \lfloor nb \rfloor$. Also $a \leq \frac{k}{n} \leq b$, if and only if $\lceil na \rceil \leq k \leq \lfloor nb \rfloor$. In general it holds (by (5))*

$$\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \psi(nx-k) \leq 1. \tag{10}$$

Let $(X, \|\cdot\|)$ be a Banach space.

**Definition 7.** *([14]) Let $f \in C([a,b], X)$ and $n \in \mathbb{N} : \lceil na \rceil \leq \lfloor nb \rfloor$. We introduce and define the $X$-valued linear neural network operators*

$$A_n(f,x) := \frac{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} f\left(\frac{k}{n}\right) \psi(nx-k)}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \psi(nx-k)}, \quad x \in [a,b]. \tag{11}$$

Clearly here $A_n(f,x) \in C([a,b], X)$. For convenience we use the same $A_n$ for real valued function when needed. We mention here the pointwise and uniform convergence of $A_n(f,x)$ to $f(x)$ with rates. For convenience also we call

$$A_n^*(f,x) := \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} f\left(\frac{k}{n}\right) \psi(nx-k), \tag{12}$$

(similarly $A_n^*$ can be defined for real valued function) that is

$$A_n(f,x) = \frac{A_n^*(f,x)}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \psi(nx-k)}. \tag{13}$$

So that

$$\begin{aligned} A_n(f,x) - f(x) &= \frac{A_n^*(f,x)}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \psi(nx-k)} - f(x) \\ &= \frac{A_n^*(f,x) - f(x)\left(\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \psi(nx-k)\right)}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \psi(nx-k)}. \end{aligned} \tag{14}$$

Consequently we derive

$$\|A_n(f,x) - f(x)\| \leq \frac{1}{\psi(1)} \left\| A_n^*(f,x) - f(x)\left(\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \psi(nx-k)\right) \right\|. \tag{15}$$

That is

$$\left\| A_n(f,x) - f(x) \right\| \leq \frac{1}{\psi(1)} \left\| \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \left( f\left(\frac{k}{n}\right) - f(x) \right) \psi(nx-k) \right\|. \tag{16}$$

We will estimate the right hand side of (16).

For that we need, for $f \in C([a,b],X)$ the first modulus of continuity

$$\omega_1(f,\delta)_{[a,b]} := \omega_1(f,\delta) := \sup_{\substack{x,y \in [a,b] \\ |x-y| \leq \delta}} \|f(x) - f(y)\|, \quad \delta > 0. \tag{17}$$

Similarly, it is defined $\omega_1$ for $f \in C_{uB}(\mathbb{R},X)$ (uniformly continuous and bounded functions from $\mathbb{R}$ into $X$), for $f \in C_B(\mathbb{R},X)$ (continuous and bounded $X$-valued) and for $f \in C_u(\mathbb{R},X)$ (uniformly continuous). The fact $f \in C([a,b],X)$ or $f \in C_u(\mathbb{R},X)$, is equivalent to $\lim\limits_{\delta \to 0} \omega_1(f,\delta) = 0$, see [11].

**Definition 8.** *([14]) When $f \in C_{uB}(\mathbb{R},X)$, or $f \in C_B(\mathbb{R},X)$, we define*

$$\overline{A}_n(f,x) := \sum_{k=-\infty}^{\infty} f\left(\frac{k}{n}\right) \psi(nx-k), \quad n \in \mathbb{N}, \ x \in \mathbb{R}, \tag{18}$$

*the $X$-valued quasi-interpolation neural network operator.*

**Remark 9.** *([14]) We have that the series*

$$\sum_{k=-\infty}^{+\infty} f\left(\frac{k}{n}\right) \psi(nx-k)$$

*is absolutely convergent in $X$, hence it is convergent in $X$ and $\overline{A}_n(f,x) \in X$.*

We denote by $\|f\|_\infty := \sup\limits_{x \in [a,b]} \|f(x)\|$, for $f \in C([a,b],X)$, similarly is defined for $f \in C_B(\mathbb{R},X)$. We mention a series of $X$-valued neural network approximations to a function given with rates. We first give

**Theorem 10.** *([14]). Let $f \in C([a,b],X)$, $0 < \alpha < 1$, $n \in \mathbb{N} : n^{1-\alpha} > 2$, $x \in [a,b]$. Then*

*i)*

$$\|A_n(f,x) - f(x)\| \leq \frac{1}{\psi(1)} \left[ \omega_1\left(f, \frac{1}{n^\alpha}\right) + \left(1 - h\left(n^{1-\alpha} - 2\right)\right) \|f\|_\infty \right] =: \rho, \tag{19}$$

*and*

*ii)*

$$\|A_n(f) - f\|_\infty \leq \rho. \tag{20}$$

*We notice $\lim\limits_{n \to \infty} A_n(f) = f$, pointwise and uniformly.*
*The speed of convergence is* $\max\left(\frac{1}{n^\alpha}, \left(1 - h\left(n^{1-\alpha} - 2\right)\right)\right)$.

Next we give

**Theorem 11.** *([14]). Let $f \in C_B(\mathbb{R}, X)$, $0 < \alpha < 1$, $n \in \mathbb{N} : n^{1-\alpha} > 2$, $x \in \mathbb{R}$. Then*

*i)*

$$\left\| \overline{A}_n(f,x) - f(x) \right\| \leq \omega_1\left(f, \frac{1}{n^\alpha}\right) + \left(1 - h\left(n^{1-\alpha} - 2\right)\right) \|f\|_\infty =: \mu, \tag{21}$$

*and*

*ii)*

$$\left\| \overline{A}_n(f) - f \right\|_\infty \leq \mu. \tag{22}$$

*For $f \in C_{uB}(\mathbb{R}, X)$ we get $\lim_{n\to\infty} \overline{A}_n(f) = f$, pointwise and uniformly. The speed of convergence is* $\max\left(\frac{1}{n^\alpha}, \left(1 - h\left(n^{1-\alpha} - 2\right)\right)\right)$.

In the next we discuss high order neural network $X$-valued approximation by using the smoothness of $f$. The $X$-valued derivatives are as the numerical ones, see ([24]).

**Theorem 12.** *([14])Let $f \in C^N([a,b], X)$, $n, N \in \mathbb{N}$, $0 < \alpha < 1$, $x \in [a,b]$ and $n^{1-\alpha} > 2$. Then*

*i)*

$$\|A_n(f,x) - f(x)\| \leq \frac{1}{\psi(1)} \left\{ \sum_{j=1}^{N} \frac{\left\|f^{(j)}(x)\right\|}{j!} \left[ \frac{1}{n^{\alpha j}} + \frac{\left(1 - h\left(n^{1-\alpha} - 2\right)\right)}{2} (b-a)^j \right] + \right. \tag{23}$$

$$\left. \left[ \omega_1\left(f^{(N)}, \frac{1}{n^\alpha}\right) \frac{1}{n^{\alpha N} N!} + \frac{\left(1 - h\left(n^{1-\alpha} - 2\right)\right) \left\|f^{(N)}\right\|_\infty (b-a)^N}{N!} \right] \right\}.$$

*ii) Assume further $f^{(j)}(x_0) = 0$, $j = 1, ..., N$, for some $x_0 \in [a,b]$, it holds*

$$\|A_n(f,x_0) - f(x_0)\| \leq \frac{1}{\psi(1)}$$

$$\left\{ \omega_1\left(f^{(N)}, \frac{1}{n^\alpha}\right) \frac{1}{n^{\alpha N} N!} + \frac{\left(1 - h\left(n^{1-\alpha} - 2\right)\right) \left\|f^{(N)}\right\|_\infty (b-a)^N}{N!} \right\}, \tag{24}$$

*and*

*iii)*

$$\|A_n(f) - f\|_\infty \leq \frac{1}{\psi(1)} \left\{ \sum_{j=1}^{N} \frac{\left\|f^{(j)}\right\|_\infty}{j!} \left[ \frac{1}{n^{\alpha j}} + \frac{\left(1 - h\left(n^{1-\alpha} - 2\right)\right)}{2} (b-a)^j \right] + \right.$$

$$\left. \left[ \omega_1\left(f^{(N)}, \frac{1}{n^\alpha}\right) \frac{1}{n^{\alpha N} N!} + \frac{\left(1 - h\left(n^{1-\alpha} - 2\right)\right) \left\|f^{(N)}\right\|_\infty (b-a)^N}{N!} \right] \right\}. \tag{25}$$

*Again we obtain $\lim_{n\to\infty} A_n(f) = f$, pointwise and uniformly.*

All integrals from now on are of Bochner type [22].
We need

**Definition 13.** *([12]) Let $[a,b] \subset \mathbb{R}$, $X$ be a Banach space, $\alpha > 0$; $m = \lceil \alpha \rceil \in \mathbb{N}$, ($\lceil \cdot \rceil$ is the ceiling of the number), $f : [a,b] \to X$. We assume that $f^{(m)} \in L_1([a,b], X)$. We call the Caputo-Bochner left fractional derivative of order $\alpha$:*

$$\left( D_{*a}^{\alpha} f \right)(x) := \frac{1}{\Gamma(m-\alpha)} \int_a^x (x-t)^{m-\alpha-1} f^{(m)}(t)\, dt, \quad \forall\, x \in [a,b]. \tag{26}$$

*If $\alpha \in \mathbb{N}$, we set $D_{*a}^{\alpha} f := f^{(m)}$ the ordinary $X$-valued derivative (defined similar to numerical one, see [24], p. 83), and also set $D_{*a}^{0} f := f$.*

By [12], $\left( D_{*a}^{\alpha} f \right)(x)$ exists almost everywhere in $x \in [a,b]$ and $D_{*a}^{\alpha} f \in L_1([a,b], X)$.

If $\left\| f^{(m)} \right\|_{L_\infty([a,b],X)} < \infty$, then by [12], $D_{*a}^{\alpha} f \in C([a,b], X)$, hence $\left\| D_{*a}^{\alpha} f \right\| \in C([a,b])$.

**Definition 14.** *([10]) Let $[a,b] \subset \mathbb{R}$, $X$ be a Banach space, $\alpha > 0$, $m := \lceil \alpha \rceil$. We assume that $f^{(m)} \in L_1([a,b], X)$, where $f : [a,b] \to X$. We call the Caputo-Bochner right fractional derivative of order $\alpha$:*

$$\left( D_{b-}^{\alpha} f \right)(x) := \frac{(-1)^m}{\Gamma(m-\alpha)} \int_x^b (z-x)^{m-\alpha-1} f^{(m)}(z)\, dz, \quad \forall\, x \in [a,b]. \tag{27}$$

*We observe that $\left( D_{b-}^{m} f \right)(x) = (-1)^m f^{(m)}(x)$, for $m \in \mathbb{N}$, and $\left( D_{b-}^{0} f \right)(x) = f(x)$.*

By [10], $\left( D_{b-}^{\alpha} f \right)(x)$ exists almost everywhere on $[a,b]$ and $\left( D_{b-}^{\alpha} f \right) \in L_1([a,b], X)$. If $\left\| f^{(m)} \right\|_{L_\infty([a,b],X)} < \infty$, and $\alpha \notin \mathbb{N}$, by [10], $D_{b-}^{\alpha} f \in C([a,b], X)$, hence $\left\| D_{b-}^{\alpha} f \right\| \in C([a,b])$.
In the next $\omega_1(f, \delta)_{[a,b]}$ refers to a modulus of continuity. $\omega_1$ defined over $[a,b]$.
We mention the following $X-$valued fractional approximation result by neural networks.

**Theorem 15.** *([14]). Let $\alpha > 0$, $N = \lceil \alpha \rceil$, $\alpha \notin \mathbb{N}$, $f \in C^N([a,b], X)$, $0 < \beta < 1$, $x \in [a,b]$, $n \in \mathbb{N} : n^{1-\beta} > 2$. Then*

*i)*

$$\left\| A_n(f,x) - \sum_{j=1}^{N-1} \frac{f^{(j)}(x)}{j!} A_n\left( (\cdot - x)^j \right)(x) - f(x) \right\| \leq$$

$$\frac{(\psi(1))^{-1}}{\Gamma(\alpha+1)} \left\{ \frac{\left( \omega_1\left( D_{x-}^{\alpha} f, \frac{1}{n^\beta} \right)_{[a,x]} + \omega_1\left( D_{*x}^{\alpha} f, \frac{1}{n^\beta} \right)_{[x,b]} \right)}{n^{\alpha\beta}} + \right.$$

$$\left. \left( \frac{1 - h\left( n^{1-\beta} - 2 \right)}{2} \right) \left( \left\| D_{x-}^{\alpha} f \right\|_{\infty,[a,x]} (x-a)^\alpha + \left\| D_{*x}^{\alpha} f \right\|_{\infty,[x,b]} (b-x)^\alpha \right) \right\}, \tag{28}$$

*ii) if $f^{(j)}(x) = 0$, for $j = 1, ..., N-1$, we have*

$$\left\| A_n(f,x) - f(x) \right\| \leq \frac{(\psi(1))^{-1}}{\Gamma(\alpha+1)}$$

$$\left\{ \frac{\left( \omega_1\left( D_{x-}^{\alpha} f, \frac{1}{n^\beta} \right)_{[a,x]} + \omega_1\left( D_{*x}^{\alpha} f, \frac{1}{n^\beta} \right)_{[x,b]} \right)}{n^{\alpha\beta}} + \right.$$

$$\left. \left( \frac{1 - h\left( n^{1-\beta} - 2 \right)}{2} \right) \left( \left\| D_{x-}^{\alpha} f \right\|_{\infty,[a,x]} (x-a)^\alpha + \left\| D_{*x}^{\alpha} f \right\|_{\infty,[x,b]} (b-x)^\alpha \right) \right\}, \tag{29}$$

*iii)*
$$\|A_n(f,x) - f(x)\| \le (\psi(1))^{-1}$$
$$\left\{\sum_{j=1}^{N-1} \frac{\|f^{(j)}(x)\|}{j!}\left\{\frac{1}{n^{\beta j}} + (b-a)^j\left(\frac{1-h\left(n^{1-\beta}-2\right)}{2}\right)\right\} +\right.$$
$$\frac{1}{\Gamma(\alpha+1)}\left\{\frac{\left(\omega_1\left(D_{x-}^{\alpha}f, \frac{1}{n^{\beta}}\right)_{[a,x]} + \omega_1\left(D_{*x}^{\alpha}f, \frac{1}{n^{\beta}}\right)_{[x,b]}\right)}{n^{\alpha\beta}} +\right.$$
$$\left.\left.\left(\frac{1-h\left(n^{1-\beta}-2\right)}{2}\right)\left(\left\|D_{x-}^{\alpha}f\right\|_{\infty,[a,x]}(x-a)^{\alpha} + \left\|D_{*x}^{\alpha}f\right\|_{\infty,[x,b]}(b-x)^{\alpha}\right)\right\}\right\}, \tag{30}$$
$\forall\ x \in [a,b]$,

*and*

*iv)*
$$\|A_nf - f\|_{\infty} \le (\psi(1))^{-1}$$
$$\left\{\sum_{j=1}^{N-1} \frac{\|f^{(j)}\|_{\infty}}{j!}\left\{\frac{1}{n^{\beta j}} + (b-a)^j\left(\frac{1-h\left(n^{1-\beta}-2\right)}{2}\right)\right\} +\right.$$
$$\frac{1}{\Gamma(\alpha+1)}\left\{\frac{\left(\sup_{x\in[a,b]}\omega_1\left(D_{x-}^{\alpha}f, \frac{1}{n^{\beta}}\right)_{[a,x]} + \sup_{x\in[a,b]}\omega_1\left(D_{*x}^{\alpha}f, \frac{1}{n^{\beta}}\right)_{[x,b]}\right)}{n^{\alpha\beta}} +\right.$$
$$\left.\left.\left(\frac{1-h\left(n^{1-\beta}-2\right)}{2}\right)(b-a)^{\alpha}\left(\sup_{x\in[a,b]}\left\|D_{x-}^{\alpha}f\right\|_{\infty,[a,x]} + \sup_{x\in[a,b]}\left\|D_{*x}^{\alpha}f\right\|_{\infty,[x,b]}\right)\right\}\right\}. \tag{31}$$
*Above, when $N=1$ the sum $\sum_{j=1}^{N-1}\cdot = 0$.*

*As we see here we obtain $X$-valued fractionally type pointwise and uniform convergence with rates of $A_n \to I$ the unit operator, as $n \to \infty$.*

Next we apply Theorem 15 for $N = 1$.

**Corollary 16.** *([14]) Let $0 < \alpha, \beta < 1$, $f \in C^1([a,b], X)$, $x \in [a,b]$, $n \in \mathbb{N} : n^{1-\beta} > 2$. Then*

*i)*
$$\|A_n(f,x) - f(x)\| \le$$
$$\frac{(\psi(1))^{-1}}{\Gamma(\alpha+1)}\left\{\frac{\left(\omega_1\left(D_{x-}^{\alpha}f, \frac{1}{n^{\beta}}\right)_{[a,x]} + \omega_1\left(D_{*x}^{\alpha}f, \frac{1}{n^{\beta}}\right)_{[x,b]}\right)}{n^{\alpha\beta}} +\right.$$
$$\left.\left(\frac{1-h\left(n^{1-\beta}-2\right)}{2}\right)\left(\left\|D_{x-}^{\alpha}f\right\|_{\infty,[a,x]}(x-a)^{\alpha} + \left\|D_{*x}^{\alpha}f\right\|_{\infty,[x,b]}(b-x)^{\alpha}\right)\right\}, \tag{32}$$

*and*

*ii)*

$$\|A_n f - f\|_\infty \leq \frac{(\psi(1))^{-1}}{\Gamma(\alpha+1)}$$

$$\left\{ \frac{\left( \sup_{x\in[a,b]} \omega_1 \left(D_{x-}^{\alpha} f, \frac{1}{n^\beta}\right)_{[a,x]} + \sup_{x\in[a,b]} \omega_1 \left(D_{*x}^{\alpha} f, \frac{1}{n^\beta}\right)_{[x,b]} \right)}{n^{\alpha\beta}} + \right.$$

$$\left. \left( \frac{1 - h\left(n^{1-\beta} - 2\right)}{2} \right) (b-a)^\alpha \left( \sup_{x\in[a,b]} \left\|D_{x-}^{\alpha} f\right\|_{\infty,[a,x]} + \sup_{x\in[a,b]} \left\|D_{*x}^{\alpha} f\right\|_{\infty,[x,b]} \right) \right\}. \tag{33}$$

When $\alpha = \frac{1}{2}$ we derive

**Corollary 17.** *([14]) Let $0 < \beta < 1$, $f \in C^1([a,b], X)$, $x \in [a,b]$, $n \in \mathbb{N} : n^{1-\beta} > 2$. Then*

*i)*

$$\|A_n(f,x) - f(x)\| \leq$$

$$\frac{2(\psi(1))^{-1}}{\sqrt{\pi}} \left\{ \frac{\left( \omega_1 \left(D_{x-}^{\frac{1}{2}} f, \frac{1}{n^\beta}\right)_{[a,x]} + \omega_1 \left(D_{*x}^{\frac{1}{2}} f, \frac{1}{n^\beta}\right)_{[x,b]} \right)}{n^{\frac{\beta}{2}}} + \right.$$

$$\left. \left( \frac{1 - h\left(n^{1-\beta} - 2\right)}{2} \right) \left( \left\|D_{x-}^{\frac{1}{2}} f\right\|_{\infty,[a,x]} \sqrt{(x-a)} + \left\|D_{*x}^{\frac{1}{2}} f\right\|_{\infty,[x,b]} \sqrt{(b-x)} \right) \right\}, \tag{34}$$

*and*

*ii)*

$$\|A_n f - f\|_\infty \leq \frac{2(\psi(1))^{-1}}{\sqrt{\pi}}$$

$$\left\{ \frac{\left( \sup_{x\in[a,b]} \omega_1 \left(D_{x-}^{\frac{1}{2}} f, \frac{1}{n^\beta}\right)_{[a,x]} + \sup_{x\in[a,b]} \omega_1 \left(D_{*x}^{\frac{1}{2}} f, \frac{1}{n^\beta}\right)_{[x,b]} \right)}{n^{\frac{\beta}{2}}} + \right.$$

$$\left. \left( \frac{1 - h\left(n^{1-\beta} - 2\right)}{2} \right) \sqrt{(b-a)} \left( \sup_{x\in[a,b]} \left\|D_{x-}^{\frac{1}{2}} f\right\|_{\infty,[a,x]} + \sup_{x\in[a,b]} \left\|D_{*x}^{\frac{1}{2}} f\right\|_{\infty,[x,b]} \right) \right\} < \infty. \tag{35}$$

From now on we set $X = \mathbb{R}$.

## 2.2 Time Seperating Stochastic Processes

Let $(\Omega, \mathcal{F}, P)$ be a probability space, $\omega \in \Omega; Y_1, Y_2, \ldots, Y_m, m \in \mathbb{N}$, be real-valued random variables on $\Omega$ with finite expectations, and $h_1(t), h_2(t), \ldots h_m(t) : I \to \mathbb{R}$, where $I$ is an infinite subset of $\mathbb{R}$. Typically here $I$ is an infinite length interval of $\mathbb{R}$, usualy $I = \mathbb{R}$ or

$I = \mathbb{R}_+$.
Clearly, then

$$Y(t,\omega) := \sum_{i=1}^{m} h_i(t) Y_i(\omega), t \in I, \tag{36}$$

is a quite common stochastic process separating time.
We can assume that $h_i \in C^r(I), i = 1, 2, ..., m; r \in \mathbb{N}$. Consequently, we have that the expectation

$$(EY)(t) = \sum_{i=1}^{m} h_i(t) EY_i \in C(I) \text{ or } C^r(I). \tag{37}$$

A classical example of a stochastic process separating time is

$$(\sin t) Y_1(\omega) + (\cos t) Y_2(\omega), t \in I.$$

Notice that $|\sin t| \leq 1$ and $|\cos t| \leq 1$.
Another typical example is

$$\sinh(t) Y_1(\omega) + \cosh(t) Y_2(\omega), t \in I. \tag{38}$$

In this article we will apply the main results of section 2.1, to $f(t) = (EY)(t)$. We will finish with several applications. See the related [19], [20].

# 3 Main Results

We present the following general approximation of the seperating stochastic processes by neural network operators.

**Theorem 18.** *Let $(EY)(t)$ as in (37), $t \in [t_1, t_2]$, where $t_1, t_2 \in \mathbb{R}$, with $t_1 < t_2, h_i \in C([t_1, t_2])$ for every $i = 1, 2, ..., m, 0 < \alpha < 1, n \in \mathbb{N} : n^{1-\alpha} > 2$. Then*

*i)*

$$|A_n((EY), t) - (EY)(t)| \leq \frac{1}{\psi(1)} \left[ \omega_1 \left( EY, \frac{1}{n^{\alpha}} \right) + \left(1 - h\left(n^{1-\alpha} - 2\right)\right) \|EY\|_{\infty} \right] =: \rho, \tag{39}$$

*and*

*ii)*

$$\|A_n(EY) - EY\|_{\infty} \leq \rho. \tag{40}$$

*We have that $\lim_{n\to\infty} A_n(EY) = EY$, pointwise and uniformly.*
*The speed of convergence is $\max\left(\frac{1}{n^{\alpha}}, \left(1 - h\left(n^{1-\alpha} - 2\right)\right)\right)$.*

**Proof.** Notice that when $h_i \in C([t_1, t_2])$ for every $i = 1, 2, ..., m$, then $(EY)(t) \in C([t_1, t_2])$. Thus, the conclusion comes from Theorem 10. □
We continue with,

**Theorem 19.** *Let $(EY)(t)$ as in (37), $h_i \in C_B(\mathbb{R})$ for every $i = 1,2,...,m, 0 < \alpha < 1$, $n \in \mathbb{N} : n^{1-\alpha} > 2$, $t \in \mathbb{R}$. Then*

*i)*

$$\left|\overline{A}_n(EY,t) - (EY)(t)\right| \le \omega_1\left(EY, \frac{1}{n^\alpha}\right) + \left(1 - h\left(n^{1-\alpha} - 2\right)\right)\|EY\|_\infty =: \mu, \tag{41}$$

*and*

*ii)*

$$\left\|\overline{A}_n(EY) - EY\right\|_\infty \le \mu. \tag{42}$$

*For $EY \in C_{uB}(\mathbb{R})$ we get $\lim_{n\to\infty} \overline{A}_n(EY) = EY$, pointwise and uniformly. The speed of convergence is* $\max\left(\frac{1}{n^\alpha}, \left(1 - h\left(n^{1-\alpha} - 2\right)\right)\right)$.

**Proof.** Since that $h_i \in C_B(\mathbb{R})$ for every $i = 1,2,...,m$ and $t \in \mathbb{R}$, then $EX \in C_B(\mathbb{R})$. Therefore the results come from Theorem 11. □

Furthermore, we have

**Theorem 20.** *Let $(EY)(t)$ as in (37), $t \in [t_1, t_2]$, where $t_1, t_2 \in \mathbb{R}$, with $t_1 < t_2, h_i \in C^N([t_1, t_2])$ for every $i = 1,2,...,m, n, N \in \mathbb{N}$, $0 < \alpha < 1$, and $n^{1-\alpha} > 2$. Then*

*i)*

$$\left|A_n(EY,t) - (EY)(t)\right| \le \frac{1}{\psi(1)}\left\{\sum_{j=1}^{N} \frac{\left|(EY)^{(j)}(t)\right|}{j!}\left[\frac{1}{n^{\alpha j}} + \frac{\left(1 - h\left(n^{1-\alpha} - 2\right)\right)}{2}(t_2 - t_1)^j\right] + \right. \tag{43}$$

$$\left.\left[\omega_1\left((EY)^{(N)}, \frac{1}{n^\alpha}\right)\frac{1}{n^{\alpha N} N!} + \frac{\left(1 - h\left(n^{1-\alpha} - 2\right)\right)\left\|(EY)^{(N)}\right\|_\infty (t_2 - t_1)^N}{N!}\right]\right\}.$$

*ii) Assume further $(EY)^{(j)}(t_0) = 0$, $j = 1,...,N$, for some $t_0 \in [t_1, t_2]$, it holds*

$$\left|A_n(EY,t_0) - (EY)(t_0)\right| \le \frac{1}{\psi(1)}$$

$$\left\{\omega_1\left((EY)^{(N)}, \frac{1}{n^\alpha}\right)\frac{1}{n^{\alpha N} N!} + \frac{\left(1 - h\left(n^{1-\alpha} - 2\right)\right)\left\|(EY)^{(N)}\right\|_\infty (t_2 - t_1)^N}{N!}\right\}, \tag{44}$$

*and*

*iii)*

$$\left\|A_n(EY) - EY\right\|_\infty \le \frac{1}{\psi(1)}\left\{\sum_{j=1}^{N} \frac{\left\|(EY)^{(j)}\right\|_\infty}{j!}\left[\frac{1}{n^{\alpha j}} + \frac{\left(1 - h\left(n^{1-\alpha} - 2\right)\right)}{2}(t_2 - t_1)^j\right] + \right.$$

$$\left.\left[\omega_1\left((EY)^{(N)}, \frac{1}{n^\alpha}\right)\frac{1}{n^{\alpha N} N!} + \frac{\left(1 - h\left(n^{1-\alpha} - 2\right)\right)\left\|(EY)^{(N)}\right\|_\infty (t_2 - t_1)^N}{N!}\right]\right\}. \tag{45}$$

*Again we obtain $\lim_{n\to\infty} A_n(EY) = EY$, pointwise and uniformly.*

**Proof.** By Theorem 12. □

We also present

**Theorem 21.** *Let* $\alpha > 0, N = \lceil \alpha \rceil, \alpha \notin \mathbb{N}, 0 < \beta < 1, t \in [t_1, t_2]$ *where* $t_1, t_2 \in \mathbb{R}$, *with* $t_1 < t_2$, $n \in \mathbb{N} : n^{1-\beta} > 2$. *Then*

*i)*

$$\left| A_n(EY, t) - \sum_{j=1}^{N-1} \frac{(EY)^{(j)}(t)}{j!} A_n\left((\cdot - t)^j\right)(t) - (EY)(t) \right| \leq$$

$$\frac{(\psi(1))^{-1}}{\Gamma(\alpha+1)} \left\{ \frac{\left(\omega_1\left(D_{t-}^{\alpha}(EY), \frac{1}{n^\beta}\right)_{[t_1,t]} + \omega_1\left(D_{*t}^{\alpha}(EY), \frac{1}{n^\beta}\right)_{[t,t_2]}\right)}{n^{\alpha\beta}} + \right.$$

$$\left. \left(\frac{1 - h(n^{1-\beta} - 2)}{2}\right) \left(\left\|D_{t-}^{\alpha}(EY)\right\|_{\infty,[t_1,t]} (t - t_1)^\alpha + \left\|D_{*t}^{\alpha}(EY)\right\|_{\infty,[t,t_2]} (t_2 - t)^\alpha\right) \right\}, \tag{46}$$

*ii) if* $(EY)^{(j)}(t) = 0$, *for* $j = 1, ..., N-1$, *we have*

$$|A_n(EY, t) - (EY)(t)| \leq \frac{(\psi(1))^{-1}}{\Gamma(\alpha+1)}$$

$$\left\{ \frac{\left(\omega_1\left(D_{t-}^{\alpha}(EY), \frac{1}{n^\beta}\right)_{[t_1,t]} + \omega_1\left(D_{*t}^{\alpha}(EY), \frac{1}{n^\beta}\right)_{[t,t_2]}\right)}{n^{\alpha\beta}} + \right.$$

$$\left. \left(\frac{1 - h(n^{1-\beta} - 2)}{2}\right) \left(\left\|D_{t-}^{\alpha}(EY)\right\|_{\infty,[t_1,t]} (t - t_1)^\alpha + \left\|D_{*t}^{\alpha}(EY)\right\|_{\infty,[t,t_2]} (t_2 - t)^\alpha\right) \right\}, \tag{47}$$

*iii)*

$$\|A_n(EY, t) - (EY)(t)\| \leq (\psi(1))^{-1}$$

$$\left\{ \sum_{j=1}^{N-1} \frac{\left|(EY)^{(j)}(t)\right|}{j!} \left\{ \frac{1}{n^{\beta j}} + (t_2 - t_1)^j \left(\frac{1 - h(n^{1-\beta} - 2)}{2}\right) \right\} + \right.$$

$$\frac{1}{\Gamma(\alpha+1)} \left\{ \frac{\left(\omega_1\left(D_{t-}^{\alpha}(EY), \frac{1}{n^\beta}\right)_{[t_1,t]} + \omega_1\left(D_{*t}^{\alpha}(EY), \frac{1}{n^\beta}\right)_{[t,t_2]}\right)}{n^{\alpha\beta}} + \right.$$

$$\left. \left. \left(\frac{1 - h(n^{1-\beta} - 2)}{2}\right) \left(\left\|D_{t-}^{\alpha}(EY)\right\|_{\infty,[t_1,t]} (t - t_1)^\alpha + \left\|D_{*t}^{\alpha}(EY)\right\|_{\infty,[t,t_2]} (t_2 - t)^\alpha\right) \right\} \right\}, \tag{48}$$

$\forall\ t \in [t_1, t_2]$,

*and*

*iv)*

$$\|A_n(EY) - EY\|_\infty \leq (\psi(1))^{-1}$$

$$\left\{ \sum_{j=1}^{N-1} \frac{\left\|(EY)^{(j)}\right\|_\infty}{j!} \left\{ \frac{1}{n^{\beta j}} + (t_2 - t_1)^j \left(\frac{1 - h(n^{1-\beta} - 2)}{2}\right) \right\} + \right.$$

$$\frac{1}{\Gamma(\alpha+1)}\left\{\frac{\left(\sup_{x\in[t_1,t_2]}\omega_1\left(D_{t-}^{\alpha}(EY),\frac{1}{n^{\beta}}\right)_{[t_1,t]}+\sup_{t\in[t_1,t_2]}\omega_1\left(D_{*t}^{\alpha}(EY),\frac{1}{n^{\beta}}\right)_{[t,t_2]}\right)}{n^{\alpha\beta}}+\right.$$

$$\left.\left(\frac{1-h\left(n^{1-\beta}-2\right)}{2}\right)(t_2-t_1)^{\alpha}\left(\sup_{t\in[t_1,t_2]}\left\|D_{t-}^{\alpha}(EY)\right\|_{\infty,[t_1,t]}+\sup_{t\in[t_1,t_2]}\left\|D_{*t}^{\alpha}(EY)\right\|_{\infty,[t,t_2]}\right)\right\}\Bigg\}. \tag{49}$$

*Above, when* $N=1$ *the sum* $\sum_{j=1}^{N-1}\cdot=0$.

*As we see here we obtain t-valued fractionally type pointwise and uniform convergence with rates of* $A_n\to I$ *the unit operator, as* $n\to\infty$.

**Proof.** By Theorem 15. □

Next we apply Theorem 21 for $N=1$.

**Corollary 22.** *Let* $(EY)(t)$ *as in (37),* $0<\alpha,\beta<1$, $t\in[t_1,t_2]$, *where* $t_1,t_2\in\mathbb{R}$, *with* $t_1<t_2$,$n\in\mathbb{N}:n^{1-\beta}>2$. *and* $h_i\in C([t_1,t_2])$ *for every* $i=1,2,...,m$. *Then*

*i)*

$$|A_n(EY,t)-(EY)(t)|\leq$$

$$\frac{(\psi(1))^{-1}}{\Gamma(\alpha+1)}\left\{\frac{\left(\omega_1\left(D_{t-}^{\alpha}(EY),\frac{1}{n^{\beta}}\right)_{[t_1,t]}+\omega_1\left(D_{*t}^{\alpha}(EY),\frac{1}{n^{\beta}}\right)_{[t,t_2]}\right)}{n^{\alpha\beta}}+\right.$$

$$\left.\left(\frac{1-h\left(n^{1-\beta}-2\right)}{2}\right)\left(\left\|D_{t-}^{\alpha}(EY)\right\|_{\infty,[t_1,t]}(t-t_1)^{\alpha}+\left\|D_{*t}^{\alpha}(EY)\right\|_{\infty,[t,t_2]}(t_2-t)^{\alpha}\right)\right\}, \tag{50}$$

*and*

*ii)*

$$\|A_n(EY)-(EY)\|_{\infty}\leq\frac{(\psi(1))^{-1}}{\Gamma(\alpha+1)}$$

$$\left\{\frac{\left(\sup_{t\in[t_1,t_2]}\omega_1\left(D_{t-}^{\alpha}(EY),\frac{1}{n^{\beta}}\right)_{[t_1,t]}+\sup_{t\in[t_1,t_2]}\omega_1\left(D_{*t}^{\alpha}(EY),\frac{1}{n^{\beta}}\right)_{[t,t_2]}\right)}{n^{\alpha\beta}}+\right.$$

$$\left.\left(\frac{1-h\left(n^{1-\beta}-2\right)}{2}\right)(t_2-t_1)^{\alpha}\left(\sup_{t\in[t_1,t_2]}\left\|D_{t-}^{\alpha}(EY)\right\|_{\infty,[t_1,t]}+\sup_{x\in[t_1,t_2]}\left\|D_{*t}^{\alpha}(EY)\right\|_{\infty,[t,t_2]}\right)\right\}. \tag{51}$$

When $\alpha=\frac{1}{2}$ we derive

**Corollary 23.** *Assume again* $(EY)(t)$ *as in (37). Let* $0<\beta<1$, $t\in[t_1,t_2]$, *where* $t_1,t_2\in\mathbb{R}$, *with* $t_1<t_2$, $n\in\mathbb{N}:n^{1-\beta}>2$ *and* $h_i\in C([t_1,t_2])$ *for every* $i=1,2,...,m$. *Then* *i)*

$$|A_n(EY,t)-(EY)(t)| \leq$$

$$\frac{2(\psi(1))^{-1}}{\sqrt{\pi}} \left\{ \frac{\left( \omega_1\left(D_{t-}^{\frac{1}{2}}(EY), \frac{1}{n^{\beta}}\right)_{[t_1,t]} + \omega_1\left(D_{*t}^{\frac{1}{2}}(EY), \frac{1}{n^{\beta}}\right)_{[t,t_2]} \right)}{n^{\frac{\beta}{2}}} + \right.$$

$$\left. \left(\frac{1-h\left(n^{1-\beta}-2\right)}{2}\right)\left( \left\| D_{t-}^{\frac{1}{2}}(EY)\right\|_{\infty,[t_1,t]} \sqrt{(t-t_1)} + \left\| D_{*t}^{\frac{1}{2}}(EY)\right\|_{\infty,[t,t_2]} \sqrt{(t_2-t)} \right) \right\}, \tag{52}$$

*and*

*ii)*

$$\|A_n(EY)-(EY)\|_{\infty} \leq \frac{2(\psi(1))^{-1}}{\sqrt{\pi}}$$

$$\left\{ \frac{\left( \sup_{t\in[t_1,t_2]} \omega_1\left(D_{t-}^{\frac{1}{2}}(EY), \frac{1}{n^{\beta}}\right)_{[t_1,t]} + \sup_{t\in[t_1,t_2]} \omega_1\left(D_{*t}^{\frac{1}{2}}(EY), \frac{1}{n^{\beta}}\right)_{[t,t_2]} \right)}{n^{\frac{\beta}{2}}} + \right.$$

$$\left. \left(\frac{1-h\left(n^{1-\beta}-2\right)}{2}\right)\sqrt{(t_2-t_1)}\left( \sup_{t\in[t_1,t_2]} \left\| D_{t-}^{\frac{1}{2}}(EY)\right\|_{\infty,[t_1,t]} + \sup_{t\in[t_1,t_2]} \left\| D_{*t}^{\frac{1}{2}}(EY)\right\|_{\infty,[t,t_2]} \right) \right\} < \infty. \tag{53}$$

# 4 Applications

For the next applications we consider $(\Omega, F, P)$ be a probability space and $Y_0, Y_1, Y_2$ be real valued random variables on $\Omega$ with finite expectations. We consider the stochastic processes $Z_i(t,\omega)$ for $i = 1, 2, \ldots, 9$, where $t \in \mathbb{R}$ and $\omega \in \Omega$ as follows:

$$Z_1(t,\omega) = \left[(t-t_0)^{\mu+1}+1\right] Y_0(\omega), \tag{54}$$

where $t_0 \in \mathbb{R}$ and $\mu \in \mathbb{N}$ are fixed;

$$Z_2(t,\omega) = \sin(\xi t)\, Y_1(\omega) + \cos(\xi t)\, Y_2(\omega), \tag{55}$$

where $\xi > 0$ is fixed;

$$Z_3(t,\omega) = \sinh(\mu t)\, Y_1(\omega) + \cosh(\mu t)\, Y_2(\omega), \tag{56}$$

where $\mu > 0$ is fixed;

$$Z_4(t,\omega) = sech(\mu t)\, Y_1(\omega) + \tanh(\mu t)\, Y_2(\omega), \tag{57}$$

where $\mu > 0$ is fixed.
Here $sechx := \frac{1}{\cosh x} = \frac{2}{e^x+e^{-x}}, x \in \mathbb{R}$.

$$Z_5(t,\omega) = e^{-\ell_1 t} Y_1(\omega) + e^{-\ell_2 t} Y_2(\omega), \tag{58}$$

where $\ell_1, \ell_2 > 0$ are fixed;

$$Z_6(t,\omega) = \frac{1}{1+e^{-\ell_1 t}} Y_1(\omega) + \frac{1}{1+e^{-\ell_2 t}} Y_2(\omega), \tag{59}$$

where $\ell_1, \ell_2 > 0$ are fixed;

$$Z_7(t,\omega) = e^{-e^{-\mu_1 t}} Y_1(\omega) + e^{-e^{-\mu_2 t}} Y_2(\omega), \tag{60}$$

where $\mu_1, \mu_2 > 0$ are fixed;

$$Z_8(t,\omega) = P_m(\ell_1 t) Y_1(\omega) + P_m(\ell_2 t) Y_2(\omega), \tag{61}$$

where $\ell_1, \ell_2 > 0$ and $m \in \mathbb{N}$ are fixed.
Here $P_m(x)$ is the Legendre Polynomial of degree $m \in \mathbb{N}$, i.e

$$P_m : [-1,1] \longrightarrow [-1,1]$$

such that,

$$P_m(x) = \frac{1}{2^m m!} \frac{d^m}{dx^m} \left(x^2-1\right)^m = \frac{1}{2^m} \sum_{k=0}^{m} \binom{m}{k}^2 (x-1)^{m-k} (x+1)^k, x \in [-1,1].$$

To define the stochastic process $Z_9(t,\omega)$, we consider the Cauchy function

$$\hat{f}(x) = \begin{cases} e^{-\frac{1}{x^2}}, & x \neq 0 \\ 0, & x = 0 \end{cases}.$$

Notice that, $\hat{f} \in C^\infty(\mathbb{R})$ and it has $\hat{f}^{(i)}(0) = 0$, for all $i = 1, 2, \ldots, \infty$.
We set,

$$Z_9(t,\omega) = \hat{f}(t) Y_0(\omega), \tag{62}$$

The expectations of $Z_i, i = 1, 2, ..., 9$ are

$$(EZ_1)(t) = \left[(t-t_0)^{\mu+1} + 1\right] E(Y_0), \tag{63}$$

$$(EZ_2)(t) = \sin(\xi t) E(Y_1) + \cos(\xi t) E(Y_2), \tag{64}$$

$$(EZ_3)(t) = \sinh(\mu t) E(Y_1) + \cosh(\mu t) E(Y_2), \tag{65}$$

$$(EZ_4)(t) = sech(\mu t) E(Y_1) + \tanh(\mu t) E(Y_2), \tag{66}$$

$$(EZ_5)(t) = e^{-\ell_1 t} E(Y_1) + e^{-\ell_2 t} E(Y_2), \tag{67}$$

$$(EZ_6)(t) = \frac{1}{1+e^{-\ell_1 t}} E(Y_1) + \frac{1}{1+e^{-\ell_2 t}} E(Y_2), \tag{68}$$

$$(EZ_7)(t) = e^{-e^{-\mu_1 t}} E(Y_1) + e^{-e^{-\mu_2 t}} E(Y_2), \tag{69}$$

$$(EZ_8)(t) = P_m(\ell_1 t) E(Y_1) + P_m(\ell_2 t) E(Y_2), \tag{70}$$

$$(EZ_9)(t) = \hat{f}(t) E(Y_0), \tag{71}$$

For the next $(EZ_i)(t), i = 1, 2, \ldots, 9$ are as defined in relations between (63) and (71) respectively.
We present the following result.

**Proposition 24.** *Let* $t \in [t_1, t_2]$, *where* $t_1, t_2 \in \mathbb{R}$, *with* $t_1 < t_2, 0 < \alpha < 1, n \in \mathbb{N} : n^{1-\alpha} > 2$. *Then for* $i = 1, 2, \dots, 9$
*i)*

$$\left|A_n\left(\left(EZ_i\right), t\right) - \left(EZ_i\right)(t)\right| \leq \frac{1}{\psi(1)}\left[\omega_1\left(EZ_i, \frac{1}{n^\alpha}\right) + \left(1 - h\left(n^{1-\alpha} - 2\right)\right)\left\|EZ_i\right\|_\infty\right] =: \rho, \tag{72}$$

*and*

*ii)*

$$\left\|A_n\left(EZ_i\right) - EZ_i\right\|_\infty \leq \rho. \tag{73}$$

*We have that* $\lim_{n\to\infty} A_n\left(EZ_i\right) = EZ_i$, *pointwise and uniformly.*
*The speed of convergence is* $\max\left(\frac{1}{n^\alpha}, \left(1 - h\left(n^{1-\alpha} - 2\right)\right)\right)$.

**Proof.** From Theorem 18. □
In the cases of stochastic processes $Z_i(t, \omega)$, for $i = 2, 4, 6, 7$ we have the next

**Proposition 25.** *Let* $i \in \{2, 4, 6, 7\}, 0 < \alpha < 1, n \in \mathbb{N} : n^{1-\alpha} > 2, t \in \mathbb{R}$. *Then*

*i)*

$$\left|\overline{A}_n\left(EZ_i, t\right) - \left(EZ_i\right)(t)\right| \leq \omega_1\left(EZ_i, \frac{1}{n^\alpha}\right) + \left(1 - h\left(n^{1-\alpha} - 2\right)\right)\left\|EZ_i\right\|_\infty =: \mu, \tag{74}$$

*and*

*ii)*

$$\left\|\overline{A}_n\left(EZ_i\right) - EZ_i\right\|_\infty \leq \mu. \tag{75}$$

*For* $EZ_i \in C_{uB}(\mathbb{R})$ *we get* $\lim_{n\to\infty} \overline{A}_n\left(EZ_i\right) = EZ_i$, *pointwise and uniformly.*
*The speed of convergence is* $\max\left(\frac{1}{n^\alpha}, \left(1 - h\left(n^{1-\alpha} - 2\right)\right)\right)$.

**Proof.** Notice that for every $t \in \mathbb{R}$ we have that:

for $Z_2(t, \omega)$, $|\sin(\xi t)| \leq 1$ and $|\cos(\xi t)| \leq 1$,

for $Z_4(t, \omega)$, $|sech(\mu t)| \leq 1$ and $|tanh(\mu t)| \leq 1$,

for $Z_6(t, \omega)$, $0 < \frac{1}{1+e^{-\ell_1 t}} < 1$ and $0 < \frac{1}{1+e^{-\ell_2 t}} < 1$,

for $Z_7(t, \omega)$, $0 < e^{-e^{-\mu_1 t}} < 1$ and $0 < e^{-e^{-\mu_2 t}} < 1$.

Thus, the results come from Theorem 19. □
Moreover, we present the next

**Proposition 26.** *Let* $i = 1, 2, \dots, 9, t \in [t_1, t_2]$, *where* $t_1, t_2 \in \mathbb{R}$, *with* $t_1 < t_2, 0 < \alpha < 1$, *and* $n^{1-\alpha} > 2$. *Then* *i)*

$$\left|A_n\left(EZ_i,t\right)-\left(EZ_i\right)(t)\right|\leq\frac{1}{\psi(1)}\left\{\sum_{j=1}^{N}\frac{\left|\left(EZ_i\right)^{(j)}(t)\right|}{j!}\left[\frac{1}{n^{\alpha j}}+\frac{\left(1-h\left(n^{1-\alpha}-2\right)\right)}{2}\left(t_2-t_1\right)^j\right]+\right. \tag{76}$$

$$\left.\left[\omega_1\left(\left(EZ_i\right)^{(N)},\frac{1}{n^{\alpha}}\right)\frac{1}{n^{\alpha N}N!}+\frac{\left(1-h\left(n^{1-\alpha}-2\right)\right)\left\|\left(EZ_i\right)^{(N)}\right\|_{\infty}\left(t_2-t_1\right)^N}{N!}\right]\right\}.$$

*ii) Assume further* $\left(EZ_i\right)^{(j)}(t_a)=0$*,* $j=1,...,N$*, for some* $t_a\in[t_1,t_2]$*, it holds*

$$\left|A_n\left(EZ_i,t_a\right)-\left(EZ_i\right)(t_a)\right|\leq\frac{1}{\psi(1)}$$

$$\left\{\omega_1\left(\left(EZ_i\right)^{(N)},\frac{1}{n^{\alpha}}\right)\frac{1}{n^{\alpha N}N!}+\frac{\left(1-h\left(n^{1-\alpha}-2\right)\right)\left\|\left(EZ_i\right)^{(N)}\right\|_{\infty}\left(t_2-t_1\right)^N}{N!}\right\}, \tag{77}$$

*and*

*iii)*

$$\left\|A_n\left(EZ_i\right)-EZ_i\right\|_{\infty}\leq\frac{1}{\psi(1)}\left\{\sum_{j=1}^{N}\frac{\left\|\left(EZ_i\right)^{(j)}\right\|_{\infty}}{j!}\left[\frac{1}{n^{\alpha j}}+\frac{\left(1-h\left(n^{1-\alpha}-2\right)\right)}{2}\left(t_2-t_1\right)^j\right]+\right.$$

$$\left.\left[\omega_1\left(\left(EZ_i\right)^{(N)},\frac{1}{n^{\alpha}}\right)\frac{1}{n^{\alpha N}N!}+\frac{\left(1-h\left(n^{1-\alpha}-2\right)\right)\left\|\left(EZ_i\right)^{(N)}\right\|_{\infty}\left(t_2-t_1\right)^N}{N!}\right]\right\}. \tag{78}$$

*Again we obtain* $\lim_{n\to\infty}A_n\left(EZ_i\right)=EZ_i$*, pointwise and uniformly.*

**Proof.** By Theorem 20. □

We also present

**Proposition 27.** *Let* $i=1,2,\ldots,9,\alpha>0,N=\lceil\alpha\rceil,\alpha\notin\mathbb{N},0<\beta<1,t\in[t_1,t_2]$ *where* $t_1,t_2\in\mathbb{R}$*, with* $t_1<t_2$*,* $n\in\mathbb{N}:n^{1-\beta}>2$*. Then*

*i)*

$$\left|A_n\left(EZ_i,t\right)-\sum_{j=1}^{N-1}\frac{\left(EZ_i\right)^{(j)}(t)}{j!}A_n\left(\left(\cdot-t\right)^j\right)(t)-\left(EZ_i\right)(t)\right|\leq$$

$$\frac{\left(\psi(1)\right)^{-1}}{\Gamma(\alpha+1)}\left\{\frac{\left(\omega_1\left(D_{t-}^{\alpha}\left(EZ_i\right),\frac{1}{n^{\beta}}\right)_{[t_1,t]}+\omega_1\left(D_{*t}^{\alpha}\left(EZ_i\right),\frac{1}{n^{\beta}}\right)_{[t,t_2]}\right)}{n^{\alpha\beta}}+\right.$$

$$\left.\left(\frac{1-h\left(n^{1-\beta}-2\right)}{2}\right)\left(\left\|D_{t-}^{\alpha}\left(EZ_i\right)\right\|_{\infty,[t_1,t]}\left(t-t_1\right)^{\alpha}+\left\|D_{*t}^{\alpha}\left(EZ_i\right)\right\|_{\infty,[t,t_2]}\left(t_2-t\right)^{\alpha}\right)\right\}, \tag{79}$$

*ii) if* $\left(EZ_i\right)^{(j)}(t)=0$*, for* $j=1,...,N-1$*, we have*

$$\left|A_n\left(EZ_i,t\right)-\left(EZ_i\right)(t)\right|\leq\frac{\left(\psi(1)\right)^{-1}}{\Gamma(\alpha+1)}$$

$$\left\{\frac{\left(\omega_1\left(D_{t-}^{\alpha}\left(EZ_i\right),\frac{1}{n^{\beta}}\right)_{[t_1,t]}+\omega_1\left(D_{*t}^{\alpha}\left(EZ_i\right),\frac{1}{n^{\beta}}\right)_{[t,t_2]}\right)}{n^{\alpha\beta}}+\right.$$

$$\left.\left(\frac{1-h\left(n^{1-\beta}-2\right)}{2}\right)\left(\left\|D_{t-}^{\alpha}\left(EZ_i\right)\right\|_{\infty,[t_1,t]}\left(t-t_1\right)^{\alpha}+\left\|D_{*t}^{\alpha}\left(EZ_i\right)\right\|_{\infty,[t,t_2]}\left(t_2-t\right)^{\alpha}\right)\right\}, \quad (80)$$

*iii)*

$$\left\|A_n\left(EZ_i,t\right)-\left(EZ_i\right)(t)\right\|\leq\left(\psi(1)\right)^{-1}$$

$$\left\{\sum_{j=1}^{N-1}\frac{\left|\left(EZ_i\right)^{(j)}(t)\right|}{j!}\left\{\frac{1}{n^{\beta j}}+\left(t_2-t_1\right)^j\left(\frac{1-h\left(n^{1-\beta}-2\right)}{2}\right)\right\}+\right.$$

$$\frac{1}{\Gamma(\alpha+1)}\left\{\frac{\left(\omega_1\left(D_{t-}^{\alpha}\left(EZ_i\right),\frac{1}{n^{\beta}}\right)_{[t_1,t]}+\omega_1\left(D_{*t}^{\alpha}\left(EZ_i\right),\frac{1}{n^{\beta}}\right)_{[t,t_2]}\right)}{n^{\alpha\beta}}+\right.$$

$$\left.\left.\left(\frac{1-h\left(n^{1-\beta}-2\right)}{2}\right)\left(\left\|D_{t-}^{\alpha}\left(EZ_i\right)\right\|_{\infty,[t_1,t]}\left(t-t_1\right)^{\alpha}+\left\|D_{*t}^{\alpha}\left(EZ_i\right)\right\|_{\infty,[t,t_2]}\left(t_2-t\right)^{\alpha}\right)\right\}\right\}, \quad (81)$$

$\forall\ t\in[t_1,t_2]$,

*and*

*iv)*

$$\left\|A_n\left(EZ_i\right)-EZ_i\right\|_{\infty}\leq\left(\psi(1)\right)^{-1}$$

$$\left\{\sum_{j=1}^{N-1}\frac{\left\|\left(EZ_i\right)^{(j)}\right\|_{\infty}}{j!}\left\{\frac{1}{n^{\beta j}}+\left(t_2-t_1\right)^j\left(\frac{1-h\left(n^{1-\beta}-2\right)}{2}\right)\right\}+\right.$$

$$\frac{1}{\Gamma(\alpha+1)}\left\{\frac{\left(\sup_{x\in[t_1,t_2]}\omega_1\left(D_{t-}^{\alpha}\left(EZ_i\right),\frac{1}{n^{\beta}}\right)_{[t_1,t]}+\sup_{t\in[t_1,t_2]}\omega_1\left(D_{*t}^{\alpha}\left(EZ_i\right),\frac{1}{n^{\beta}}\right)_{[t,t_2]}\right)}{n^{\alpha\beta}}+\right.$$

$$\left.\left.\left(\frac{1-h\left(n^{1-\beta}-2\right)}{2}\right)\left(t_2-t_1\right)^{\alpha}\left(\sup_{t\in[t_1,t_2]}\left\|D_{t-}^{\alpha}\left(EZ_i\right)\right\|_{\infty,[t_1,t]}+\sup_{t\in[t_1,t_2]}\left\|D_{*t}^{\alpha}\left(EZ_i\right)\right\|_{\infty,[t,t_2]}\right)\right\}\right\}. \quad (82)$$

*Above, when $N=1$ the sum $\sum_{j=1}^{N-1}\cdot=0$.*

*As we see here we obtain t-valued fractionally type pointwise and uniform convergence with rates of $A_n\to I$ the unit operator, as $n\to\infty$.*

**Proof.** By Theorem 21. □

Next we apply Proposition 27 for $N=1$.

**Corollary 28.** *Let $i=1,2,\ldots,9, 0<\alpha,\beta<1$, $t\in[t_1,t_2]$, where $t_1,t_2\in\mathbb{R}$, with $t_1<t_2$,$n\in\mathbb{N}:n^{1-\beta}>2$. Then*

*i)*

$$\left|A_n\left(EZ_i,t\right)-\left(EZ_i\right)(t)\right|\leq$$

$$\frac{(\psi(1))^{-1}}{\Gamma(\alpha+1)}\left\{\frac{\left(\omega_1\left(D_{t-}^{\alpha}(EZ_i),\frac{1}{n^\beta}\right)_{[t_1,t]}+\omega_1\left(D_{*t}^{\alpha}(EZ_i),\frac{1}{n^\beta}\right)_{[t,t_2]}\right)}{n^{\alpha\beta}}+\right.$$

$$\left.\left(\frac{1-h\left(n^{1-\beta}-2\right)}{2}\right)\left(\left\|D_{t-}^{\alpha}(EZ_i)\right\|_{\infty,[t_1,t]}(t-t_1)^{\alpha}+\left\|D_{*t}^{\alpha}(EZ_i)\right\|_{\infty,[t,t_2]}(t_2-t)^{\alpha}\right)\right\}, \quad (83)$$

*and*

*ii)*

$$\|A_n(EZ_i)-(EZ_i)\|_\infty\le\frac{(\psi(1))^{-1}}{\Gamma(\alpha+1)}$$

$$\left\{\frac{\left(\sup_{t\in[t_1,t_2]}\omega_1\left(D_{t-}^{\alpha}(EZ_i),\frac{1}{n^\beta}\right)_{[t_1,t]}+\sup_{t\in[t_1,t_2]}\omega_1\left(D_{*t}^{\alpha}(EZ_i),\frac{1}{n^\beta}\right)_{[t,t_2]}\right)}{n^{\alpha\beta}}+\right.$$

$$\left.\left(\frac{1-h\left(n^{1-\beta}-2\right)}{2}\right)(t_2-t_1)^{\alpha}\left(\sup_{t\in[t_1,t_2]}\left\|D_{t-}^{\alpha}(EZ_i)\right\|_{\infty,[t_1,t]}+\sup_{x\in[t_1,t_2]}\left\|D_{*t}^{\alpha}(EZ_i)\right\|_{\infty,[t,t_2]}\right)\right\}. \quad (84)$$

When $\alpha=\frac{1}{2}$ we derive

**Corollary 29.** *Assume $i=1,2,\ldots,9$. Let $0<\beta<1$, $t\in[t_1,t_2]$, where $t_1,t_2\in\mathbb{R}$, with $t_1<t_2$, and $n\in\mathbb{N}:n^{1-\beta}>2$. Then*

*i)*

$$|A_n(EZ_i,t)-(EZ_i)(t)|\le$$

$$\frac{2(\psi(1))^{-1}}{\sqrt{\pi}}\left\{\frac{\left(\omega_1\left(D_{t-}^{\frac{1}{2}}(EZ_i),\frac{1}{n^\beta}\right)_{[t_1,t]}+\omega_1\left(D_{*t}^{\frac{1}{2}}(EZ_i),\frac{1}{n^\beta}\right)_{[t,t_2]}\right)}{n^{\frac{\beta}{2}}}+\right.$$

$$\left.\left(\frac{1-h\left(n^{1-\beta}-2\right)}{2}\right)\left(\left\|D_{t-}^{\frac{1}{2}}(EZ_i)\right\|_{\infty,[t_1,t]}\sqrt{(t-t_1)}+\left\|D_{*t}^{\frac{1}{2}}(EZ_i)\right\|_{\infty,[t,t_2]}\sqrt{(t_2-t)}\right)\right\}, \quad (85)$$

*and*

*ii)*

$$\|A_n(EZ_i)-(EZ_i)\|_\infty\le\frac{2(\psi(1))^{-1}}{\sqrt{\pi}}$$

$$\left\{\frac{\left(\sup_{t\in[t_1,t_2]}\omega_1\left(D_{t-}^{\frac{1}{2}}(EZ_i),\frac{1}{n^\beta}\right)_{[t_1,t]}+\sup_{t\in[t_1,t_2]}\omega_1\left(D_{*t}^{\frac{1}{2}}(EZ_i),\frac{1}{n^\beta}\right)_{[t,t_2]}\right)}{n^{\frac{\beta}{2}}}+\right.$$

$$\left(\frac{1-h\left(n^{1-\beta}-2\right)}{2}\right)\sqrt{(t_2-t_1)}\left(\sup_{t\in[t_1,t_2]}\left\|D_{t-}^{\frac{1}{2}}\left(EZ_i\right)\right\|_{\infty,[t_1,t]}+\sup_{t\in[t_1,t_2]}\left\|D_{*t}^{\frac{1}{2}}\left(EZ_i\right)\right\|_{\infty,[t,t_2]}\right)\Bigg\}<\infty. \tag{86}$$

# 5 Specific Applications

Let $(\Omega,\mathcal{F},P)$, where $\Omega$ is the set of non-negative integers, be a probability space, $Y_{1,1},Y_{2,1}$ be real-valued random variables on $\Omega$ following Poisson distributions with parameters $\lambda_1,\lambda_2\in(0,\infty)$ respectively.
We consider the stochastic processes $Z_{i,1}(t,\omega)$ for $i=1,2,3,5$, where $t\in\mathbb{R}$ and $\omega\in\Omega$ as follows:

$$Z_{1,1}(t,\omega)=\left[(t-t_0)^{\mu+1}+1\right]Y_{1,1}(\omega), \tag{87}$$

where $t_0\in\mathbb{R}$ and $\mu\in\mathbb{N}$ are fixed;

$$Z_{2,1}(t,\omega)=\sin(\xi t)\,Y_{1,1}(\omega)+\cos(\xi t)\,Y_{2,1}(\omega), \tag{88}$$

where $\xi>0$ is fixed;

$$Z_{3,1}(t,\omega)=\sinh(\mu t)\,Y_{1,1}(\omega)+\cosh(\mu t)\,Y_{2,1}(\omega), \tag{89}$$

where $\mu>0$ is fixed;

$$Z_{5,1}(t,\omega)=e^{-\ell_1 t}Y_{1,1}(\omega)+e^{-\ell_2 t}Y_{2,1}(\omega), \tag{90}$$

where $\ell_1,\ell_2>0$ are fixed.
Since $E\left(Y_{1,1}\right)=\lambda_1$ and $E\left(Y_{2,1}\right)=\lambda_2$ , the expectations of $Z_{i,1}, i=1,2,3,5$, are

$$\left(EZ_{1,1}\right)(t)=\lambda_1\left[(t-t_0)^{\mu+1}+1\right], \tag{91}$$

$$\left(EZ_{2,1}\right)(t)=\lambda_1\sin(\xi t)+\lambda_2\cos(\xi t), \tag{92}$$

$$\left(EZ_{3,1}\right)(t)=\lambda_1\sinh(\mu t)+\lambda_2\cosh(\mu t), \tag{93}$$

$$\left(EZ_{5,1}\right)(t)=\lambda_1 e^{-\ell_1 t}+\lambda_2 e^{-\ell_2 t}, \tag{94}$$

For the next we consider $(\Omega,\mathcal{F},P)$, where $\Omega=\mathbb{R}$, be a probability space, $Y_{1,2},Y_{2,2}$ be real-valued random variables on $\Omega$ following Gaussian distributions with expectations $\hat{\mu}_1,\hat{\mu}_2\in\mathbb{R}$ respectively.
We consider the stochastic processes $Z_{i,2}(t,\omega)$ for $i=1,2,3,5$, where $t\in\mathbb{R}$ and $\omega\in\Omega$ as follows:

$$Z_{1,2}(t,\omega)=\left[(t-t_0)^{\mu+1}+1\right]Y_{1,2}(\omega), \tag{95}$$

where $t_0\in\mathbb{R}$ and $\mu\in\mathbb{N}$ are fixed;

$$Z_{2,2}(t,\omega)=\sin(\xi t)\,Y_{1,2}(\omega)+\cos(\xi t)\,Y_{2,2}(\omega), \tag{96}$$

where $\xi>0$ is fixed;

$$Z_{3,2}(t,\omega)=\sinh(\mu t)\,Y_{1,2}(\omega)+\cosh(\mu t)\,Y_{2,2}(\omega), \tag{97}$$

where $\mu > 0$ is fixed;

$$Z_{5,2}(t,\omega) = e^{-\ell_1 t} Y_{1,2}(\omega) + e^{-\ell_2 t} Y_{2,2}(\omega), \tag{98}$$

where $\ell_1, \ell_2 > 0$ are fixed.
Since $E\left(Y_{1,2}\right) = \hat{\mu}_1$ and $E\left(Y_{2,2}\right) = \hat{\mu}_2$ , The expectations of $Z_{i,2}, i = 1, 2, 3, 5$ are

$$\left(EZ_{1,2}\right)(t) = \hat{\mu}_1 \left[(t - t_0)^{\mu+1} + 1\right], \tag{99}$$

$$\left(EZ_{2,2}\right)(t) = \hat{\mu}_1 \sin\left(\xi t\right) + \hat{\mu}_2 \cos\left(\xi t\right), \tag{100}$$

$$\left(EZ_{3,2}\right)(t) = \hat{\mu}_1 \sinh\left(\mu t\right) + \hat{\mu}_2 \cosh\left(\mu t\right), \tag{101}$$

$$\left(EZ_{5,2}\right)(t) = \hat{\mu}_1 e^{-\ell_1 t} + \hat{\mu}_2 e^{-\ell_2 t}. \tag{102}$$

Furthermore, we consider $(\Omega, \mathcal{F}, P)$, where $\Omega = [0, \infty)$, be a probability space, $Y_{1,3}, Y_{2,3}$ be real-valued random variables on $\Omega$ following Weibull distributions with scale parameters 1 and shape parameters $\gamma_1, \gamma_2 \in (0, \infty)$ respectively.
We consider the stochastic processes $Z_{i,3}(t,\omega)$ for $i = 1, 2, 3, 5$, where $t \in \mathbb{R}$ and $\omega \in \Omega$ as follows:

$$Z_{1,3}(t,\omega) = \left[(t - t_0)^{\mu+1} + 1\right] Y_{1,3}(\omega), \tag{103}$$

where $t_0 \in \mathbb{R}$ and $\mu \in \mathbb{N}$ are fixed;

$$Z_{2,3}(t,\omega) = \sin\left(\xi t\right) Y_{1,3}(\omega) + \cos\left(\xi t\right) Y_{2,3}(\omega), \tag{104}$$

where $\xi > 0$ is fixed;

$$Z_{3,3}(t,\omega) = \sinh\left(\mu t\right) Y_{1,3}(\omega) + \cosh\left(\mu t\right) Y_{2,3}(\omega), \tag{105}$$

where $\mu > 0$ is fixed;

$$Z_{5,3}(t,\omega) = e^{-\ell_1 t} Y_{1,3}(\omega) + e^{-\ell_2 t} Y_{2,3}(\omega), \tag{106}$$

where $\ell_1, \ell_2 > 0$ are fixed.
Since $E\left(Y_{1,3}\right) = \Gamma\left(1 + \frac{1}{\gamma_1}\right)$ and $E\left(Y_{2,3}\right) = \Gamma\left(1 + \frac{1}{\gamma_2}\right)$ , where $\Gamma\left(\cdot\right)$ is the Gamma function, The expectations of $Z_{i,3}, i = 1, 2, 3, 5$, are

$$\left(EZ_{1,3}\right)(t) = \Gamma\left(1 + \frac{1}{\gamma_1}\right)\left[(t - t_0)^{\mu+1} + 1\right], \tag{107}$$

$$\left(EZ_{2,3}\right)(t) = \Gamma\left(1 + \frac{1}{\gamma_1}\right)\sin\left(\xi t\right) + \Gamma\left(1 + \frac{1}{\gamma_2}\right)\cos\left(\xi t\right), \tag{108}$$

$$\left(EZ_{3,3}\right)(t) = \Gamma\left(1 + \frac{1}{\gamma_1}\right)\sinh\left(\mu t\right) + \Gamma\left(1 + \frac{1}{\gamma_2}\right)\cosh\left(\mu t\right), \tag{109}$$

$$\left(EZ_{5,3}\right)(t) = \Gamma\left(1 + \frac{1}{\gamma_1}\right) e^{-\ell_1 t} + \Gamma\left(1 + \frac{1}{\gamma_2}\right) e^{-\ell_2 t}, \tag{110}$$

We present the following result.

**Proposition 30.** *Let $t \in [t_1, t_2]$, where $t_1, t_2 \in \mathbb{R}$, with $t_1 < t_2, 0 < \alpha < 1, n \in \mathbb{N} : n^{1-\alpha} > 2$. Then for $i = 1, 2, 3, 5$ and $k = 1, 2, 3$*
*i)*

$$\left| A_n \left( \left( EZ_{i,k} \right), t \right) - \left( EZ_{i,k} \right)(t) \right| \leq \frac{1}{\psi(1)} \left[ \omega_1 \left( EZ_{i,k}, \frac{1}{n^\alpha} \right) + \left( 1 - h \left( n^{1-\alpha} - 2 \right) \right) \left\| EZ_{i,k} \right\|_\infty \right] =: \rho, \tag{111}$$

*and*

*ii)*

$$\left\| A_n \left( EZ_{i,k} \right) - EZ_{i,k} \right\|_\infty \leq \rho. \tag{112}$$

*We have that* $\lim_{n \to \infty} A_n \left( EZ_{i,k} \right) = EZ_{i,k}$*, pointwise and uniformly.*
*The speed of convergence is* $\max \left( \frac{1}{n^\alpha}, \left( 1 - h \left( n^{1-\alpha} - 2 \right) \right) \right)$.

**Proof.** From Proposition 24. □
In the cases of stochastic processes $Z_{2,k}(t, \omega)$, for $k = 1, 2, 3$ we have the next

**Proposition 31.** *Let $k \in \{1, 2, 3\}, 0 < \alpha < 1, n \in \mathbb{N} : n^{1-\alpha} > 2, t \in \mathbb{R}$. Then*

*i)*

$$\left| \overline{A}_n \left( EZ_{2,k}, t \right) - \left( EZ_{2,k} \right)(t) \right| \leq \omega_1 \left( EZ_{2,k}, \frac{1}{n^\alpha} \right) + \left( 1 - h \left( n^{1-\alpha} - 2 \right) \right) \left\| EZ_{2,k} \right\|_\infty =: \mu, \tag{113}$$

*and*

*ii)*

$$\left\| \overline{A}_n \left( EZ_{2,k} \right) - EZ_{2,k} \right\|_\infty \leq \mu. \tag{114}$$

*For $EZ_{2,k} \in C_{uB}(\mathbb{R})$ we get* $\lim_{n \to \infty} \overline{A}_n \left( EZ_{2,k} \right) = EZ_{2,k}$*, pointwise and uniformly.*
*The speed of convergence is* $\max \left( \frac{1}{n^\alpha}, \left( 1 - h \left( n^{1-\alpha} - 2 \right) \right) \right)$.

**Proof.** The results come from Proposition 25. □
Moreover, we present the next

**Corollary 32.** *Assume $i = 1, 2, 3, 5$ and $k = 1, 2, 3$. Let $0 < \beta < 1$, $t \in [t_1, t_2]$, where $t_1, t_2 \in \mathbb{R}$, with $t_1 < t_2$, and $n \in \mathbb{N} : n^{1-\beta} > 2$. Then*

*i)*

$$\left| A_n \left( EZ_{i,k}, t \right) - \left( EZ_{i,k} \right)(t) \right| \leq$$

$$\frac{2 \left( \psi(1) \right)^{-1}}{\sqrt{\pi}} \left\{ \frac{\left( \omega_1 \left( D_{t-}^{\frac{1}{2}} \left( EZ_{i,k} \right), \frac{1}{n^\beta} \right)_{[t_1, t]} + \omega_1 \left( D_{*t}^{\frac{1}{2}} \left( EZ_{i,k} \right), \frac{1}{n^\beta} \right)_{[t, t_2]} \right)}{n^{\frac{\beta}{2}}} + \right.$$

$$\left. \left( \frac{1 - h \left( n^{1-\beta} - 2 \right)}{2} \right) \left( \left\| D_{t-}^{\frac{1}{2}} \left( EZ_{i,k} \right) \right\|_{\infty, [t_1, t]} \sqrt{(t - t_1)} + \left\| D_{*t}^{\frac{1}{2}} \left( EZ_{i,k} \right) \right\|_{\infty, [t, t_2]} \sqrt{(t_2 - t)} \right) \right\}, \tag{115}$$

*and*

*ii)*

$$\left\| A_n\left(EZ_{i,k}\right) - \left(EZ_{i,k}\right)\right\|_\infty \leq \frac{2\left(\psi\left(1\right)\right)^{-1}}{\sqrt{\pi}}$$

$$\left\{ \frac{\left( \sup_{t\in[t_1,t_2]} \omega_1\left( D_{t-}^{\frac{1}{2}}\left(EZ_{i,k}\right), \frac{1}{n^\beta}\right)_{[t_1,t]} + \sup_{t\in[t_1,t_2]} \omega_1\left( D_{*t}^{\frac{1}{2}}\left(EZ_{i,k}\right), \frac{1}{n^\beta}\right)_{[t,t_2]}\right)}{n^{\frac{\beta}{2}}} + \right.$$

$$\left.\left(\frac{1-h\left(n^{1-\beta}-2\right)}{2}\right)\sqrt{(t_2-t_1)}\left(\sup_{t\in[t_1,t_2]}\left\| D_{t-}^{\frac{1}{2}}\left(EZ_{i,k}\right)\right\|_{\infty,[t_1,t]} + \sup_{t\in[t_1,t_2]}\left\| D_{*t}^{\frac{1}{2}}\left(EZ_{i,k}\right)\right\|_{\infty,[t,t_2]}\right)\right\} < \infty. \tag{116}$$

**Proof.** From Corollary 29. □

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