SOME PROPERTIES OF TRACE CLASS P-DETERMINANT OF POSITIVE OPERATORS IN HILBERT SPACES

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Abstract. Let H be a complex Hilbert space. For a given operator $P \geq 0$ with $P \in \mathcal{B}_1(H)$, the trace class associated to $\mathcal{B}(H)$ and $\operatorname{tr}(P) = 1$, we define the *P-determinant* of the positive invertible operator A by

$$\Delta_P(A) := \exp \operatorname{tr}(P \ln A).$$

In this paper we present some fundamental properties of this determinant. Among others we show that

$$1 \leq \frac{\operatorname{tr}(PA)}{\Delta_P(A)} \leq \exp\left[\operatorname{tr}(PA)\operatorname{tr}\left(PA^{-1}\right) - 1\right]$$

and

$$1 \leq \frac{\Delta_P(A)}{\left[\operatorname{tr}\left(PA^{-1}\right)\right]^{-1}} \leq \exp\left[\operatorname{tr}\left(PA^{-1}\right)\operatorname{tr}(PA) - 1\right],$$

for $A > 0$ and $P \geq 0$ with $P \in \mathcal{B}_1(H)$ and $\operatorname{tr}(P) = 1$.

1. Introduction

In 1952, in the paper [5], B. Fuglede and R. V. Kadison introduced the determinant of a (invertible) operator and established its fundamental properties. The notion generalizes the usual determinant and can be considered for any operator in a finite von Neumann algebra (M, τ) with a faithful normal trace.

Let $T \in M$ be normal and $|T| := (T^*T)^{1/2}$ its modulus. By the spectral theorem one can represent T as an integral

$$T = \int_{\operatorname{Sp}(T)} \lambda dE(\lambda),$$

where $E(\lambda)$ is a projection valued measure and $\operatorname{Sp}(T)$ is the spectrum of T. The measure $\mu_T := \tau \circ E$ becomes a probability measure on the complex plane and has the support in the spectrum $\operatorname{Sp}(T)$.

For any $T \in M$ the Fuglede-Kadison determinant (*FK-determinant*) is defined by

$$\Delta_{FK}(T) := \exp\left(\int_0^\infty \ln t d\mu_{|T|}\right).$$

If T is invertible, then

$$\Delta_{FK}(T) := \exp(\tau(\ln(|T|))),$$

where $\ln(|T|)$ is defined by the use of functional calculus.

Let $B(H)$ be the space of all bounded linear operators on a Hilbert space H, and I stands for the identity operator on H. An operator A in $B(H)$ is said to

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be positive (in symbol: $A \geq 0$) if $\langle Ax, x\rangle \geq 0$ for all $x \in H$. In particular, $A > 0$ means that A is positive and invertible. For a pair A, B of selfadjoint operators the order relation $A \geq B$ means as usual that $A - B$ is positive.

In 1998, Fujii et al. [6], [7], introduced the *normalized determinant* $\Delta_x(A)$ for positive invertible operators A on a Hilbert space H and a fixed unit vector $x \in H$, namely $\|x\| = 1$, defined by

$$\Delta_x(A) := \exp \langle \ln Ax, x\rangle$$

and discussed it as a continuous geometric mean and observed some inequalities around the determinant from this point of view. For some recent results, see [8].

We need now some preparations for trace of operators in Hilbert spaces.

Let $(H, \langle \cdot, \cdot \rangle)$ be a complex Hilbert space and $\{e_i\}_{i\in I}$ an *orthonormal basis* of H. We say that $A \in \mathcal{B}(H)$ is a *Hilbert-Schmidt operator* if

$$\sum_{i\in I} \|Ae_i\|^2 < \infty. \tag{1.1}$$

It is well know that, if $\{e_i\}_{i\in I}$ and $\{f_j\}_{j\in J}$ are orthonormal bases for H and $A \in \mathcal{B}(H)$ then

$$\sum_{i\in I} \|Ae_i\|^2 = \sum_{j\in I} \|Af_j\|^2 = \sum_{j\in I} \|A^* f_j\|^2 \tag{1.2}$$

showing that the definition (1.1) is independent of the orthonormal basis and A is a Hilbert-Schmidt operator iff A^* is a Hilbert-Schmidt operator.

Let $\mathcal{B}_2(H)$ the set of *Hilbert-Schmidt operators* in $\mathcal{B}(H)$. For $A \in \mathcal{B}_2(H)$ we define

$$\|A\|_2 := \left(\sum_{i\in I} \|Ae_i\|^2\right)^{1/2} \tag{1.3}$$

for $\{e_i\}_{i\in I}$ an orthonormal basis of H.

Using the triangle inequality in $l^2(I)$, one checks that $\mathcal{B}_2(H)$ is a *vector space* and that $\|\cdot\|_2$ is a norm on $\mathcal{B}_2(H)$, which is usually called in the literature as the *Hilbert-Schmidt norm.*

Denote *the modulus* of an operator $A \in \mathcal{B}(H)$ by $|A| := (A^*A)^{1/2}$.

Because $\||A|x\| = \|Ax\|$ for all $x \in H$, A is Hilbert-Schmidt iff $|A|$ is Hilbert-Schmidt and $\|A\|_2 = \||A|\|_2$. From (1.2) we have that if $A \in \mathcal{B}_2(H)$, then $A^* \in \mathcal{B}_2(H)$ and $\|A\|_2 = \|A^*\|_2$.

The following theorem collects some of the most important properties of Hilbert-Schmidt operators:

Theorem 1. *We have:*

(i) $(\mathcal{B}_2(H), \|\cdot\|_2)$ *is a Hilbert space with inner product*

$$\langle A, B\rangle_2 := \sum_{i\in I} \langle Ae_i, Be_i\rangle = \sum_{i\in I} \langle B^*Ae_i, e_i\rangle \tag{1.4}$$

and the definition does not depend on the choice of the orthonormal basis $\{e_i\}_{i\in I}$*;*

(ii) We have the inequalities

$$\|A\| \leq \|A\|_2 \tag{1.5}$$

for any $A\in\mathcal{B}_2(H)$ *and, if* $A\in\mathcal{B}_2(H)$ *and* $T\in\mathcal{B}(H)$, *then* AT, $TA\in\mathcal{B}_2(H)$ *with*

$$\|AT\|_2\,,\ \ \|TA\|_2\leq\|T\|\,\|A\|_2 \tag{1.6}$$

(iii) $\mathcal{B}_2(H)$ *is an operator ideal in* $\mathcal{B}(H)$, *i.e.*

$$\mathcal{B}(H)\,\mathcal{B}_2(H)\,\mathcal{B}(H)\subseteq\mathcal{B}_2(H)\,.$$

If $\{e_i\}_{i\in I}$ an orthonormal basis of H, we say that $A\in\mathcal{B}(H)$ is *trace class* if

$$\|A\|_1:=\sum_{i\in I}\langle|A|\,e_i,e_i\rangle<\infty. \tag{1.7}$$

The definition of $\|A\|_1$ does not depend on the choice of the orthonormal basis $\{e_i\}_{i\in I}$. We denote by $\mathcal{B}_1(H)$ the set of trace class operators in $\mathcal{B}(H)$.

The following proposition holds:

Proposition 1. *If* $A\in\mathcal{B}(H)$, *then the following are equivalent:*

(i) $A\in\mathcal{B}_1(H)$;

(ii) $|A|^{1/2}\in\mathcal{B}_2(H)$.

The following properties are also well known:

Theorem 2. *With the above notations:*

(i) We have

$$\|A\|_1=\|A^*\|_1\ \ \text{and}\ \ \|A\|_2\leq\|A\|_1 \tag{1.8}$$

for any $A\in\mathcal{B}_1(H)$;

(ii) $\mathcal{B}_1(H)$ *is an operator ideal in* $\mathcal{B}(H)$, *i.e.*

$$\mathcal{B}(H)\,\mathcal{B}_1(H)\,\mathcal{B}(H)\subseteq\mathcal{B}_1(H)\,;$$

(iii) We have

$$\mathcal{B}_2(H)\,\mathcal{B}_2(H)=\mathcal{B}_1(H)\,;$$

(iv) We have

$$\|A\|_1=\sup\left\{\langle A,B\rangle_2\ \mid\ B\in\mathcal{B}_2(H)\,,\ \ \|B\|_2\leq 1\right\};$$

(v) $(\mathcal{B}_1(H)\,,\|\cdot\|_1)$ *is a Banach space.*

We define the *trace* of a trace class operator $A\in\mathcal{B}_1(H)$ to be

$$\operatorname{tr}(A):=\sum_{i\in I}\langle Ae_i,e_i\rangle\,, \tag{1.9}$$

where $\{e_i\}_{i\in I}$ an orthonormal basis of H. Note that this coincides with the usual definition of the trace if H is finite-dimensional. We observe that the series (1.9) converges absolutely and it is independent from the choice of basis.

The following result collects some properties of the trace:

Theorem 3. *We have:*

(i) If $A\in\mathcal{B}_1(H)$ *then* $A^*\in\mathcal{B}_1(H)$ *and*

$$\operatorname{tr}(A^*)=\overline{\operatorname{tr}(A)}; \tag{1.10}$$

(ii) If $A\in\mathcal{B}_1(H)$ *and* $T\in\mathcal{B}(H)$, *then* AT, $TA\in\mathcal{B}_1(H)$,

$$\operatorname{tr}(AT)=\operatorname{tr}(TA)\ \ \text{and}\ \ |\operatorname{tr}(AT)|\leq\|A\|_1\,\|T\|\,; \tag{1.11}$$

(iii) $\operatorname{tr}(\cdot)$ *is a bounded linear functional on* $\mathcal{B}_1(H)$ *with* $\|\operatorname{tr}\|=1$;

(iv) If A, $B\in\mathcal{B}_2(H)$ *then* AB, $BA\in\mathcal{B}_1(H)$ *and* $\operatorname{tr}(AB)=\operatorname{tr}(BA)$.

Now, if we assume that $P \geq 0$ and $P \in \mathcal{B}_1(H)$, then for all $T \in \mathcal{B}(H)$, PT, $TP \in \mathcal{B}_1(H)$ and $\operatorname{tr}(PT) = \operatorname{tr}(TP)$. Also, since $P^{1/2} \in \mathcal{B}_2(H)$, $TP^{1/2} \in \mathcal{B}_2(H)$, hence $P^{1/2}TP^{1/2}$ and $TP^{1/2}P^{1/2} = TP \in \mathcal{B}_1(H)$ with $\operatorname{tr}\left(P^{1/2}TP^{1/2}\right) = \operatorname{tr}(TP)$. Therefore, if $P \geq 0$ and $P \in \mathcal{B}_1(H)$,

$$\operatorname{tr}(PT) = \operatorname{tr}(TP) = \operatorname{tr}\left(P^{1/2}TP^{1/2}\right)$$

for all $T \in \mathcal{B}(H)$.

If $T \geq 0$, then $P^{1/2}TP^{1/2} \geq 0$, which implies that $\operatorname{tr}(PT) \geq 0$ that shows that the functional $\mathcal{B}(H) \ni T \longmapsto \operatorname{tr}(PT)$ is linear and isotonic functional. Also, by (1.11), if $T_n \to T$ for $n \to \infty$ in $\mathcal{B}(H)$ then $\lim_{n\to\infty} \operatorname{tr}(PT_n) = \operatorname{tr}(PT)$, namely $\mathcal{B}(H) \ni T \longmapsto \operatorname{tr}(PT)$ is also continuous in the norm topology.

For a survey on recent trace inequalities see [4] and the references therein.

Now, for a given $P \geq 0$ with $P \in \mathcal{B}_1(H)$ and $\operatorname{tr}(P) = 1$, we define the *P-determinant* of the positive invertible operator A by

$$\Delta_P(A) := \exp\operatorname{tr}(P\ln A) = \exp\operatorname{tr}((\ln A)P) = \exp\operatorname{tr}\left(P^{1/2}(\ln A)P^{1/2}\right). \tag{1.12}$$

In this paper we present some fundamental properties of this determinant. Among others we show that

$$1 \leq \frac{\operatorname{tr}(PA)}{\Delta_P(A)} \leq \exp\left[\operatorname{tr}(PA)\operatorname{tr}\left(PA^{-1}\right) - 1\right]$$

and

$$1 \leq \frac{\Delta_P(A)}{\left[\operatorname{tr}(PA^{-1})\right]^{-1}} \leq \exp\left[\operatorname{tr}\left(PA^{-1}\right)\operatorname{tr}(PA) - 1\right],$$

for $A > 0$ and $P \geq 0$ with $P \in \mathcal{B}_1(H)$ and $\operatorname{tr}(P) = 1$.

2. Main Properties

Assume that $P \geq 0$ with $P \in \mathcal{B}_1(H)$ and $\operatorname{tr}(P) = 1$. We observe that we have the following elementary properties

(i) *continuity*: the map $A \to \Delta_P(A)$ is norm continuous;
(ii) *power equality:* $\Delta_P(A^t) = \Delta_P(A)^t$ for all $t > 0$;
(iii) *homogeneity*: $\Delta_P(tA) = t\Delta_x(A)$ and $\Delta_P(tI) = t$ for all $t > 0$;
(iv) *monotonicity*: $0 < A \leq B$ implies $\Delta_P(A) \leq \Delta_P(B)$.

We have the following result:

Theorem 4. *Let $P \geq 0$ with $P \in \mathcal{B}_1(H)$ and $\operatorname{tr}(P) = 1$, then for all A, $B > 0$ and $t \in [0,1]$,*

$$\Delta_P((1-t)A + tB) \geq [\Delta_P(A)]^{1-t}[\Delta_P(B)]^t. \tag{2.1}$$

Proof. By the operator concavity of ln on $(0,\infty)$ we have

$$\ln((1-t)A + tB) \geq (1-t)\ln A + t\ln B$$

for all A, $B > 0$ and $t \in [0,1]$.

If we multiply both sides by $P^{1/2}$ then we get

$$P^{1/2}[\ln((1-t)A + tB)]P^{1/2} \geq (1-t)P^{1/2}(\ln A)P^{1/2} + tP^{1/2}(\ln B)P^{1/2}$$

for all A, $B > 0$ and $t \in [0,1]$.

By taking the trace, we derive

$$\operatorname{tr}(P\ln((1-t)A + tB)) \geq (1-t)\operatorname{tr}(P\ln A) + t\operatorname{tr}(P\ln B)$$

for all $A,\ B>0$ and $t\in[0,1]$.

Now, if we take the exponential, then we obtain

$$\begin{aligned}\exp\left[\operatorname{tr}\left(P\ln\left((1-t)A+tB\right)\right)\right] &\geq \exp\left[(1-t)\operatorname{tr}(P\ln A)+t\operatorname{tr}(P\ln B)\right]\\ &= \left[\exp\left(\operatorname{tr}(P\ln A)\right)\right]^{1-t}\left[\exp(P\ln B)\right]^{t}\end{aligned}$$

for all $A,\ B>0$ and $t\in[0,1]$, and the inequality (2.1) is proved. □

We define the *logarithmic mean* of two positive numbers $a,\ b$ by

$$L(a,b):=\begin{cases}\frac{b-a}{\ln b-\ln a} \text{ if } b\neq a,\\ \\ a \text{ if } b=a.\end{cases}$$

The following Hermite-Hadamard type integral inequalities hold:

Corollary 1. *Let $P\geq 0$ with $P\in\mathcal{B}_1(H)$ and $\operatorname{tr}(P)=1$, then for all $A,\ B>0$*

$$\int_0^1 \Delta_P((1-t)A+tB)dt\geq L\left(\Delta_P(A),\Delta_P(B)\right). \tag{2.2}$$

Proof. If we take the integral over $t\in[0,1]$ in (2.1), then we get

$$\begin{aligned}\int_0^1 \Delta_P((1-t)A+tB)dt &\geq \int_0^1 \left[\Delta_P(A)\right]^{1-t}\left[\Delta_P(B)\right]^{t}dt\\ &= L\left(\Delta_P(A),\Delta_P(B)\right)\end{aligned}$$

for all $A,\ B>0$, which proves (2.2). □

Corollary 2. *Let $P\geq 0$ with $P\in\mathcal{B}_1(H)$ and $\operatorname{tr}(P)=1$, then for all $A,\ B>0$*

$$\Delta_P\left(\frac{A+B}{2}\right)\geq\int_0^1\left[\Delta_P\left((1-t)A+tB\right)\right]^{1/2}\left[\Delta_P\left(tA+(1-t)B\right)\right]^{1/2}dt. \tag{2.3}$$

Proof. We get from (2.1) for $t=1/2$ that

$$\Delta_P\left(\frac{A+B}{2}\right)\geq\left[\Delta_P(A)\right]^{1/2}\left[\Delta_P(B)\right]^{1/2}.$$

If we replace A by $(1-t)A+tB$ and B by $tA+(1-t)B$ we obtain

$$\Delta_P\left(\frac{A+B}{2}\right)\geq\left[\Delta_P\left((1-t)A+tB\right)\right]^{1/2}\left[\Delta_P\left(tA+(1-t)B\right)\right]^{1/2}.$$

By taking the integral, we derive the desired result (2.3). □

We can provide some upper and lower bounds for $\Delta_P(A)$ as follows:

Theorem 5. *Let $P\geq 0$ with $P\in\mathcal{B}_1(H)$ and $\operatorname{tr}(P)=1$, then for all $A>0$ and $a>0$ we have the double inequality*

$$a\exp\left[1-a\operatorname{tr}\left(PA^{-1}\right)\right]\leq\Delta_P(A)\leq a\exp\left[a^{-1}\operatorname{tr}(PA)-1\right]. \tag{2.4}$$

In particular

$$1\leq\frac{\operatorname{tr}(PA)}{\Delta_P(A)}\leq\exp\left[\operatorname{tr}(PA)\operatorname{tr}\left(PA^{-1}\right)-1\right] \tag{2.5}$$

and

$$1\leq\frac{\Delta_P(A)}{\left[\operatorname{tr}\left(PA^{-1}\right)\right]^{-1}}\leq\exp\left[\operatorname{tr}\left(PA^{-1}\right)\operatorname{tr}(PA)-1\right]. \tag{2.6}$$

The first inequalities in (2.5) and 2.6) are best possible from (2.4).

Proof. It is well know that, if f is differentiable convex on an interval I, then for all u, $v \in I$ we have

$$f'(u)(u-v) \geq f(u) - f(v) \geq f'(v)(u-v).$$

If we write this inequality for $f(t) = -\ln t$, we get for all $u, v > 0$ that

$$-\frac{1}{u}(u-v) \geq \ln v - \ln u \geq -\frac{1}{v}(u-v),$$

namely

$$\frac{1}{u}(u-v) \leq \ln u - \ln v \leq \frac{1}{v}(u-v) \tag{2.7}$$

for $u, v > 0$.

If we use the continuous functional calculus for selfadjoint operators, we have from (2.7) that

$$I - aA^{-1} \leq \ln A - \ln a \leq a^{-1}A - I \tag{2.8}$$

for the operator $A > 0$ and positive number a.

If we multiply both sides of (2.8) by $P^{1/2}$ we get

$$P - aP^{1/2}A^{-1}P^{1/2} \leq P^{1/2}(\ln A)P^{1/2} - (\ln a)P \leq a^{-1}P^{1/2}AP^{1/2} - P. \tag{2.9}$$

for the operator $A > 0$ and positive number a.

If we take the trace in (2.9), then we get

$$\begin{aligned} \operatorname{tr} P - a \operatorname{tr}\left(P^{1/2}A^{-1}P^{1/2}\right) &\leq \operatorname{tr}\left(P^{1/2}(\ln A)P^{1/2}\right) - (\ln a)\operatorname{tr} P \\ &\leq a^{-1}\operatorname{tr}\left(P^{1/2}AP^{1/2}\right) - \operatorname{tr} P, \end{aligned}$$

namely

$$1 - a\operatorname{tr}\left(PA^{-1}\right) \leq \operatorname{tr}((\ln A)P) - \ln a \leq a^{-1}\operatorname{tr}(PA) - 1. \tag{2.10}$$

Now, if we take the exponential in (2.10) we derive

$$\exp\left[1 - a\operatorname{tr}\left(PA^{-1}\right)\right] \leq \frac{\exp\left[\operatorname{tr}((\ln A)P)\right]}{a} \leq \exp\left[a^{-1}\operatorname{tr}(PA) - 1\right],$$

for the operator $A > 0$ and positive number a, which is the desired inequalities (2.4).

The inequality (2.5) follows by (2.4) on taking $a = \operatorname{tr}(PA)$ while (2.6) follows by (2.4) for $a = \left[\operatorname{tr}\left(PA^{-1}\right)\right]^{-1}$.

Now, consider the function $f(t) = t\exp\left[t^{-1}\operatorname{tr}(PA) - 1\right]$, $t > 0$, then

$$\begin{aligned} f'(t) &= \exp\left[t^{-1}\operatorname{tr}(PA) - 1\right] + t\exp\left[t^{-1}\operatorname{tr}(PA) - 1\right]\left(-\frac{\operatorname{tr}(PA)}{t^2}\right) \\ &= \exp\left[t^{-1}\operatorname{tr}(PA) - 1\right]\left(1 - \frac{\operatorname{tr}(PA)}{t}\right). \end{aligned}$$

We have that $f'(t_0) = 0$ for $t_0 = \operatorname{tr}(PA)$ which shows that f is strictly decreasing on $(0, \operatorname{tr}(PA))$ and strictly increasing on $(\operatorname{tr}(PA), \infty)$. Therefore

$$\inf_{t\in(0,\infty)} f(t) = f(\operatorname{tr}(PA)) = \operatorname{tr}(PA),$$

which proves that the first inequality in (2.5) is best possible.

Also if we take the function $g(t) = t\exp\left[1 - t\operatorname{tr}\left(PA^{-1}\right)\right]$, $t > 0$, then

$$\begin{aligned} g'(t) &= \exp\left[1 - t\operatorname{tr}\left(PA^{-1}\right)\right] - t\operatorname{tr}\left(PA^{-1}\right)\exp\left[1 - t\operatorname{tr}\left(PA^{-1}\right)\right] \\ &= \exp\left[1 - t\operatorname{tr}\left(PA^{-1}\right)\right]\left(1 - t\operatorname{tr}\left(PA^{-1}\right)\right), \end{aligned}$$

which shows that g is strictly increasing on $\left(0, \left[\operatorname{tr}\left(PA^{-1}\right)\right]^{-1}\right)$ and strictly decreasing on $\left(\left[\operatorname{tr}\left(PA^{-1}\right)\right]^{-1}, \infty\right)$, therefore

$$\sup_{t\in(0,\infty)} g(t) = g\left(\left[\operatorname{tr}\left(PA^{-1}\right)\right]^{-1}\right) = \left[\operatorname{tr}\left(PA^{-1}\right)\right]^{-1},$$

which shows that the first inequality is best possible in (2.6) □

Corollary 3. *For any* A, $B > 0$ *and* $a > 0$, *we have*

$$\begin{aligned} a\exp\left(1 - \frac{1}{2}a\operatorname{tr}\left[P\left(A^{-1} + B^{-1}\right)\right]\right) \\ \le \int_0^1 \Delta_P((1-t)A + tB)\,dt \le a\psi(a, P, A, B), \end{aligned} \tag{2.11}$$

where

$$\psi(a,P,A,B) := \begin{cases} \frac{\exp\left(a^{-1}(\operatorname{tr}(PB)-\operatorname{tr}(PA))+\operatorname{tr}(PA)-1\right)-\exp(\operatorname{tr}(PA)-1)}{a^{-1}(\operatorname{tr}(PB)-\operatorname{tr}(PA))} \\ \text{if } \operatorname{tr}(PB) \ne \operatorname{tr}(PA), \\ \\ \exp(\operatorname{tr}(PA) - 1) \quad \text{if } \operatorname{tr}(PB) \ne \operatorname{tr}(PA). \end{cases}$$

In particular, we have

$$\left(\operatorname{tr}\left[P\left(\frac{A^{-1} + B^{-1}}{2}\right)\right]\right)^{-1} \le \int_0^1 \Delta_P((1-t)A + tB)\,dt. \tag{2.12}$$

Proof. From (2.4) we have

$$\begin{aligned} &a\exp\left[1 - a\operatorname{tr}\left(P\left[(1-t)A + tB\right]^{-1}\right)\right] \\ &\le \Delta_P((1-t)A + tB) \\ &\le a\exp\left[a^{-1}\operatorname{tr}\left(P\left[(1-t)A + tB\right]\right) - 1\right], \end{aligned}$$

for all $t \in [0,1]$, which gives by integration that

$$\begin{aligned} &a\int_0^1 \exp\left[1 - a\operatorname{tr}\left(P\left[(1-t)A + tB\right]^{-1}\right)\right]dt \\ &\le \int_0^1 \Delta_P((1-t)A + tB)\,dt \\ &\le a\int_0^1 \exp\left[a^{-1}\operatorname{tr}\left(P\left[(1-t)A + tB\right]\right) - 1\right]dt. \end{aligned}$$

Now, observe that

$$\begin{aligned}
&\int_0^1 \exp\left[a^{-1}\operatorname{tr}\left(P\left[(1-t)A+tB\right]\right)-1\right]dt\\
&=\int_0^1 \exp\left[a^{-1}\left[(1-t)\operatorname{tr}(PA)+t\operatorname{tr}(PB)\right]-1\right]dt\\
&=\int_0^1 \exp\left[a^{-1}\left(\operatorname{tr}(PB)-\operatorname{tr}(PA)\right)t+\operatorname{tr}(PA)-1\right]dt.
\end{aligned}$$

Since

$$\int_0^1 \exp(\alpha t+\beta)\,dt=\begin{cases}\frac{\exp(\alpha+\beta)-\exp\beta}{\alpha} \text{ if } \alpha\neq 0,\\ \\ \exp\beta \text{ if } \alpha=0\end{cases}$$

hence

$$\int_0^1 \exp\left[a^{-1}\left(\operatorname{tr}(PB)-\operatorname{tr}(PA)\right)t+\operatorname{tr}(PA)-1\right]dt=\psi(a,P,A,B),$$

which proves the second inequality in (2.11).

Using the Jensen inequality for exp we have

$$\begin{aligned}
&\int_0^1 \exp\left[1-a\operatorname{tr}\left(P\left[(1-t)A+tB\right]^{-1}\right)\right]dt\\
&\geq \exp\left(\int_0^1\left[1-a\operatorname{tr}\left(P\left[(1-t)A+tB\right]^{-1}\right)\right]dt\right)\\
&=\exp\left(\left[1-a\operatorname{tr}\left(P\left(\int_0^1\left[(1-t)A+tB\right]^{-1}dt\right)\right)\right]\right),
\end{aligned}$$

where for the last equality we used the continuity of the map $\mathcal{B}(H)\ni T\longmapsto \operatorname{tr}(PT)$.

The function $g(u)=u^{-1}$, $u>0$ is operator convex and by Hermite-Hadamard inequality for operator convex functions, we have, see [1]

$$\left(\frac{A+B}{2}\right)^{-1}\leq\int_0^1\left((1-t)A+tB\right)^{-1}dt\leq\frac{1}{2}\left(A^{-1}+B^{-1}\right)$$

for A, $B>0$.

By multiplying both sides by $P^{1/2}$ we get

$$\begin{aligned}
P^{1/2}\left(\frac{A+B}{2}\right)^{-1}P^{1/2}&\leq\int_0^1 P^{1/2}\left((1-t)A+tB\right)^{-1}P^{1/2}dt\\
&\leq\frac{1}{2}P^{1/2}\left(A^{-1}+B^{-1}\right)P^{1/2}
\end{aligned}$$

and by taking the trace, we get

(2.13)
$$\operatorname{tr}\left[P\left(\frac{A+B}{2}\right)^{-1}\right]\leq\int_0^1\operatorname{tr}\left[P\left((1-t)A+tB\right)^{-1}\right]dt\leq\frac{1}{2}\operatorname{tr}\left[P\left(A^{-1}+B^{-1}\right)\right].$$

This gives,

$$1-a\operatorname{tr}\left(P\left(\int_0^1\left[(1-t)A+tB\right]^{-1}dt\right)\right)\geq 1-\frac{1}{2}a\operatorname{tr}\left[P\left(A^{-1}+B^{-1}\right)\right],$$

which implies that

$$\exp\left(\left[1-a\operatorname{tr}\left(P\left(\int_0^1\left[(1-t)A+tB\right]^{-1}dt\right)\right)\right]\right)$$
$$\geq\exp\left(1-\frac{1}{2}a\operatorname{tr}\left[P\left(A^{-1}+B^{-1}\right)\right]\right)$$

and the first inequality is also obtained. □

3. Related Results

We also have the following upper bounds:

Proposition 2. *Let $P\geq 0$ with $P\in\mathcal{B}_1(H)$ and $\operatorname{tr}(P)=1$. If A satisfies the condition*

$$0<mI\leq A\leq MI \tag{3.1}$$

for some constants $0<m<M$, then

$$\begin{aligned}1\leq\frac{\operatorname{tr}(PA)}{\Delta_P(A)}&\leq\exp\left[\operatorname{tr}(PA)\operatorname{tr}\left(PA^{-1}\right)-1\right]\\&\leq\exp\left(\frac{1}{2}\frac{M-m}{mM}\operatorname{tr}\left(\left|A-\operatorname{tr}(PA)\right|P\right)\right)\\&\leq\exp\left(\frac{1}{2}\frac{M-m}{mM}\left[\operatorname{tr}\left(PA^2\right)-\left(\operatorname{tr}(PA)\right)^2\right]^{1/2}\right)\\&\leq\exp\left(\frac{1}{4}\frac{(M-m)^2}{mM}\right)\end{aligned}\tag{3.2}$$

and

$$\begin{aligned}1\leq\frac{\Delta_P(A)}{\left[\operatorname{tr}\left(PA^{-1}\right)\right]^{-1}}&\leq\exp\left[\operatorname{tr}\left(PA^{-1}\right)\operatorname{tr}(PA)-1\right]\\&\leq\exp\left(\frac{1}{2}\frac{M-m}{mM}\operatorname{tr}\left(\left|A-\operatorname{tr}(PA)\right|P\right)\right)\\&\leq\exp\left(\frac{1}{2}\frac{M-m}{mM}\left[\operatorname{tr}\left(PA^2\right)-\left(\operatorname{tr}(PA)\right)^2\right]^{1/2}\right)\\&\leq\exp\left(\frac{1}{4}\frac{(M-m)^2}{mM}\right).\end{aligned}\tag{3.3}$$

Proof. In [2] we proved among others that, if $\operatorname{Sp}(S)\subseteq[m,M]\subset(0,\infty)$ and $Q\in\mathcal{B}_1(H)$ and $Q>0$, then

$$\begin{aligned}0&\leq\frac{\operatorname{tr}(QS)\operatorname{tr}\left(QS^{-1}\right)}{\left[\operatorname{tr}(Q)\right]^2}-1\\&\leq\frac{1}{2}\frac{M-m}{mM}\frac{1}{\operatorname{tr}(Q)}\operatorname{tr}\left(\left|S-\frac{\operatorname{tr}(QS)}{\operatorname{tr}(Q)}\right|Q\right)\\&\leq\frac{1}{2}\frac{M-m}{mM}\left[\frac{\operatorname{tr}\left(QS^2\right)}{\operatorname{tr}(Q)}-\left(\frac{\operatorname{tr}(QS)}{\operatorname{tr}(Q)}\right)^2\right]^{1/2}\\&\leq\frac{1}{4}\frac{(M-m)^2}{mM}.\end{aligned}$$

By taking $S = A$ and $Q = P$ we obtain the desired bounds in (3.2) and (3.3). $\square$

Corollary 4. *Let $P \geq 0$ with $P \in \mathcal{B}_1(H)$ and $\operatorname{tr}(P) = 1$. If A, B satisfy the conditions $0 < m \leq A$, $B \leq M$, then*

$$\exp\left(-\frac{1}{4}\frac{(M-m)^2}{mM}\right)\operatorname{tr}\left(P\left(\frac{A+B}{2}\right)\right) \leq \int_0^1 \Delta_P\left((1-t)A + tB\right)dt \leq \operatorname{tr}\left(P\left(\frac{A+B}{2}\right)\right) \tag{3.4}$$

and

$$\exp\left(-\frac{1}{4}\frac{(M-m)^2}{mM}\right)\operatorname{tr}\left[P\left(\frac{A+B}{2}\right)^{-1}\right] \leq \int_0^1 \left[\Delta_P\left((1-t)A + tB\right)\right]^{-1}dt \leq \frac{1}{2}\operatorname{tr}\left[P\left(A^{-1}+B^{-1}\right)\right]. \tag{3.5}$$

Proof. From (3.2) we get

$$\begin{aligned}\Delta_P\left((1-t)A + tB\right) &\leq \operatorname{tr}\left(P\left((1-t)A + tB\right)\right) \\ &\leq \exp\left(\frac{1}{4}\frac{(M-m)^2}{mM}\right)\Delta_P\left((1-t)A + tB\right)\end{aligned}$$

for all $t \in [0,1]$.

If we take the integral, then we get

$$\begin{aligned}&\int_0^1 \Delta_P\left((1-t)A + tB\right)dt \\ &\leq \operatorname{tr}\left(P\left(\frac{A+B}{2}\right)\right) \\ &\leq \exp\left(\frac{1}{4}\frac{(M-m)^2}{mM}\right)\int_0^1 \Delta_P\left((1-t)A + tB\right)dt,\end{aligned}$$

which proves (3.4).

From (3.3) we get

$$\left[\operatorname{tr}\left(PA^{-1}\right)\right]^{-1} \leq \Delta_P(A) \leq \exp\left(\frac{1}{4}\frac{(M-m)^2}{mM}\right)\left[\operatorname{tr}\left(PA^{-1}\right)\right]^{-1},$$

namely

$$\operatorname{tr}\left(PA^{-1}\right) \geq \left[\Delta_P(A)\right]^{-1} \geq \exp\left(-\frac{1}{4}\frac{(M-m)^2}{mM}\right)\operatorname{tr}\left(PA^{-1}\right),$$

which implies that

$$\begin{aligned}\operatorname{tr}\left(P\left((1-t)A + tB\right)^{-1}\right) &\geq \left[\Delta_P\left((1-t)A + tB\right)\right]^{-1} \\ &\geq \exp\left(-\frac{1}{4}\frac{(M-m)^2}{mM}\right)\operatorname{tr}\left(P\left((1-t)A + tB\right)^{-1}\right).\end{aligned}$$

By taking the integral, we get

$$\begin{aligned}
&\int_0^1 \operatorname{tr}\left(P\left((1-t)A+tB\right)^{-1}\right)dt \\
&\geq \int_0^1 \left[\Delta_P\left((1-t)A+tB\right)\right]^{-1}dt \\
&\geq \exp\left(-\frac{1}{4}\frac{(M-m)^2}{mM}\right)\int_0^1 \operatorname{tr}\left(P\left((1-t)A+tB\right)^{-1}\right)dt.
\end{aligned} \tag{3.6}$$

and since

$$\operatorname{tr}\left[P\left(\frac{A+B}{2}\right)^{-1}\right] \leq \int_0^1 \operatorname{tr}\left[P\left((1-t)A+tB\right)^{-1}\right]dt \leq \frac{1}{2}\operatorname{tr}\left[P\left(A^{-1}+B^{-1}\right)\right],$$

hence by (3.6) we get (3.5). □

Some different upper bounds are as follows:

Proposition 3. *With the assumptions of Proposition 2 we have*

$$\begin{aligned}
1 &\leq \frac{\operatorname{tr}(PA)}{\Delta_P(A)} \leq \exp\left[\operatorname{tr}(PA)\operatorname{tr}\left(PA^{-1}\right)-1\right] \\
&\leq \exp\left[\operatorname{tr}(PA)\operatorname{tr}\left(PA^{-2}\right)-\operatorname{tr}\left(PA^{-1}\right)\right] \\
&\leq \exp\left(\operatorname{tr}(PA)\times\begin{cases}\frac{1}{2}\frac{M^2-m^2}{m^2M^2}\operatorname{tr}\left(P\left|A-\operatorname{tr}(PA)1_H\right|\right) \\ \\ \frac{1}{2}(M-m)\operatorname{tr}\left(P\left|A^{-2}-\operatorname{tr}\left(PA^{-2}\right)1_H\right|\right)\end{cases}\right) \\
&\leq \exp\left(\operatorname{tr}(PA)\times\begin{cases}\frac{1}{2}\frac{M^2-m^2}{m^2M^2}\left[\operatorname{tr}\left(PA^2\right)-\left(\operatorname{tr}(PA)\right)^2\right]^{1/2} \\ \\ \frac{1}{2}(M-m)\left[\operatorname{tr}\left(PA^{-4}\right)-\left(\operatorname{tr}\left(PA^{-2}\right)\right)^2\right]^{1/2}\end{cases}\right) \\
&\leq \exp\left[\frac{1}{4}\left(1+\frac{m}{M}\right)\left(\frac{M}{m}-1\right)^2\frac{\operatorname{tr}(PA)}{M}\right] \\
&\leq \exp\left[\frac{1}{4}\left(1+\frac{m}{M}\right)\left(\frac{M}{m}-1\right)^2\right]
\end{aligned}$$

and.

$$\begin{aligned}
1 &\leq \frac{\Delta_P(A)}{\left[\operatorname{tr}\left(PA^{-1}\right)\right]^{-1}} \leq \exp\left[\operatorname{tr}\left(PA^{-1}\right)\operatorname{tr}(PA)-1\right] \\
&\leq \exp\left[\operatorname{tr}(PA)\operatorname{tr}\left(PA^{-2}\right)-\operatorname{tr}\left(PA^{-1}\right)\right] \\
&\leq \exp\left(\operatorname{tr}(PA)\times\begin{cases}\frac{1}{2}\frac{M^2-m^2}{m^2M^2}\operatorname{tr}\left(P\left|A-\operatorname{tr}(PA)1_H\right|\right) \\ \\ \frac{1}{2}(M-m)\operatorname{tr}\left(P\left|A^{-2}-\operatorname{tr}\left(PA^{-2}\right)1_H\right|\right)\end{cases}\right)
\end{aligned}$$

$$\leq \exp\left(\operatorname{tr}(PA)\times\left\{\begin{array}{l}\frac{1}{2}\frac{M^2-m^2}{m^2M^2}\left[\operatorname{tr}\left(PA^2\right)-\left(\operatorname{tr}(PA)\right)^2\right]^{1/2}\\ \\ \frac{1}{2}(M-m)\left[\operatorname{tr}\left(PA^{-4}\right)-\left(\operatorname{tr}\left(PA^{-2}\right)\right)^2\right]^{1/2}\end{array}\right.\right)$$
$$\leq \exp\left[\frac{1}{4}\left(1+\frac{m}{M}\right)\left(\frac{M}{m}-1\right)^2\frac{\operatorname{tr}(PA)}{M}\right]$$
$$\leq \exp\left[\frac{1}{4}\left(1+\frac{m}{M}\right)\left(\frac{M}{m}-1\right)^2\right]$$

Proof. Let A be a selfadjoint operator on the Hilbert space H and assume that $Sp(A)\subseteq[m,M]$ for some scalars m, M with $m<M$. If f is a continuously differentiable convex function on $[m,M]$ and $P\in\mathcal{B}_1(H)\setminus\{0\}$, $P\geq 0$ and $\operatorname{tr}(P)=1$, then we have [3]

$$\begin{aligned}0&\leq\operatorname{tr}(Pf(A))-f(\operatorname{tr}(PA))\\&\leq\operatorname{tr}(Pf'(A)A)-\operatorname{tr}(PA)\operatorname{tr}(Pf'(A))\end{aligned}\tag{3.7}$$

$$\leq\left\{\begin{array}{l}\frac{1}{2}[f'(M)-f'(m)]\operatorname{tr}(P|A-\operatorname{tr}(PA)1_H|)\\ \\ \frac{1}{2}(M-m)\operatorname{tr}(P|f'(A)-\operatorname{tr}(Pf'(A))1_H|)\end{array}\right.$$
$$\leq\left\{\begin{array}{l}\frac{1}{2}[f'(M)-f'(m)]\left[\operatorname{tr}\left(PA^2\right)-\left(\operatorname{tr}(PA)\right)^2\right]^{1/2}\\ \\ \frac{1}{2}(M-m)\left[\operatorname{tr}\left(P[f'(A)]^2\right)-\left(\operatorname{tr}(Pf'(A))\right)^2\right]^{1/2}\end{array}\right.$$
$$\leq\frac{1}{4}[f'(M)-f'(m)](M-m).$$

Now, if we take $f(t)=t^{-1}$, $t>0$, then we get

$$\begin{aligned}0&\leq\operatorname{tr}\left(PA^{-1}\right)-\frac{1}{\operatorname{tr}(PA)}\\&\leq\operatorname{tr}(PA)\operatorname{tr}\left(PA^{-2}\right)-\operatorname{tr}\left(PA^{-1}\right)\\&\leq\left\{\begin{array}{l}\frac{1}{2}\frac{M^2-m^2}{m^2M^2}\operatorname{tr}(P|A-\operatorname{tr}(PA)1_H|)\\ \\ \frac{1}{2}(M-m)\operatorname{tr}\left(P\left|A^{-2}-\operatorname{tr}\left(PA^{-2}\right)1_H\right|\right)\end{array}\right.\\&\leq\left\{\begin{array}{l}\frac{1}{2}\frac{M^2-m^2}{m^2M^2}\left[\operatorname{tr}\left(PA^2\right)-\left(\operatorname{tr}(PA)\right)^2\right]^{1/2}\\ \\ \frac{1}{2}(M-m)\left[\operatorname{tr}\left(PA^{-4}\right)-\left(\operatorname{tr}\left(PA^{-2}\right)\right)^2\right]^{1/2}\end{array}\right.\\&\leq\frac{1}{4}\frac{(M+m)(M-m)^2}{m^2M^2},\end{aligned}$$

which gives that

$$\begin{aligned}0&\leq\operatorname{tr}\left(PA^{-1}\right)\operatorname{tr}(PA)-1\\&\leq\left[\operatorname{tr}(PA)\operatorname{tr}\left(PA^{-2}\right)-\operatorname{tr}\left(PA^{-1}\right)\right]\end{aligned}$$

$$\leq \operatorname{tr}(PA) \times \begin{cases} \frac{1}{2}\frac{M^2-m^2}{m^2M^2} \operatorname{tr}\left(P\left|A - \operatorname{tr}(PA) 1_H\right|\right) \\ \\ \frac{1}{2}(M-m) \operatorname{tr}\left(P\left|A^{-2} - \operatorname{tr}\left(PA^{-2}\right) 1_H\right|\right) \end{cases}$$

$$\leq \operatorname{tr}(PA) \times \begin{cases} \frac{1}{2}\frac{M^2-m^2}{m^2M^2} \left[\operatorname{tr}\left(PA^2\right) - \left(\operatorname{tr}(PA)\right)^2\right]^{1/2} \\ \\ \frac{1}{2}(M-m) \left[\operatorname{tr}\left(PA^{-4}\right) - \left(\operatorname{tr}\left(PA^{-2}\right)\right)^2\right]^{1/2} \end{cases}$$

$$\leq \frac{1}{4}\frac{M^2-m^2}{m^2M^2}(M-m)\operatorname{tr}(PA) = \frac{1}{4}\frac{M+m}{m^2M^2}(M-m)^2 \operatorname{tr}(PA)$$

$$\leq \frac{1}{4}\left(1+\frac{m}{M}\right)\left(\frac{M}{m}-1\right)^2 \frac{\operatorname{tr}(PA)}{M}$$

$$\leq \frac{1}{4}\left(1+\frac{m}{M}\right)\left(\frac{M}{m}-1\right)^2.$$

□

Remark 1. *Observe that for $0 < m < M$, the difference*

$$D(m,M) := \frac{1}{4}\frac{(M-m)^2}{mM} - \frac{1}{4}\frac{(M+m)(M-m)^2}{m^2M^2} = \frac{1}{4}\frac{(M-m)^2}{mM}\left(1 - \frac{M+m}{mM}\right)$$

takes both negative and positive values showing that neither of the absolute upper bounds from Propositions 2 and 3 is always best.

Proposition 4. *With the assumptions of Proposition 2, we have*

$$1 \leq \frac{\operatorname{tr}(PA)}{\Delta_P(A)} \leq \exp\left[\operatorname{tr}(PA)\operatorname{tr}\left(PA^{-1}\right) - 1\right] \leq \exp\left[\frac{\left(\sqrt{M}-\sqrt{m}\right)^2}{mM}\operatorname{tr}(PA)\right] \leq \exp\left[\frac{\left(\sqrt{M}-\sqrt{m}\right)^2}{m}\right] \tag{3.8}$$

and

$$1 \leq \frac{\Delta_P(A)}{\left[\operatorname{tr}\left(PA^{-1}\right)\right]^{-1}} \leq \exp\left[\operatorname{tr}\left(PA^{-1}\right)\operatorname{tr}(PA) - 1\right] \leq \exp\left[\frac{\left(\sqrt{M}-\sqrt{m}\right)^2}{mM}\operatorname{tr}(PA)\right] \leq \exp\left[\frac{\left(\sqrt{M}-\sqrt{m}\right)^2}{m}\right]. \tag{3.9}$$

Proof. If $t \in [m, M] \subset (0, \infty)$, then obviously

$$(M-t)\left(m^{-1} - t^{-1}\right) \geq 0,$$

which is equivalent to

$$m + M \geq mMt^{-1} + t$$

for all $t \in [m, M]$.

Using the functional calculus for selfadjoint operators, we then get

$$(m+M) I \geq mMA^{-1} + A$$

for $0 < mI \le A \le MI$.

If we multiply both sides with $P^{1/2}$ we get

$$(m+M)\, P \ge mMP^{1/2}A^{-1}P^{1/2} + P^{1/2}AP^{1/2}$$

and is we take the trace, then we get

$$m + M \ge mM \operatorname{tr}\left(PA^{-1}\right) + \operatorname{tr}(PA),$$

namely

$$\frac{m+M}{mM} \ge \operatorname{tr}\left(PA^{-1}\right) + \frac{\operatorname{tr}(PA)}{mM}.$$

This gives

$$\begin{aligned} &\operatorname{tr}\left(PA^{-1}\right) - [\operatorname{tr}(PA)]^{-1} \\ &\le \frac{1}{m} + \frac{1}{M} - \frac{1}{mM}\operatorname{tr}(PA) - [\operatorname{tr}(PA)]^{-1} \\ &= \left(\frac{1}{\sqrt{m}} - \frac{1}{\sqrt{M}}\right)^2 - \left(\frac{1}{\sqrt{mM}}[\operatorname{tr}(PA)]^{1/2} - [\operatorname{tr}(PA)]^{-1/2}\right)^2 \le \left(\frac{1}{\sqrt{m}} - \frac{1}{\sqrt{M}}\right)^2, \end{aligned}$$

which implies, by multiplying with $\operatorname{tr}(PA)$ that

$$\operatorname{tr}\left(PA^{-1}\right)\operatorname{tr}(PA) - 1 \le \left(\frac{1}{\sqrt{m}} - \frac{1}{\sqrt{M}}\right)^2 \operatorname{tr}(PA) \le \frac{\left(\sqrt{M} - \sqrt{m}\right)^2}{m}.$$

The proposition is thus proved. □

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