UPPER AND LOWER BOUNDS FOR TRACE CLASS P-DETERMINANT OF POSITIVE OPERATORS IN HILBERT SPACES

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Abstract. Let H be a complex Hilbert space. For a given operator $P \geq 0$ with $P \in \mathcal{B}_1(H)$, the trace class associated to $\mathcal{B}(H)$ and $\operatorname{tr}(P) = 1$, we define the P-*determinant* of the positive invertible operator A by

$$\Delta_P(A) := \exp \operatorname{tr}(P \ln A).$$

In this paper we showed among others that

$$1 \leq \exp\left[\frac{1}{2M}\left(\operatorname{tr}(PA^2) - [\operatorname{tr}(PA)]^2\right)\right] \leq \frac{\operatorname{tr}(PA)}{\Delta_P(A)} \leq \exp\left[\frac{1}{2m}\left(\operatorname{tr}(PA^2) - [\operatorname{tr}(PA)]^2\right)\right]$$

and

$$1 \leq \exp\left[\frac{1}{2M}\left(\operatorname{tr}(PA^2) - [\operatorname{tr}(PA)]^2\right)\right] \leq \frac{\Delta_P(A)}{\operatorname{tr}(PA)\exp\left(\operatorname{tr}(PA)\operatorname{tr}(PA^{-1}) - 1\right)} \leq \exp\left[\frac{1}{2m}\left(\operatorname{tr}(PA^2) - [\operatorname{tr}(PA)]^2\right)\right],$$

where A is satisfying the condition $0 < mI \leq A \leq MI$ for some constants $m < M$ and $P \geq 0$ with $P \in \mathcal{B}_1(H)$ and $\operatorname{tr}(P) = 1$.

1. Introduction

In 1952, in the paper [6], B. Fuglede and R. V. Kadison introduced the determinant of a (invertible) operator and established its fundamental properties. The notion generalizes the usual determinant and can be considered for any operator in a finite von Neumann algebra (M, τ) with a faithful normal trace.

Let $T \in M$ be normal and $|T| := (T^*T)^{1/2}$ its modulus. By the spectral theorem one can represent T as an integral

$$T = \int_{\operatorname{Sp}(T)} \lambda dE(\lambda),$$

where $E(\lambda)$ is a projection valued measure and $\operatorname{Sp}(T)$ is the spectrum of T. The measure $\mu_T := \tau \circ E$ becomes a probability measure on the complex plane and has the support in the spectrum $\operatorname{Sp}(T)$.

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For any $T \in M$ the Fuglede-Kadison determinant (*FK-determinant*) is defined by

$$\Delta_{FK}(T) := \exp\left(\int_0^\infty \ln t d\mu_{|T|}\right).$$

If T is invertible, then

$$\Delta_{FK}(T) := \exp\left(\tau\left(\ln\left(|T|\right)\right)\right),$$

where $\ln(|T|)$ is defined by the use of functional calculus.

Let $B(H)$ be the space of all bounded linear operators on a Hilbert space H, and I stands for the identity operator on H. An operator A in $B(H)$ is said to be positive (in symbol: $A \geq 0$) if $\langle Ax, x\rangle \geq 0$ for all $x \in H$. In particular, $A > 0$ means that A is positive and invertible. For a pair A, B of selfadjoint operators the order relation $A \geq B$ means as usual that $A - B$ is positive.

In 1998, Fujii et al. [7], [8], introduced the *normalized determinant* $\Delta_x(A)$ for positive invertible operators A on a Hilbert space H and a fixed unit vector $x \in H$, namely $\|x\| = 1$, defined by

$$\Delta_x(A) := \exp\langle \ln Ax, x\rangle$$

and discussed it as a continuous geometric mean and observed some inequalities around the determinant from this point of view. For some recent results, see [9].

We need now some preparations for trace of operators in Hilbert spaces.

Let $(H, \langle\cdot,\cdot\rangle)$ be a complex Hilbert space and $\{e_i\}_{i\in I}$ an *orthonormal basis* of H. We say that $A \in \mathcal{B}(H)$ is a *Hilbert-Schmidt operator* if

$$\sum_{i\in I} \|Ae_i\|^2 < \infty. \tag{1.1}$$

It is well know that, if $\{e_i\}_{i\in I}$ and $\{f_j\}_{j\in J}$ are orthonormal bases for H and $A \in \mathcal{B}(H)$ then

$$\sum_{i\in I} \|Ae_i\|^2 = \sum_{j\in I} \|Af_j\|^2 = \sum_{j\in I} \|A^* f_j\|^2 \tag{1.2}$$

showing that the definition (1.1) is independent of the orthonormal basis and A is a Hilbert-Schmidt operator iff A^* is a Hilbert-Schmidt operator.

Let $\mathcal{B}_2(H)$ the set of *Hilbert-Schmidt operators* in $\mathcal{B}(H)$. For $A \in \mathcal{B}_2(H)$ we define

$$\|A\|_2 := \left(\sum_{i\in I} \|Ae_i\|^2\right)^{1/2} \tag{1.3}$$

for $\{e_i\}_{i\in I}$ an orthonormal basis of H.

Using the triangle inequality in $l^2(I)$, one checks that $\mathcal{B}_2(H)$ is a *vector space* and that $\|\cdot\|_2$ is a norm on $\mathcal{B}_2(H)$, which is usually called in the literature as the *Hilbert-Schmidt norm*.

Denote *the modulus* of an operator $A \in \mathcal{B}(H)$ by $|A| := (A^*A)^{1/2}$.

Because $\||A|x\| = \|Ax\|$ for all $x \in H$, A is Hilbert-Schmidt iff $|A|$ is Hilbert-Schmidt and $\|A\|_2 = \||A|\|_2$. From (1.2) we have that if $A \in \mathcal{B}_2(H)$, then $A^* \in \mathcal{B}_2(H)$ and $\|A\|_2 = \|A^*\|_2$.

The following theorem collects some of the most important properties of Hilbert-Schmidt operators:

Theorem 1. *We have:*

(i) $(\mathcal{B}_2(H), \|\cdot\|_2)$ *is a Hilbert space with inner product*

$$\langle A, B\rangle_2 := \sum_{i\in I} \langle Ae_i, Be_i\rangle = \sum_{i\in I} \langle B^* Ae_i, e_i\rangle \tag{1.4}$$

and the definition does not depend on the choice of the orthonormal basis $\{e_i\}_{i\in I}$;

(ii) We have the inequalities

$$\|A\| \le \|A\|_2 \tag{1.5}$$

for any $A \in \mathcal{B}_2(H)$ *and, if* $A \in \mathcal{B}_2(H)$ *and* $T \in \mathcal{B}(H)$, *then* AT, $TA \in \mathcal{B}_2(H)$ *with*

$$\|AT\|_2, \ \|TA\|_2 \le \|T\| \|A\|_2 \tag{1.6}$$

(iii) $\mathcal{B}_2(H)$ *is an operator ideal in* $\mathcal{B}(H)$, *i.e.*

$$\mathcal{B}(H) \mathcal{B}_2(H) \mathcal{B}(H) \subseteq \mathcal{B}_2(H).$$

If $\{e_i\}_{i\in I}$ an orthonormal basis of H, we say that $A \in \mathcal{B}(H)$ is *trace class* if

$$\|A\|_1 := \sum_{i\in I} \langle |A| e_i, e_i\rangle < \infty. \tag{1.7}$$

The definition of $\|A\|_1$ does not depend on the choice of the orthonormal basis $\{e_i\}_{i\in I}$. We denote by $\mathcal{B}_1(H)$ the set of trace class operators in $\mathcal{B}(H)$.

The following proposition holds:

Proposition 1. *If* $A \in \mathcal{B}(H)$, *then the following are equivalent:*

(i) $A \in \mathcal{B}_1(H)$;

(ii) $|A|^{1/2} \in \mathcal{B}_2(H)$.

The following properties are also well known:

Theorem 2. *With the above notations:*

(i) We have

$$\|A\|_1 = \|A^*\|_1 \ \text{ and } \ \|A\|_2 \le \|A\|_1 \tag{1.8}$$

for any $A \in \mathcal{B}_1(H)$;

(ii) $\mathcal{B}_1(H)$ *is an operator ideal in* $\mathcal{B}(H)$, *i.e.*

$$\mathcal{B}(H) \mathcal{B}_1(H) \mathcal{B}(H) \subseteq \mathcal{B}_1(H);$$

(iii) We have

$$\mathcal{B}_2(H) \mathcal{B}_2(H) = \mathcal{B}_1(H);$$

(iv) We have

$$\|A\|_1 = \sup\{\langle A, B\rangle_2 \mid B \in \mathcal{B}_2(H), \ \|B\|_2 \le 1\};$$

(v) $(\mathcal{B}_1(H), \|\cdot\|_1)$ *is a Banach space.*

We define the *trace* of a trace class operator $A \in \mathcal{B}_1(H)$ to be

$$\operatorname{tr}(A) := \sum_{i\in I} \langle Ae_i, e_i\rangle, \tag{1.9}$$

where $\{e_i\}_{i\in I}$ an orthonormal basis of H. Note that this coincides with the usual definition of the trace if H is finite-dimensional. We observe that the series (1.9) converges absolutely and it is independent from the choice of basis.

The following result collects some properties of the trace:

Theorem 3. *We have:*

(i) If $A\in\mathcal{B}_1(H)$ *then* $A^*\in\mathcal{B}_1(H)$ *and*

$$\operatorname{tr}(A^*)=\overline{\operatorname{tr}(A)}; \tag{1.10}$$

(ii) If $A\in\mathcal{B}_1(H)$ *and* $T\in\mathcal{B}(H)$, *then* $AT,\ TA\in\mathcal{B}_1(H)$,

$$\operatorname{tr}(AT)=\operatorname{tr}(TA) \text{ and } |\operatorname{tr}(AT)|\le\|A\|_1\|T\|; \tag{1.11}$$

(iii) $\operatorname{tr}(\cdot)$ *is a bounded linear functional on* $\mathcal{B}_1(H)$ *with* $\|\operatorname{tr}\|=1$;

(iv) If $A,\ B\in\mathcal{B}_2(H)$ *then* $AB,\ BA\in\mathcal{B}_1(H)$ *and* $\operatorname{tr}(AB)=\operatorname{tr}(BA)$.

Now, if we assume that $P\ge 0$ and $P\in\mathcal{B}_1(H)$, then for all $T\in\mathcal{B}(H)$, PT, $TP\in\mathcal{B}_1(H)$ and $\operatorname{tr}(PT)=\operatorname{tr}(TP)$. Also, since $P^{1/2}\in\mathcal{B}_2(H)$, $TP^{1/2}\in\mathcal{B}_2(H)$, hence $P^{1/2}TP^{1/2}$ and $TP^{1/2}P^{1/2}=TP\in\mathcal{B}_1(H)$ with $\operatorname{tr}\left(P^{1/2}TP^{1/2}\right)=\operatorname{tr}(TP)$. Therefore, if $P\ge 0$ and $P\in\mathcal{B}_1(H)$,

$$\operatorname{tr}(PT)=\operatorname{tr}(TP)=\operatorname{tr}\left(P^{1/2}TP^{1/2}\right)$$

for all $T\in\mathcal{B}(H)$.

If $T\ge 0$, then $P^{1/2}TP^{1/2}\ge 0$, which implies that $\operatorname{tr}(PT)\ge 0$ that shows that the functional $\mathcal{B}(H)\ni T\longmapsto\operatorname{tr}(PT)$ is linear and isotonic functional. Also, by (1.11), if $T_n\to T$ for $n\to\infty$ in $\mathcal{B}(H)$ then $\lim_{n\to\infty}\operatorname{tr}(PT_n)=\operatorname{tr}(PT)$, namely $\mathcal{B}(H)\ni T\longmapsto\operatorname{tr}(PT)$ is also continuous in the norm topology.

For a survey on recent trace inequalities see [4] and the references therein.

Now, for a given $P\ge 0$ with $P\in\mathcal{B}_1(H)$ and $\operatorname{tr}(P)=1$, we define the P-*determinant* of the positive invertible operator A by

$$\Delta_P(A):=\exp\operatorname{tr}(P\ln A)=\exp\operatorname{tr}((\ln A)P)=\exp\operatorname{tr}\left(P^{1/2}(\ln A)P^{1/2}\right). \tag{1.12}$$

Assume that $P\ge 0$ with $P\in\mathcal{B}_1(H)$ and $\operatorname{tr}(P)=1$. We observe that we have the following elementary properties:

(i) *continuity*: the map $A\to\Delta_P(A)$ is norm continuous;
(ii) *power equality:* $\Delta_P(A^t)=\Delta_P(A)^t$ for all $t>0$;
(iii) *homogeneity*: $\Delta_P(tA)=t\Delta_x(A)$ and $\Delta_P(tI)=t$ for all $t>0$;
(iv) *monotonicity*: $0<A\le B$ implies $\Delta_P(A)\le\Delta_P(B)$.

In the recent paper [5] we obtained the following results:

Theorem 4. *Let* $P\ge 0$ *with* $P\in\mathcal{B}_1(H)$ *and* $\operatorname{tr}(P)=1$, *then for all* $A,\ B>0$ *and* $t\in[0,1]$,

$$\Delta_P((1-t)A+tB)\ge[\Delta_P(A)]^{1-t}[\Delta_P(B)]^t. \tag{1.13}$$

and

Theorem 5. *Let* $P\ge 0$ *with* $P\in\mathcal{B}_1(H)$ *and* $\operatorname{tr}(P)=1$, *then for all* $A>0$ *and* $a>0$ *we have the double inequality*

$$a\exp\left[1-a\operatorname{tr}\left(PA^{-1}\right)\right]\le\Delta_P(A)\le a\exp\left[a^{-1}\operatorname{tr}(PA)-1\right]. \tag{1.14}$$

In particular

$$1\le\frac{\operatorname{tr}(PA)}{\Delta_P(A)}\le\exp\left[\operatorname{tr}(PA)\operatorname{tr}\left(PA^{-1}\right)-1\right] \tag{1.15}$$

and

$$1\le\frac{\Delta_P(A)}{\left[\operatorname{tr}\left(PA^{-1}\right)\right]^{-1}}\le\exp\left[\operatorname{tr}\left(PA^{-1}\right)\operatorname{tr}(PA)-1\right]. \tag{1.16}$$

The first inequalities in (1.15) and 1.16) are best possible from (1.14).

Motivated by the above results, in this paper we showed among others that

$$1 \le \exp\left[\frac{1}{2M}\left(\operatorname{tr}\left(PA^2\right) - \left[\operatorname{tr}(PA)\right]^2\right)\right] \le \frac{\operatorname{tr}(PA)}{\Delta_P(A)} \le \exp\left[\frac{1}{2m}\left(\operatorname{tr}\left(PA^2\right) - \left[\operatorname{tr}(PA)\right]^2\right)\right]$$

and

$$1 \le \exp\left[\frac{1}{2M}\left(\operatorname{tr}\left(PA^2\right) - \left[\operatorname{tr}(PA)\right]^2\right)\right] \le \frac{\Delta_P(A)}{\operatorname{tr}(PA)\exp\left(\operatorname{tr}(PA)\operatorname{tr}\left(PA^{-1}\right) - 1\right)} \le \exp\left[\frac{1}{2m}\left(\operatorname{tr}\left(PA^2\right) - \left[\operatorname{tr}(PA)\right]^2\right)\right],$$

where A is satisfying the condition $0 < mI \le A \le MI$ for some constants $m < M$ and $P \ge 0$ with $P \in \mathcal{B}_1(H)$ and $\operatorname{tr}(P) = 1$.

2. Main Results

We also have the following lower and upper bounds:

Theorem 6. *Let $P \ge 0$ with $P \in \mathcal{B}_1(H)$ and $\operatorname{tr}(P) = 1$, then for the operator A satisfying the condition $0 < mI \le A \le MI$ for some constants $m < M$ and $a > 0$ we have the inequalities*

$$1 \le \exp\left[\frac{1}{2}\min\left\{a^2, m^2\right\}\left(a^{-2} - 2a^{-1}\operatorname{tr}\left(PA^{-1}\right) + \operatorname{tr}\left(PA^{-2}\right)\right)\right] \le \frac{a\exp\left[a^{-1}\operatorname{tr}(PA) - 1\right]}{\Delta_P(A)} \le \exp\left[\frac{1}{2}\max\left\{a^2, M^2\right\}\left(a^{-2} - 2a^{-1}\operatorname{tr}\left(PA^{-1}\right) + \operatorname{tr}\left(PA^{-2}\right)\right)\right] \tag{2.1}$$

and

$$1 \le \exp\left[\frac{1}{2}\min\left\{a^2, m^2\right\}\left(a^{-2} - 2a^{-1}\operatorname{tr}\left(PA^{-1}\right) + \operatorname{tr}\left(PA^{-2}\right)\right)\right] \le \frac{\Delta_P(A)}{a\exp\left(a\operatorname{tr}\left(PA^{-1}\right) - 1\right)} \le \exp\left[\frac{1}{2}\max\left\{a^2, M^2\right\}\left(a^{-2} - 2a^{-1}\operatorname{tr}\left(PA^{-1}\right) + \operatorname{tr}\left(PA^{-2}\right)\right)\right]. \tag{2.2}$$

Proof. Observe that

$$\int_a^b \frac{b-t}{t^2}dt = b\int_a^b t^{-2}dt - \int_a^b \frac{1}{t}dt = \frac{b-a}{a} - \ln b + \ln a \tag{2.3}$$

for any $a,\ b > 0$.

If $b > a$, then

$$\frac{1}{2}\frac{(b-a)^2}{a^2} \ge \int_a^b \frac{b-t}{t^2}dt \ge \frac{1}{2}\frac{(b-a)^2}{b^2}. \tag{2.4}$$

If $a>b$ then

$$\int_a^b \frac{b-t}{t^2}dt = -\int_b^a \frac{b-t}{t^2}dt = \int_b^a \frac{t-b}{t^2}dt$$

and

$$\frac{1}{2}\frac{(b-a)^2}{b^2} \geq \int_b^a \frac{t-b}{t^2}dt \geq \frac{1}{2}\frac{(b-a)^2}{a^2}. \tag{2.5}$$

Therefore, by (2.4) and (2.5) we have for any a, $b>0$ that

$$\begin{aligned}\int_a^b \frac{b-t}{t^2}dt &\geq \frac{1}{2}\frac{(b-a)^2}{\max^2\{a,b\}} = \frac{1}{2}\frac{(b-a)^2 a^2b^2}{a^2b^2\max^2\{a,b\}} \\ &= \frac{1}{2}\frac{(b-a)^2\min^2\{a,b\}}{a^2b^2} \\ &= \frac{1}{2}\left(a^{-1}-b^{-1}\right)^2\min\left\{a^2,b^2\right\}\end{aligned}$$

and

$$\begin{aligned}\int_a^b \frac{b-t}{t^2}dt &\leq \frac{1}{2}\frac{(b-a)^2}{\min^2\{a,b\}} = \frac{1}{2}\frac{(b-a)^2 a^2b^2}{a^2b^2\min^2\{a,b\}} \\ &= \frac{1}{2}\frac{(b-a)^2\max^2\{a,b\}}{a^2b^2} \\ &\leq \frac{1}{2}\left(a^{-1}-b^{-1}\right)^2\max\left\{a^2,b^2\right\}\end{aligned}$$

By the representation (2.3) we then get

$$\begin{aligned}\frac{1}{2}\left(a^{-1}-b^{-1}\right)^2\min\left\{a^2,b^2\right\} &\leq \frac{b-a}{a} - \ln b + \ln a \\ &\leq \frac{1}{2}\left(a^{-1}-b^{-1}\right)^2\max\left\{a^2,b^2\right\}\end{aligned} \tag{2.6}$$

for a, $b>0$.

By swapping a with b in (2.6) we derive

$$\begin{aligned}\frac{1}{2}\left(a^{-1}-b^{-1}\right)^2\min\left\{a^2,b^2\right\} &\leq \ln b - \ln a - \frac{b-a}{b} \\ &\leq \frac{1}{2}\left(a^{-1}-b^{-1}\right)^2\max\left\{a^2,b^2\right\}\end{aligned} \tag{2.7}$$

for a, $b>0$.

Since $0<mI\leq A\leq MI$, then by (2.6),

$$\begin{aligned}\frac{1}{2}\left(a^{-1}-b^{-1}\right)^2\min\left\{a^2,m^2\right\} &\leq \frac{b-a}{a} - \ln b + \ln a \\ &\leq \frac{1}{2}\left(a^{-1}-b^{-1}\right)^2\max\left\{a^2,M^2\right\}.\end{aligned}$$

Now, if we use the functional calculus for $A>0$ and $a>0$, then we get

$$\begin{aligned}&\frac{1}{2}\min\left\{a^2,m^2\right\}\left(a^{-2}-2a^{-1}A^{-1}+A^{-2}\right) \\ &\leq a^{-1}A - \ln A + \ln a - 1 \\ &\leq \frac{1}{2}\max\left\{a^2,M^2\right\}\left(a^{-2}-2a^{-1}A^{-1}+A^{-2}\right).\end{aligned}$$

If we multiply both sides by $P^{1/2}$ and take the trace, then we get,

$$\begin{aligned}
&\frac{1}{2}\min\left\{a^2,m^2\right\}\left(a^{-2}-2a^{-1}\operatorname{tr}\left(PA^{-1}\right)+\operatorname{tr}\left(PA^{-2}\right)\right)\\
&\le a^{-1}\operatorname{tr}\left(PA\right)-\operatorname{tr}\left(P\ln A\right)+\ln a-1\\
&\le \frac{1}{2}\max\left\{a^2,M^2\right\}\left(a^{-2}-2a^{-1}\operatorname{tr}\left(PA^{-1}\right)+\operatorname{tr}\left(PA^{-2}\right)\right).
\end{aligned}$$

By taking the exponential, we get

$$\begin{aligned}
&\exp\left[\frac{1}{2}\min\left\{a^2,m^2\right\}\left(a^{-2}-2a^{-1}\operatorname{tr}\left(PA^{-1}\right)+\operatorname{tr}\left(PA^{-2}\right)\right)\right]\\
&\le \frac{a\exp\left[a^{-1}\operatorname{tr}\left(PA\right)-1\right]}{\exp\left[\operatorname{tr}\left(P\ln A\right)\right]}\\
&\le \exp\left[\frac{1}{2}\max\left\{a^2,M^2\right\}\left(a^{-2}-2a^{-1}\operatorname{tr}\left(PA^{-1}\right)+\operatorname{tr}\left(PA^{-2}\right)\right)\right],
\end{aligned}$$

which proves (2.1).

From (2.7) we get

$$\begin{aligned}
\frac{1}{2}\left(a^{-1}I-A^{-1}\right)^2\min\left\{a^2,m^2\right\} &\le \ln A-\ln a-I+aA^{-1}\\
&\le \frac{1}{2}\left(a^{-1}I-A^{-1}\right)^2\max\left\{a^2,M^2\right\}.
\end{aligned}$$

If we multiply both sides by $P^{1/2}$ and take the trace, then we get the desired result (2.2). $\square$

Corollary 1. *Let $P\ge 0$ with $P\in\mathcal{B}_1(H)$ and $\operatorname{tr}(P)=1$, then for the operator A satisfying the condition $0<mI\le A\le MI$ for some constants $m<M$,*

$$\begin{aligned}
1 &\le \exp\left[\frac{1}{2}m^2\left(\operatorname{tr}\left(PA^{-2}\right)-\left[\operatorname{tr}\left(PA^{-1}\right)\right]^2\right)\right]\\
&\le \frac{\left[\operatorname{tr}\left(PA^{-1}\right)\right]^{-1}\exp\left[\operatorname{tr}\left(PA^{-1}\right)\operatorname{tr}\left(PA\right)-1\right]}{\Delta_P(A)}\\
&\le \exp\left[\frac{1}{2}M^2\left(\operatorname{tr}\left(PA^{-2}\right)-\left[\operatorname{tr}\left(PA^{-1}\right)\right]^2\right)\right]
\end{aligned}\tag{2.8}$$

and

$$\begin{aligned}
1 &\le \exp\left[\frac{1}{2}m^2\left(\operatorname{tr}\left(PA^{-2}\right)-\left[\operatorname{tr}\left(PA^{-1}\right)\right]^2\right)\right]\\
&\le \frac{\Delta_P(A)}{\left[\operatorname{tr}\left(PA^{-1}\right)\right]^{-1}}\\
&\le \exp\left[\frac{1}{2}M^2\left(\operatorname{tr}\left(PA^{-2}\right)-\left[\operatorname{tr}\left(PA^{-1}\right)\right]^2\right)\right].
\end{aligned}\tag{2.9}$$

Proof. Since $0<M^{-1}\le A^{-1}\le m^{-1}$, hence $M^{-1}\le\operatorname{tr}\left(PA^{-1}\right)\le m^{-1}$, namely $m\le\left[\operatorname{tr}\left(PA^{-1}\right)\right]^{-1}\le M$. Then for $a=\left[\operatorname{tr}\left(PA^{-1}\right)\right]^{-1}$ we have

$$\min\left\{a^2,m^2\right\}=\min\left\{\left[\operatorname{tr}\left(PA^{-1}\right)\right]^{-2},m^2\right\}=m^2,$$

$$\max\left\{a^2,M^2\right\}=\max\left\{\left[\operatorname{tr}\left(PA^{-1}\right)\right]^{-2},M^2\right\}=M^2$$

and

$$\begin{aligned}
&a^{-2}-2a^{-1}\operatorname{tr}\left(PA^{-1}\right)+\operatorname{tr}\left(PA^{-2}\right)\\
&=\left[\operatorname{tr}\left(PA^{-1}\right)\right]^{2}-2\left[\operatorname{tr}\left(PA^{-1}\right)\right]\operatorname{tr}\left(PA^{-1}\right)+\operatorname{tr}\left(PA^{-2}\right)\\
&=\operatorname{tr}\left(PA^{-2}\right)-\left[\operatorname{tr}\left(PA^{-1}\right)\right]^{2}
\end{aligned}$$

and by (2.1) and (2.2) we derive (2.8) and (2.9). □

Theorem 7. *With the assumptions of Theorem 6 we have*

$$\begin{aligned}
1&\leq\exp\left[\frac{1}{2\max^{2}\{a,M\}}(a^{2}-2a\operatorname{tr}(PA)+\operatorname{tr}\left(PA^{2}\right))\right]\\
&\leq\frac{a\exp\left[a^{-1}\operatorname{tr}(PA)-1\right]}{\Delta_{P}(A)}\\
&\leq\exp\left[\frac{1}{2\min^{2}\{a,m\}}(a^{2}-2a\operatorname{tr}(PA)+\operatorname{tr}\left(PA^{2}\right))\right].
\end{aligned}\tag{2.10}$$

and

$$\begin{aligned}
1&\leq\exp\left[\frac{1}{2\max^{2}\{a,M\}}(a^{2}-2a\operatorname{tr}(PA)+\operatorname{tr}\left(PA^{2}\right))\right]\\
&\leq\frac{\Delta_{P}(A)}{a\exp\left(a\operatorname{tr}\left(PA^{-1}\right)-1\right)}\\
&\leq\exp\left[\frac{1}{2\min^{2}\{a,m\}}\left(a^{2}-2a\operatorname{tr}(PA)+\operatorname{tr}\left(PA^{2}\right)\right)\right].
\end{aligned}\tag{2.11}$$

Proof. From the above considerations, we also have

$$\frac{1}{2}\frac{(b-a)^{2}}{\max^{2}\{a,b\}}\leq\frac{b-a}{a}-\ln b+\ln a\leq\frac{1}{2}\frac{(b-a)^{2}}{\min^{2}\{a,b\}}\tag{2.12}$$

and

$$\frac{1}{2}\frac{(b-a)^{2}}{\max^{2}\{a,b\}}\leq\ln b-\ln a-\frac{b-a}{b}\leq\frac{1}{2}\frac{(b-a)^{2}}{\min^{2}\{a,b\}}$$

for all $a,b>0$.

If $b\in[m,M]\subset(0,\infty)$, then by (2.12) we get

$$\begin{aligned}
\frac{1}{2\max^{2}\{a,M\}}(b-a)^{2}&\leq\frac{b-a}{a}-\ln b+\ln a\\
&\leq\frac{1}{2\min^{2}\{a,m\}}(b-a)^{2}.
\end{aligned}\tag{2.13}$$

Using the functional calculus, we get

$$\begin{aligned}
&\frac{1}{2\max^{2}\{a,M\}}(a^{2}-2aA+A^{2})\\
&\leq a^{-1}A-\ln A+\ln a-1\leq\frac{1}{2\min^{2}\{a,m\}}(a^{2}-2aA+A^{2})
\end{aligned}$$

for all $a>0$ and $0<mI\leq A\leq MI$.

If we multiply both sides by $P^{1/2}$ and take the trace, then we get (2.10). □

Corollary 2. *With the assumptions of Corollary 1,*

$$1 \le \exp\left[\frac{1}{2M}\left(\operatorname{tr}\left(PA^2\right) - \left[\operatorname{tr}\left(PA\right)\right]^2\right)\right] \le \frac{\operatorname{tr}\left(PA\right)}{\Delta_P\left(A\right)} \le \exp\left[\frac{1}{2m}\left(\operatorname{tr}\left(PA^2\right) - \left[\operatorname{tr}\left(PA\right)\right]^2\right)\right] \tag{2.14}$$

and

$$1 \le \exp\left[\frac{1}{2M}\left(\operatorname{tr}\left(PA^2\right) - \left[\operatorname{tr}\left(PA\right)\right]^2\right)\right] \le \frac{\Delta_P\left(A\right)}{\operatorname{tr}\left(PA\right)\exp\left(\operatorname{tr}\left(PA\right)\operatorname{tr}\left(PA^{-1}\right) - 1\right)} \le \exp\left[\frac{1}{2m}\left(\operatorname{tr}\left(PA^2\right) - \left[\operatorname{tr}\left(PA\right)\right]^2\right)\right]. \tag{2.15}$$

Proof. Observe that $m \le \operatorname{tr}(PA) \le M$ and by taking $a = \operatorname{tr}(PA)$ in (2.10) and (2.11), we derive (2.14) and (2.15). □

3. Some Related Results

In [2] we proved among others that, if $P \in \mathcal{B}_1^+(H)$, A, $B \in \mathcal{B}(H)$ and

$$(A - mB)(MB - A) \ge 0, m, M \in \mathbb{R} \tag{3.1a}$$

with $M > m$, then

$$0 \le \operatorname{tr}\left(PA^2\right)\operatorname{tr}\left(PB^2\right) - \left[\operatorname{tr}\left(PBA\right)\right]^2 \le \left[\left(M\operatorname{tr}\left(PB^2\right) - \operatorname{tr}\left(PBA\right)\right)\left(\operatorname{tr}\left(PAB\right) - m\operatorname{tr}\left(PB^2\right)\right)\right] \le \frac{1}{4}(M-m)^2\left[\operatorname{tr}\left(PB^2\right)\right]^2 \tag{3.2}$$

and

$$0 \le \operatorname{tr}\left(PA^2\right)\operatorname{tr}\left(PB^2\right) - \left[\operatorname{tr}\left(PBA\right)\right]^2 \le \frac{1}{4}(M-m)^2\left[\operatorname{tr}\left(PB^2\right)\right]^2 - \operatorname{tr}\left(PB^2\right)\operatorname{tr}\left[P(A-mB)(MB-A)\right] \le \frac{1}{4}(M-m)^2\left[\operatorname{tr}\left(PB^2\right)\right]^2. \tag{3.3}$$

If we take $B = I$ and assume that $(A - mI)(MI - A) \ge 0$, then we get for $P \in \mathcal{B}_1^+(H)$ with $\operatorname{tr}(P) = 1$ that

$$0 \le \operatorname{tr}\left(PA^2\right) - \left[\operatorname{tr}\left(PA\right)\right]^2 \le \left[\left(M - \operatorname{tr}\left(PA\right)\right)\left(\operatorname{tr}\left(PA\right) - m\right)\right] \le \frac{1}{4}(M-m)^2 \tag{3.4}$$

and

$$0 \le \operatorname{tr}\left(PA^2\right) - \left[\operatorname{tr}\left(PA\right)\right]^2 \le \frac{1}{4}(M-m)^2 - \operatorname{tr}\left[P(A-mI)(MI-A)\right] \le \frac{1}{4}(M-m)^2. \tag{3.5}$$

It is clear that, if $0 < mI \le A \le MI$ then the inequalities (3.4) and (3.5) are valid. Since $0 < M^{-1} \le A^{-1} \le m^{-1}$, then by applying (3.4) and (3.5) we also get

$$0 \le \operatorname{tr}\left(PA^{-2}\right) - \left[\operatorname{tr}\left(PA^{-1}\right)\right]^2 \le \left[\left(m^{-1} - \operatorname{tr}\left(PA^{-1}\right)\right)\left(\operatorname{tr}\left(PA^{-1}\right) - M^{-1}\right)\right] \le \frac{1}{4}\frac{(M-m)^2}{mM} \tag{3.6}$$

and

$$\begin{aligned} 0 &\le \operatorname{tr}\left(PA^{-2}\right) - \left[\operatorname{tr}\left(PA^{-1}\right)\right]^2 \\ &\le \frac{1}{4}\frac{(M-m)^2}{mM} - \operatorname{tr}\left[P\left(A^{-1} - M^{-1}I\right)\left(m^{-1}I - A^{-1}\right)\right] \\ &\le \frac{1}{4}\frac{(M-m)^2}{mM}. \end{aligned} \tag{3.7}$$

Proposition 2. *Let $P \ge 0$ with $P \in \mathcal{B}_1(H)$ and $\operatorname{tr}(P) = 1$, then for the operator A satisfying the condition $0 < mI \le A \le MI$ for some constants $m < M$,*

$$\begin{aligned} &\frac{\left[\operatorname{tr}\left(PA^{-1}\right)\right]^{-1}\exp\left[\operatorname{tr}\left(PA^{-1}\right)\operatorname{tr}(PA) - 1\right]}{\Delta_P(A)} \\ &\le \exp\left[\frac{1}{2}M^2\left(\operatorname{tr}\left(PA^{-2}\right) - \left[\operatorname{tr}\left(PA^{-1}\right)\right]^2\right)\right] \\ &\le \exp\left[\frac{1}{2}M^2\left[\left(m^{-1} - \operatorname{tr}\left(PA^{-1}\right)\right)\left(\operatorname{tr}\left(PA^{-1}\right) - M^{-1}\right)\right]\right] \\ &\le \exp\left[\frac{1}{8}\frac{(M-m)^2 M}{m}\right] \end{aligned} \tag{3.8}$$

and

$$\begin{aligned} \frac{\Delta_P(A)}{\left[\operatorname{tr}\left(PA^{-1}\right)\right]^{-1}} &\le \exp\left[\frac{1}{2}M^2\left(\operatorname{tr}\left(PA^{-2}\right) - \left[\operatorname{tr}\left(PA^{-1}\right)\right]^2\right)\right] \\ &\le \exp\left[\frac{1}{2}M^2\left[\left(m^{-1} - \operatorname{tr}\left(PA^{-1}\right)\right)\left(\operatorname{tr}\left(PA^{-1}\right) - M^{-1}\right)\right]\right] \\ &\le \exp\left[\frac{1}{8}\frac{(M-m)^2 M}{m}\right]. \end{aligned} \tag{3.9}$$

The proof follows by Corollary 1 and the inequality (3.6). Similar upper bounds may be obtained by employing the inequality (3.7).

Proposition 3. *With the assumptions of Proposition 2,*

$$\begin{aligned} \frac{\operatorname{tr}(PA)}{\Delta_P(A)} &\le \exp\left[\frac{1}{2m}\left(\operatorname{tr}\left(PA^2\right) - \left[\operatorname{tr}(PA)\right]^2\right)\right] \\ &\le \exp\left[\frac{1}{2m}\left(M - \operatorname{tr}(PA)\right)\left(\operatorname{tr}(PA) - m\right)\right] \\ &\le \exp\left[\frac{1}{8}\frac{(M-m)^2}{m}\right] \end{aligned} \tag{3.10}$$

and

$$\frac{\Delta_P(A)}{\operatorname{tr}(PA)\exp(\operatorname{tr}(PA)\operatorname{tr}(PA^{-1})-1)} \tag{3.11}$$
$$\leq \exp\left[\frac{1}{2m}\left(\operatorname{tr}\left(PA^2\right)-[\operatorname{tr}(PA)]^2\right)\right]$$
$$\leq \exp\left[\frac{1}{2m}(M-\operatorname{tr}(PA))(\operatorname{tr}(PA)-m)\right]$$
$$\leq \exp\left[\frac{1}{8}\frac{(M-m)^2}{m}\right].$$

The proof follows by Corollary 2 and the inequality (3.4).

In [3] we also obtained the following inequality

$$0\leq \operatorname{tr}\left(PA^2\right)\operatorname{tr}\left(PB^2\right)-[\operatorname{tr}(PBA)]^2\leq \frac{1}{4}\frac{(M-m)^2}{mM}[\operatorname{tr}(PBA)]^2,$$

provided that A and B satisfy the condition (3.1a) and $P\in\mathcal{B}_1^+(H)$.

If $P\geq 0$ with $P\in\mathcal{B}_1(H)$ and $\operatorname{tr}(P)=1$, then for the operator A satisfying the condition $0<mI\leq A\leq MI$ we obtain

$$0\leq \operatorname{tr}\left(PA^2\right)-[\operatorname{tr}(PA)]^2\leq \frac{1}{4}\frac{(M-m)^2}{mM}[\operatorname{tr}(PA)]^2.$$

Therefore, from Corollary 2 we obtain the following upper bounds

$$\frac{\operatorname{tr}(PA)}{\Delta_P(A)}\leq \exp\left[\frac{1}{8}\frac{(M-m)^2}{m^2M}[\operatorname{tr}(PA)]^2\right] \tag{3.12}$$

and

$$\frac{\Delta_P(A)}{\operatorname{tr}(PA)\exp(\operatorname{tr}(PA)\operatorname{tr}(PA^{-1})-1)} \tag{3.13}$$
$$\leq \exp\left[\frac{1}{8}\frac{(M-m)^2}{m^2M}[\operatorname{tr}(PA)]^2\right].$$

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