

UPPER AND LOWER BOUNDS FOR TRACE CLASS P -DETERMINANT OF POSITIVE OPERATORS IN HILBERT SPACES VIA KANTOROVICH'S CONSTANT

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ABSTRACT. Let H be a complex Hilbert space. For a given operator $P \geq 0$ with $P \in \mathcal{B}_1(H)$, the trace class associated to $\mathcal{B}(H)$ and $\text{tr}(P) = 1$, we define the P -determinant of the positive invertible operator A by

$$\Delta_P(A) := \exp \text{tr}(P \ln A).$$

In this paper we show, among others that, if A is an operator satisfying the condition $0 < mI \leq A \leq MI$, then

$$\begin{aligned} 1 &\leq K \left(\frac{M}{m} \right)^{\left[\frac{1}{2} - \frac{1}{M-m} \text{tr}(P|A - \frac{1}{2}(m+M)I|) \right]} \\ &\leq \frac{\Delta_P(A)}{m^{\frac{M-\text{tr}(PA)}{M-m}} M^{\frac{\text{tr}(PA)-m}{M-m}}} \\ &\leq K \left(\frac{M}{m} \right)^{\left[\frac{1}{2} + \frac{1}{M-m} \text{tr}(P|A - \frac{1}{2}(m+M)I|) \right]} \leq K \left(\frac{M}{m} \right), \end{aligned}$$

where $K(\cdot)$ is Kantorovich's constant.

1. INTRODUCTION

In 1952, in the paper [3], B. Fuglede and R. V. Kadison introduced the determinant of a (invertible) operator and established its fundamental properties. The notion generalizes the usual determinant and can be considered for any operator in a finite von Neumann algebra (M, τ) with a faithful normal trace.

Let $T \in M$ be normal and $|T| := (T^*T)^{1/2}$ its modulus. By the spectral theorem one can represent T as an integral

$$T = \int_{\text{Sp}(T)} \lambda dE(\lambda),$$

where $E(\lambda)$ is a projection valued measure and $\text{Sp}(T)$ is the spectrum of T . The measure $\mu_T := \tau \circ E$ becomes a probability measure on the complex plane and has the support in the spectrum $\text{Sp}(T)$.

For any $T \in M$ the Fuglede-Kadison determinant (FK -determinant) is defined by

$$\Delta_{FK}(T) := \exp \left(\int_0^\infty \ln t d\mu_{|T|} \right).$$

If T is invertible, then

$$\Delta_{FK}(T) := \exp(\tau(\ln(|T|))),$$

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where $\ln(|T|)$ is defined by the use of functional calculus.

Let $B(H)$ be the space of all bounded linear operators on a Hilbert space H , and I stands for the identity operator on H . An operator A in $B(H)$ is said to be positive (in symbol: $A \geq 0$) if $\langle Ax, x \rangle \geq 0$ for all $x \in H$. In particular, $A > 0$ means that A is positive and invertible. For a pair A, B of selfadjoint operators the order relation $A \geq B$ means as usual that $A - B$ is positive.

In 1998, Fujii et al. [4], [5], introduced the *normalized determinant* $\Delta_x(A)$ for positive invertible operators A on a Hilbert space H and a fixed unit vector $x \in H$, namely $\|x\| = 1$, defined by

$$\Delta_x(A) := \exp \langle \ln Ax, x \rangle$$

and discussed it as a continuous geometric mean and observed some inequalities around the determinant from this point of view. For some recent results, see [6].

We need now some preparations for trace of operators in Hilbert spaces.

Let $(H, \langle \cdot, \cdot \rangle)$ be a complex Hilbert space and $\{e_i\}_{i \in I}$ an *orthonormal basis* of H . We say that $A \in \mathcal{B}(H)$ is a *Hilbert-Schmidt operator* if

$$(1.1) \quad \sum_{i \in I} \|Ae_i\|^2 < \infty.$$

It is well known that, if $\{e_i\}_{i \in I}$ and $\{f_j\}_{j \in J}$ are orthonormal bases for H and $A \in \mathcal{B}(H)$ then

$$(1.2) \quad \sum_{i \in I} \|Ae_i\|^2 = \sum_{j \in I} \|Af_j\|^2 = \sum_{j \in I} \|A^*f_j\|^2$$

showing that the definition (1.1) is independent of the orthonormal basis and A is a Hilbert-Schmidt operator iff A^* is a Hilbert-Schmidt operator.

Let $\mathcal{B}_2(H)$ the set of *Hilbert-Schmidt operators* in $\mathcal{B}(H)$. For $A \in \mathcal{B}_2(H)$ we define

$$(1.3) \quad \|A\|_2 := \left(\sum_{i \in I} \|Ae_i\|^2 \right)^{1/2}$$

for $\{e_i\}_{i \in I}$ an orthonormal basis of H .

Using the triangle inequality in $l^2(I)$, one checks that $\mathcal{B}_2(H)$ is a *vector space* and that $\|\cdot\|_2$ is a norm on $\mathcal{B}_2(H)$, which is usually called in the literature as the *Hilbert-Schmidt norm*.

Denote the *modulus* of an operator $A \in \mathcal{B}(H)$ by $|A| := (A^*A)^{1/2}$.

Because $\| |A| x \| = \|Ax\|$ for all $x \in H$, A is Hilbert-Schmidt iff $|A|$ is Hilbert-Schmidt and $\|A\|_2 = \||A|\|_2$. From (1.2) we have that if $A \in \mathcal{B}_2(H)$, then $A^* \in \mathcal{B}_2(H)$ and $\|A\|_2 = \|A^*\|_2$.

The following theorem collects some of the most important properties of Hilbert-Schmidt operators:

Theorem 1. *We have:*

(i) $(\mathcal{B}_2(H), \|\cdot\|_2)$ is a Hilbert space with inner product

$$(1.4) \quad \langle A, B \rangle_2 := \sum_{i \in I} \langle Ae_i, Be_i \rangle = \sum_{i \in I} \langle B^*Ae_i, e_i \rangle$$

and the definition does not depend on the choice of the orthonormal basis $\{e_i\}_{i \in I}$;

(ii) We have the inequalities

$$(1.5) \quad \|A\| \leq \|A\|_2$$

for any $A \in \mathcal{B}_2(H)$ and, if $A \in \mathcal{B}_2(H)$ and $T \in \mathcal{B}(H)$, then $AT, TA \in \mathcal{B}_2(H)$ with

$$(1.6) \quad \|AT\|_2, \|TA\|_2 \leq \|T\| \|A\|_2$$

(iii) $\mathcal{B}_2(H)$ is an operator ideal in $\mathcal{B}(H)$, i.e.

$$\mathcal{B}(H) \mathcal{B}_2(H) \mathcal{B}(H) \subseteq \mathcal{B}_2(H).$$

If $\{e_i\}_{i \in I}$ an orthonormal basis of H , we say that $A \in \mathcal{B}(H)$ is *trace class* if

$$(1.7) \quad \|A\|_1 := \sum_{i \in I} \langle |A| e_i, e_i \rangle < \infty.$$

The definition of $\|A\|_1$ does not depend on the choice of the orthonormal basis $\{e_i\}_{i \in I}$. We denote by $\mathcal{B}_1(H)$ the set of trace class operators in $\mathcal{B}(H)$.

The following proposition holds:

Proposition 1. *If $A \in \mathcal{B}(H)$, then the following are equivalent:*

- (i) $A \in \mathcal{B}_1(H)$;
- (ii) $|A|^{1/2} \in \mathcal{B}_2(H)$.

The following properties are also well known:

Theorem 2. *With the above notations:*

(i) We have

$$(1.8) \quad \|A\|_1 = \|A^*\|_1 \quad \text{and} \quad \|A\|_2 \leq \|A\|_1$$

for any $A \in \mathcal{B}_1(H)$;

(ii) $\mathcal{B}_1(H)$ is an operator ideal in $\mathcal{B}(H)$, i.e.

$$\mathcal{B}(H) \mathcal{B}_1(H) \mathcal{B}(H) \subseteq \mathcal{B}_1(H);$$

(iii) We have

$$\mathcal{B}_2(H) \mathcal{B}_2(H) = \mathcal{B}_1(H);$$

(iv) We have

$$\|A\|_1 = \sup \{ \langle A, B \rangle_2 \mid B \in \mathcal{B}_2(H), \|B\|_2 \leq 1 \};$$

(v) $(\mathcal{B}_1(H), \|\cdot\|_1)$ is a Banach space.

We define the *trace* of a trace class operator $A \in \mathcal{B}_1(H)$ to be

$$(1.9) \quad \text{tr}(A) := \sum_{i \in I} \langle A e_i, e_i \rangle,$$

where $\{e_i\}_{i \in I}$ an orthonormal basis of H . Note that this coincides with the usual definition of the trace if H is finite-dimensional. We observe that the series (1.9) converges absolutely and it is independent from the choice of basis.

The following result collects some properties of the trace:

Theorem 3. *We have:*

(i) If $A \in \mathcal{B}_1(H)$ then $A^* \in \mathcal{B}_1(H)$ and

$$(1.10) \quad \text{tr}(A^*) = \overline{\text{tr}(A)};$$

(ii) If $A \in \mathcal{B}_1(H)$ and $T \in \mathcal{B}(H)$, then $AT, TA \in \mathcal{B}_1(H)$,

(1.11) $\quad \text{tr}(AT) = \text{tr}(TA)$ and $|\text{tr}(AT)| \leq \|A\|_1 \|T\|$;

(iii) $\text{tr}(\cdot)$ is a bounded linear functional on $\mathcal{B}_1(H)$ with $\|\text{tr}\| = 1$;
 (iv) If $A, B \in \mathcal{B}_2(H)$ then $AB, BA \in \mathcal{B}_1(H)$ and $\text{tr}(AB) = \text{tr}(BA)$.

Now, if we assume that $P \geq 0$ and $P \in \mathcal{B}_1(H)$, then for all $T \in \mathcal{B}(H)$, $PT, TP \in \mathcal{B}_1(H)$ and $\text{tr}(PT) = \text{tr}(TP)$. Also, since $P^{1/2} \in \mathcal{B}_2(H)$, $TP^{1/2} \in \mathcal{B}_2(H)$, hence $P^{1/2}TP^{1/2}$ and $TP^{1/2}P^{1/2} = TP \in \mathcal{B}_1(H)$ with $\text{tr}(P^{1/2}TP^{1/2}) = \text{tr}(TP)$. Therefore, if $P \geq 0$ and $P \in \mathcal{B}_1(H)$,

$$\text{tr}(PT) = \text{tr}(TP) = \text{tr}\left(P^{1/2}TP^{1/2}\right)$$

for all $T \in \mathcal{B}(H)$.

If $T \geq 0$, then $P^{1/2}TP^{1/2} \geq 0$, which implies that $\text{tr}(PT) \geq 0$ that shows that the functional $\mathcal{B}(H) \ni T \mapsto \text{tr}(PT)$ is linear and isotonic functional. Also, by (1.11), if $T_n \rightarrow T$ for $n \rightarrow \infty$ in $\mathcal{B}(H)$ then $\lim_{n \rightarrow \infty} \text{tr}(PT_n) = \text{tr}(PT)$, namely $\mathcal{B}(H) \ni T \mapsto \text{tr}(PT)$ is also continuous in the norm topology.

For a survey on recent trace inequalities see [1] and the references therein.

Now, for a given $P \geq 0$ with $P \in \mathcal{B}_1(H)$ and $\text{tr}(P) = 1$, we define the P -determinant of the positive invertible operator A by

$$(1.12) \quad \Delta_P(A) := \exp \text{tr}(P \ln A) = \exp \text{tr}((\ln A)P) = \exp \text{tr}\left(P^{1/2}(\ln A)P^{1/2}\right).$$

Assume that $P \geq 0$ with $P \in \mathcal{B}_1(H)$ and $\text{tr}(P) = 1$. We observe that we have the following elementary properties:

- (i) *continuity*: the map $A \rightarrow \Delta_P(A)$ is norm continuous;
- (ii) *power equality*: $\Delta_P(A^t) = \Delta_P(A)^t$ for all $t > 0$;
- (iii) *homogeneity*: $\Delta_P(tA) = t\Delta_x(A)$ and $\Delta_P(tI) = t$ for all $t > 0$;
- (iv) *monotonicity*: $0 < A \leq B$ implies $\Delta_P(A) \leq \Delta_P(B)$.

In the recent paper [2] we obtained the following results:

Theorem 4. Let $P \geq 0$ with $P \in \mathcal{B}_1(H)$ and $\text{tr}(P) = 1$, then for all $A, B > 0$ and $t \in [0, 1]$,

$$\Delta_P((1-t)A + tB) \geq [\Delta_P(A)]^{1-t} [\Delta_P(B)]^t.$$

and

Theorem 5. Let $P \geq 0$ with $P \in \mathcal{B}_1(H)$ and $\text{tr}(P) = 1$, then for all $A > 0$ and $a > 0$ we have the double inequality

$$a \exp [1 - a \text{tr}(PA^{-1})] \leq \Delta_P(A) \leq a \exp [a^{-1} \text{tr}(PA) - 1].$$

In particular

$$1 \leq \frac{\text{tr}(PA)}{\Delta_P(A)} \leq \exp [\text{tr}(PA) \text{tr}(PA^{-1}) - 1]$$

and

$$1 \leq \frac{\Delta_P(A)}{[\text{tr}(PA^{-1})]^{-1}} \leq \exp [\text{tr}(PA^{-1}) \text{tr}(PA) - 1].$$

We consider the *Kantorovich's constant* defined by

$$(1.13) \quad K(h) := \frac{(h+1)^2}{4h}, \quad h > 0.$$

The function K is decreasing on $(0, 1)$ and increasing on $[1, \infty)$, $K(h) \geq 1$ for any $h > 0$ and $K(h) = K(\frac{1}{h})$ for any $h > 0$.

The following multiplicative refinement and reverse of Young inequality in terms of Kantorovich's constant holds

$$(1.14) \quad (a^{1-\nu}b^\nu \leq) K^r \left(\frac{a}{b}\right) a^{1-\nu}b^\nu \leq (1-\nu)a + \nu b \leq K^R \left(\frac{a}{b}\right) a^{1-\nu}b^\nu$$

where $a, b > 0$, $\nu \in [0, 1]$, $r = \min\{1-\nu, \nu\}$ and $R = \max\{1-\nu, \nu\}$.

The first inequality in (1.14) was obtained by Zou et al. in [8] while the second by Liao et al. [7].

Motivated by the above results, we provide in this paper some upper and lower in bounds in terms of *Kantorovich's constant* for the quantity

$$\frac{\Delta_P(A)}{m^{\frac{M-\text{tr}(PA)}{M-m}} M^{\frac{\text{tr}(PA)-m}{M-m}}}$$

under the assumptions that $P \geq 0$ with $P \in \mathcal{B}_1(H)$ and $\text{tr}(P) = 1$, while A is an operator satisfying the condition $0 < mI \leq A \leq MI$.

2. MAIN RESULTS

We start with the following main result:

Theorem 6. *Let $P \geq 0$ with $P \in \mathcal{B}_1(H)$ and $\text{tr}(P) = 1$. If A is an operator satisfying the condition $0 < mI \leq A \leq MI$, then*

$$(2.1) \quad \begin{aligned} 1 &\leq K\left(\frac{M}{m}\right)^{\left[\frac{1}{2}-\frac{1}{M-m}\text{tr}(P|A-\frac{1}{2}(m+M)I)\right]} \\ &\leq \frac{\Delta_P(A)}{m^{\frac{M-\text{tr}(PA)}{M-m}} M^{\frac{\text{tr}(PA)-m}{M-m}}} \\ &\leq K\left(\frac{M}{m}\right)^{\left[\frac{1}{2}+\frac{1}{M-m}\text{tr}(P|A-\frac{1}{2}(m+M)I)\right]} \leq K\left(\frac{M}{m}\right). \end{aligned}$$

Proof. Assume that $t \in [m, M]$ and consider $\nu = \frac{t-m}{M-m} \in [0, 1]$. Then

$$\begin{aligned} \min\{1-\nu, \nu\} &= \frac{1}{2} - \left|\nu - \frac{1}{2}\right| = \frac{1}{2} - \left|\frac{t-m}{M-m} - \frac{1}{2}\right| \\ &= \frac{1}{2} - \frac{1}{M-m} \left|t - \frac{1}{2}(m+M)\right|, \end{aligned}$$

$$\begin{aligned} \max\{1-\nu, \nu\} &= \frac{1}{2} + \left|\nu - \frac{1}{2}\right| = \frac{1}{2} + \left|\frac{t-m}{M-m} - \frac{1}{2}\right| \\ &= \frac{1}{2} + \frac{1}{M-m} \left|t - \frac{1}{2}(m+M)\right|, \end{aligned}$$

$$(1-\nu)m + \nu M = \frac{M-t}{M-m}m + \frac{t-m}{M-m}M = t$$

and

$$m^{1-\nu}M^\nu = m^{\frac{M-t}{M-m}} M^{\frac{t-m}{M-m}}.$$

By using (1.14) we get

$$(2.2) \quad m^{\frac{M-t}{M-m}} M^{\frac{t-m}{M-m}} \leq \left[K \left(\frac{M}{m} \right) \right]^{\frac{1}{2} - \frac{1}{M-m} \left| t - \frac{1}{2}(m+M) \right|} m^{\frac{M-t}{M-m}} M^{\frac{t-m}{M-m}}$$

$$\leq t \leq \left[K \left(\frac{M}{m} \right) \right]^{\frac{1}{2} + \frac{1}{M-m} \left| t - \frac{1}{2}(m+M) \right|} m^{\frac{M-t}{M-m}} M^{\frac{t-m}{M-m}}$$

for $t \in [m, M]$.

By taking the log in (2.2) we get

$$(2.3) \quad \frac{M-t}{M-m} \ln m + \frac{t-m}{M-m} \ln M$$

$$\leq \left[\frac{1}{2} - \frac{1}{M-m} \left| t - \frac{1}{2}(m+M) \right| \right] \ln K \left(\frac{M}{m} \right)$$

$$+ \frac{M-t}{M-m} \ln m + \frac{t-m}{M-m} \ln M$$

$$\leq \ln t \leq \left[\frac{1}{2} + \frac{1}{M-m} \left| t - \frac{1}{2}(m+M) \right| \right] \ln K \left(\frac{M}{m} \right)$$

$$+ \frac{M-t}{M-m} \ln m + \frac{t-m}{M-m} \ln M$$

$$\leq \ln K \left(\frac{M}{m} \right) + \frac{M-t}{M-m} \ln m + \frac{t-m}{M-m} \ln M$$

for $t \in [m, M]$.

If $0 < mI \leq A \leq MI$, then by using the continuous functional calculus for selfadjoint operators we get from (2.3) that

$$(2.4) \quad \ln m \frac{MI - A}{M - m} + \ln M \frac{A - mI}{M - m}$$

$$\leq \left[\frac{1}{2}I - \frac{1}{M - m} \left| A - \frac{1}{2}(m + M)I \right| \right] \ln K \left(\frac{M}{m} \right)$$

$$+ \ln m \frac{MI - A}{M - m} + \ln M \frac{A - mI}{M - m}$$

$$\leq \ln A \leq \left[\frac{1}{2}I + \frac{1}{M - m} \left| A - \frac{1}{2}(m + M)I \right| \right] \ln K \left(\frac{M}{m} \right)$$

$$+ \ln m \frac{MI - A}{M - m} + \ln M \frac{A - mI}{M - m}$$

$$\leq \ln K \left(\frac{M}{m} \right) I + \ln m \frac{MI - A}{M - m} + \ln M \frac{A - mI}{M - m}.$$

If we multiply the inequality (2.4) both sides by $P^{1/2}$ we get

$$\begin{aligned}
 (2.5) \quad & \ln m \frac{MP - P^{1/2}AP^{1/2}}{M - m} + \ln M \frac{P^{1/2}AP^{1/2} - mP}{M - m} \\
 & \leq \left[\frac{1}{2}P - \frac{1}{M - m}P^{1/2} \left| A - \frac{1}{2}(m + M)I \right| P^{1/2} \right] \ln K \left(\frac{M}{m} \right) \\
 & + \ln m \frac{MP - P^{1/2}AP^{1/2}}{M - m} + \ln M \frac{P^{1/2}AP^{1/2} - mP}{M - m} \\
 & \leq P^{1/2} (\ln A) P^{1/2} \\
 & \leq \left[\frac{1}{2}P + \frac{1}{M - m}P^{1/2} \left| A - \frac{1}{2}(m + M)I \right| P^{1/2} \right] \ln K \left(\frac{M}{m} \right) \\
 & + \ln m \frac{MP - P^{1/2}AP^{1/2}}{M - m} + \ln M \frac{P^{1/2}AP^{1/2} - mP}{M - m} \\
 & \leq \ln K \left(\frac{M}{m} \right) P + \ln m \frac{MP - P^{1/2}AP^{1/2}}{M - m} + \ln M \frac{P^{1/2}AP^{1/2} - mP}{M - m}.
 \end{aligned}$$

Now, if we take the trace and use the fact that $\text{tr}(P) = 1$, then we get

$$\begin{aligned}
 & \ln m \frac{M - \text{tr}(PA)}{M - m} + \ln M \frac{\text{tr}(PA) - m}{M - m} \\
 & \leq \left[\frac{1}{2} - \frac{1}{M - m} \text{tr} \left(P \left| A - \frac{1}{2}(m + M)I \right| \right) \right] \ln K \left(\frac{M}{m} \right) \\
 & + \ln m \frac{M - \text{tr}(PA)}{M - m} + \ln M \frac{\text{tr}(PA) - m}{M - m} \\
 & \leq \text{tr} [P (\ln A)] \\
 & \leq \left[\frac{1}{2} + \frac{1}{M - m} \text{tr} \left(P \left| A - \frac{1}{2}(m + M)I \right| \right) \right] \ln K \left(\frac{M}{m} \right) \\
 & + \ln m \frac{M - \text{tr}(PA)}{M - m} + \ln M \frac{\text{tr}(PA) - m}{M - m} \\
 & \leq \ln K \left(\frac{M}{m} \right) + \ln m \frac{M - P \text{tr}(PA)}{M - m} + \ln M \frac{\text{tr}(PA) - m}{M - m},
 \end{aligned}$$

namely

$$\begin{aligned}
 (2.6) \quad & \ln \left(m \frac{M - \text{tr}(PA)}{M - m} M \frac{\text{tr}(PA) - m}{M - m} \right) \\
 & \leq \ln K \left(\frac{M}{m} \right)^{\left[\frac{1}{2} - \frac{1}{M - m} \text{tr} \left(P \left| A - \frac{1}{2}(m + M)I \right| \right) \right]} + \ln \left(m \frac{M - \text{tr}(PA)}{M - m} M \frac{\text{tr}(PA) - m}{M - m} \right) \\
 & \leq \text{tr} [P (\ln A)] \\
 & \leq \ln K \left(\frac{M}{m} \right)^{\left[\frac{1}{2} + \frac{1}{M - m} \text{tr} \left(P \left| A - \frac{1}{2}(m + M)I \right| \right) \right]} + \ln \left(m \frac{M - \text{tr}(PA)}{M - m} M \frac{\text{tr}(PA) - m}{M - m} \right) \\
 & \leq \ln K \left(\frac{M}{m} \right) + \ln \left(m \frac{M - \text{tr}(PA)}{M - m} M \frac{\text{tr}(PA) - m}{M - m} \right).
 \end{aligned}$$

Now, if we take the exp in (2.6), then we get the desired result (2.1). \square

Corollary 1. *With the assumption of Theorem 6, we have the alternative inequality*

$$\begin{aligned}
 (2.7) \quad 1 &\leq K \left(\frac{M}{m} \right)^{\left[\frac{1}{2} - \frac{1}{m^{-1} - M^{-1}} \operatorname{tr}(P|A^{-1} - \frac{1}{2}(M^{-1} + m^{-1})I|) \right]} \\
 &\leq \frac{M^{-\frac{m^{-1} - \operatorname{tr}(PA^{-1})}{m^{-1} - M^{-1}}} m^{\frac{\operatorname{tr}(PA^{-1}) - M^{-1}}{m^{-1} - M^{-1}}}}{\Delta_P(A)} \\
 &\leq \left[K \left(\frac{M}{m} \right) \right]^{\left[\frac{1}{2} + \frac{1}{m^{-1} - M^{-1}} \operatorname{tr}(P|A^{-1} - \frac{1}{2}(M^{-1} + m^{-1})I|) \right]} \leq K \left(\frac{M}{m} \right).
 \end{aligned}$$

Proof. If we write the inequality for A^{-1} that satisfies the condition $0 < M^{-1}I \leq A^{-1} \leq m^{-1}I$, then

$$\begin{aligned}
 1 &\leq K \left(\frac{m^{-1}}{M^{-1}} \right)^{\left[\frac{1}{2} - \frac{1}{m^{-1} - M^{-1}} \operatorname{tr}(P|A^{-1} - \frac{1}{2}(M^{-1} + m^{-1})I|) \right]} \\
 &\leq \frac{\Delta_P(A^{-1})}{M^{-\frac{m^{-1} - \operatorname{tr}(PA^{-1})}{m^{-1} - M^{-1}}} m^{-\frac{\operatorname{tr}(PA^{-1}) - M^{-1}}{m^{-1} - M^{-1}}}} \\
 &\leq \left[K \left(\frac{m^{-1}}{M^{-1}} \right) \right]^{\left[\frac{1}{2} + \frac{1}{m^{-1} - M^{-1}} \operatorname{tr}(P|A^{-1} - \frac{1}{2}(M^{-1} + m^{-1})I|) \right]} \\
 &\leq K \left(\frac{m^{-1}}{M^{-1}} \right),
 \end{aligned}$$

namely

$$\begin{aligned}
 1 &\leq K \left(\frac{M}{m} \right)^{\left[\frac{1}{2} - \frac{1}{m^{-1} - M^{-1}} \operatorname{tr}(P|A^{-1} - \frac{1}{2}(M^{-1} + m^{-1})I|) \right]} \\
 &\leq \frac{[\Delta_P(A)]^{-1}}{\left(M^{-\frac{m^{-1} - \operatorname{tr}(PA^{-1})}{m^{-1} - M^{-1}}} m^{-\frac{\operatorname{tr}(PA^{-1}) - M^{-1}}{m^{-1} - M^{-1}}} \right)^{-1}} \\
 &\leq \left[K \left(\frac{M}{m} \right) \right]^{\left[\frac{1}{2} + \frac{1}{m^{-1} - M^{-1}} \operatorname{tr}(P|A^{-1} - \frac{1}{2}(M^{-1} + m^{-1})I|) \right]} \leq K \left(\frac{M}{m} \right),
 \end{aligned}$$

which is equivalent to the desired result (2.7). \square

Corollary 2. *Let $P \geq 0$ with $P \in \mathcal{B}_1(H)$ and $\operatorname{tr}(P) = 1$. If $0 < mI \leq A, B \leq MI$ for positive numbers m, M , then*

$$\begin{aligned}
 (2.8) \quad \Theta(A, B, m, M, P) &\leq \frac{\int_0^1 \Delta_P((1-t)A + tB) dt}{\frac{m^{\frac{M-1}{M-m}} M^{\frac{1-m}{M-m}}}{\ln\left(\frac{M}{m}\right)}} \\
 &\leq K \left(\frac{M}{m} \right) \Theta(A, B, m, M, P),
 \end{aligned}$$

where

$$\Theta(A, B, m, M, P) := \begin{cases} \left(\frac{M}{m} \right)^{\frac{\text{tr}[P(B-A)]}{M-m}} - 1 & \text{if } \text{tr}[P(B-A)] \neq 0, \\ 1 & \text{if } \text{tr}[P(B-A)] = 0. \end{cases}$$

Proof. From (2.7) we get

$$\begin{aligned} & m^{\frac{M-\text{tr}(P[(1-t)A+tB])}{M-m}} M^{\frac{\text{tr}(P[(1-t)A+tB])-m}{M-m}} \\ & \leq \Delta_P((1-t)A+tB) \\ & \leq K \left(\frac{M}{m} \right) m^{\frac{M-\text{tr}(P[(1-t)A+tB])}{M-m}} M^{\frac{\text{tr}(P[(1-t)A+tB])-m}{M-m}} \end{aligned}$$

for $t \in [0, 1]$.

If we take the integral over $t \in [0, 1]$, then we get

$$\begin{aligned} (2.9) \quad & \int_0^1 m^{\frac{M-\text{tr}(P[(1-t)A+tB])}{M-m}} M^{\frac{\text{tr}(P[(1-t)A+tB])-m}{M-m}} dt \\ & \leq \int_0^1 \Delta_P((1-t)A+tB) dt \\ & \leq K \left(\frac{M}{m} \right) \int_0^1 m^{\frac{M-\text{tr}(P[(1-t)A+tB])}{M-m}} M^{\frac{\text{tr}(P[(1-t)A+tB])-m}{M-m}} dt. \end{aligned}$$

Observe that

$$\begin{aligned} & \int_0^1 m^{\frac{\text{tr}(P[(1-t)A+tB])}{M-m}} M^{\frac{\text{tr}(P[(1-t)A+tB])-m}{M-m}} dt \\ & = m^{\frac{M}{M-m}} M^{\frac{-m}{M-m}} \int_0^1 \left(\frac{M}{m} \right)^{\frac{\text{tr}(P[(1-t)A+tB])}{M-m}} dt \\ & = m^{\frac{M}{M-m}} M^{\frac{-m}{M-m}} \left(\frac{M}{m} \right)^{\frac{1}{M-m}} \int_0^1 \left(\frac{M}{m} \right)^{t \frac{\text{tr}[P(B-A)]}{M-m}} dt \\ & = m^{\frac{M-1}{M-m}} M^{\frac{1-m}{M-m}} \int_0^1 \left(\frac{M}{m} \right)^{t \frac{\text{tr}[P(B-A)]}{M-m}} dt. \end{aligned}$$

Since for $a > 0$, $a \neq 1$ and $b \in \mathbb{R}$ we have

$$\int_0^1 a^{bx} dx = \frac{a^b - 1}{b \ln a},$$

then for $\text{tr}[P(B-A)] \neq 0$

$$\int_0^1 \left(\frac{M}{m} \right)^{t \frac{\text{tr}[P(B-A)]}{M-m}} dt = \frac{\left(\frac{M}{m} \right)^{\frac{\text{tr}[P(B-A)]}{M-m}} - 1}{\frac{\text{tr}[P(B-A)]}{M-m} \ln \left(\frac{M}{m} \right)}$$

and by (2.9) we derive (2.8). □

3. RELATED RESULTS

We also have:

Theorem 7. *Let $P \geq 0$ with $P \in \mathcal{B}_1(H)$ and $\text{tr}(P) = 1$. If A is an operator satisfying the condition $0 < mI \leq A \leq MI$, then*

$$\begin{aligned}
 (3.1) \quad 1 &\leq \left[K \left(\frac{M}{m} \right) \right]^{\frac{1}{2} - \frac{1}{\ln M - \ln m} \left| \text{tr}(P \ln A) - \frac{\ln M + \ln m}{2} \right|} \\
 &\leq \frac{\frac{\ln M - \text{tr}(P \ln A)}{\ln M - \ln m} m + \frac{\text{tr}(P \ln A) - \ln m}{\ln M - \ln m} M}{\Delta_P(A)} \\
 &\leq \left[K \left(\frac{M}{m} \right) \right]^{\frac{1}{2} + \frac{1}{\ln M - \ln m} \left| \text{tr}(P \ln A) - \frac{\ln M + \ln m}{2} \right|} \leq K \left(\frac{M}{m} \right).
 \end{aligned}$$

Proof. Assume that $m^{1-\nu} M^\nu = \exp s$, then $s = (1-\nu) \ln m + \nu \ln M \in [\ln m, \ln M]$ which gives that

$$\nu = \frac{s - \ln m}{\ln M - \ln m}.$$

Also

$$\min \{1 - \nu, \nu\} = \frac{1}{2} - \frac{1}{\ln M - \ln m} \left| s - \frac{\ln M + \ln m}{2} \right|$$

and

$$\max \{1 - \nu, \nu\} = \frac{1}{2} + \frac{1}{\ln M - \ln m} \left| s - \frac{\ln M + \ln m}{2} \right|$$

From (1.14) we have

$$\begin{aligned}
 \exp s &\leq \exp s \left[K \left(\frac{M}{m} \right) \right]^{\frac{1}{2} - \frac{1}{\ln M - \ln m} \left| s - \frac{\ln M + \ln m}{2} \right|} \\
 &\leq \frac{\ln M - s}{\ln M - \ln m} m + \frac{s - \ln m}{\ln M - \ln m} M \\
 &\leq \exp s \left[K \left(\frac{M}{m} \right) \right]^{\frac{1}{2} + \frac{1}{\ln M - \ln m} \left| s - \frac{\ln M + \ln m}{2} \right|}
 \end{aligned}$$

namely

$$\begin{aligned}
 1 &\leq \left[K \left(\frac{M}{m} \right) \right]^{\frac{1}{2} - \frac{1}{\ln M - \ln m} \left| s - \frac{\ln M + \ln m}{2} \right|} \\
 &\leq \frac{\frac{\ln M - s}{\ln M - \ln m} m + \frac{s - \ln m}{\ln M - \ln m} M}{\exp s} \\
 &\leq \left[K \left(\frac{M}{m} \right) \right]^{\frac{1}{2} + \frac{1}{\ln M - \ln m} \left| s - \frac{\ln M + \ln m}{2} \right|}
 \end{aligned}$$

for $s \in [\ln m, \ln M]$.

If $0 < m \leq A \leq M$ and $P \in \mathcal{B}_1(H)$ with $\text{tr}(P) = 1$, then $\ln m \leq \text{tr}(P \ln A) \leq \ln M$ and for $s = \text{tr}(P \ln A)$ we deduce

$$\begin{aligned} 1 &\leq \left[K \left(\frac{M}{m} \right) \right]^{\frac{1}{2} - \frac{1}{\ln M - \ln m} \left| \text{tr}(P \ln A) - \frac{\ln M + \ln m}{2} \right|} \\ &\leq \frac{\frac{\ln M - \text{tr}(P \ln A)}{\ln M - \ln m} m + \frac{\text{tr}(P \ln A) - \ln m}{\ln M - \ln m} M}{\exp[\text{tr}(P \ln A)]} \\ &\leq \left[K \left(\frac{M}{m} \right) \right]^{\frac{1}{2} + \frac{1}{\ln M - \ln m} \left| \text{tr}(P \ln A) - \frac{\ln M + \ln m}{2} \right|} \end{aligned}$$

and the inequality (3.1) is proved. \square

Corollary 3. *With the assumption of Theorem 6, we have*

$$\begin{aligned} (3.2) \quad 1 &\leq \left[K \left(\frac{M}{m} \right) \right]^{\frac{1}{2} - \frac{1}{\ln M - \ln m} \left| \text{tr}(P \ln A) - \frac{\ln M + \ln m}{2} \right|} \\ &\leq \frac{\Delta_P(A)}{\left(\frac{\text{tr}(P \ln A) - \ln m}{\ln M - \ln m} M^{-1} + \frac{\ln M - \text{tr}(P \ln A)}{\ln M - \ln m} m^{-1} \right)^{-1}} \\ &\leq \left[K \left(\frac{M}{m} \right) \right]^{\frac{1}{2} + \frac{1}{\ln M - \ln m} \left| \text{tr}(P \ln A) - \frac{\ln M + \ln m}{2} \right|} \leq K \left(\frac{M}{m} \right). \end{aligned}$$

Proof. If we write the inequality (3.1) for A^{-1} that satisfies the condition $0 < M^{-1}I \leq A^{-1} \leq m^{-1}I$, then we obtain

$$\begin{aligned} 1 &\leq K \left(\frac{m^{-1}}{M^{-1}} \right)^{\frac{1}{2} - \frac{1}{\ln m^{-1} - \ln M^{-1}} \left| \text{tr}(P \ln A^{-1}) - \frac{\ln m^{-1} + \ln M^{-1}}{2} \right|} \\ &\leq \frac{\frac{\ln m^{-1} - \text{tr}(P \ln A^{-1})}{\ln m^{-1} - \ln M^{-1}} M^{-1} + \frac{\text{tr}(P \ln A^{-1}) - \ln M^{-1}}{\ln m^{-1} - \ln M^{-1}} m^{-1}}{\Delta_x(A^{-1})} \\ &\leq K \left(\frac{m^{-1}}{M^{-1}} \right)^{\frac{1}{2} + \frac{1}{\ln m^{-1} - \ln M^{-1}} \left| \text{tr}(P \ln A^{-1}) - \frac{\ln m^{-1} + \ln M^{-1}}{2} \right|} \leq M \left(\frac{m^{-1}}{M^{-1}} \right), \end{aligned}$$

namely

$$\begin{aligned} 1 &\leq K \left(\frac{M}{m} \right)^{\frac{1}{2} - \frac{1}{\ln M - \ln m} \left| \text{tr}(P \ln A) - \frac{\ln m + \ln M}{2} \right|} \\ &\leq \frac{\frac{\text{tr}(P \ln A) - \ln m}{\ln M - \ln m} M^{-1} + \frac{\ln M - \text{tr}(P \ln A)}{\ln M - \ln m} m^{-1}}{\Delta_x(A^{-1})} \\ &\leq K \left(\frac{M}{m} \right)^{\frac{1}{2} + \frac{1}{\ln M - \ln m} \left| \text{tr}(P \ln A) - \frac{\ln m + \ln M}{2} \right|} \leq K \left(\frac{M}{m} \right), \end{aligned}$$

for $P \in \mathcal{B}_1(H)$ with $\text{tr}(P) = 1$.

This proves (3.2). \square

REFERENCES

- [1] S. S. Dragomir, Trace inequalities for operators in Hilbert spaces: a survey of recent results, *Aust. J. Math. Anal. Appl.* Vol. **19** (2022), No. 1, Art. 1, 202 pp. [Online <https://ajmaa.org/searchroot/files/pdf/v19n1/v19i1p1.pdf>].

- [2] S. S. Dragomir, Some properties of trace class P -determinant of positive operators in Hilbert spaces, Preprint *RGMIA Res. Rep. Coll.* **25** (2022), Art.
- [3] B. Fuglede and R. V. Kadison, Determinant theory in finite factors, *Ann. of Math. (2)* **55** (1952), 520–530.
- [4] J. I. Fujii and Y. Seo, Determinant for positive operators, *Sci. Math.*, **1** (1998), 153–156.
- [5] J. I. Fujii, S. Izumino and Y. Seo, Determinant for positive operators and Specht's Theorem, *Sci. Math.*, **1** (1998), 307–310.
- [6] S. Hiramatsu and Y. Seo, Determinant for positive operators and Oppenheim's inequality, *J. Math. Inequal.*, Volume 15 (2021), Number 4, 1637–1645
- [7] W. Liao, J. Wu and J. Zhao, New versions of reverse Young and Heinz mean inequalities with the Kantorovich constant, *Taiwanese J. Math.* **19** (2015), No. 2, pp. 467-479.
- [8] G. Zuo, G. Shi and M. Fujii, Refined Young inequality with Kantorovich constant, *J. Math. Inequal.*, **5** (2011), 551-556.

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