ON SOME UPPER AND LOWER BOUNDS FOR TRACE CLASS P-DETERMINANT OF POSITIVE OPERATORS IN HILBERT SPACES

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Abstract. Let H be a complex Hilbert space. For a given operator $P \geq 0$ with $P \in \mathcal{B}_1(H)$, the trace class associated to $\mathcal{B}(H)$ and $\operatorname{tr}(P) = 1$, we define the P-*determinant* of the positive invertible operator A by

$$\Delta_P(A) := \exp \operatorname{tr}(P \ln A).$$

In this paper we show among others that, if A is an operator satisfying the condition $0 < mI \leq A \leq MI$, then

$$\begin{aligned} 1 &\leq \exp\left[\frac{1}{2M^2} \operatorname{tr}\left[P(MI - A)(A - mI)\right]\right] \\ &\leq \frac{\Delta_P(A)}{m^{\frac{M - \operatorname{tr}(PA)}{M-m}} M^{\frac{\operatorname{tr}(PA) - m}{M-m}}} \\ &\leq \exp\left[\frac{1}{2m^2} \operatorname{tr}\left[P(MI - A)(A - mI)\right]\right] \\ &\leq \exp\left[\frac{1}{2m^2}(M - \operatorname{tr}(PA))(\operatorname{tr}(PA) - m)\right] \leq \exp\left[\frac{1}{8}\left(\frac{M}{m} - 1\right)^2\right]. \end{aligned}$$

1. Introduction

In 1952, in the paper [6], B. Fuglede and R. V. Kadison introduced the determinant of a (invertible) operator and established its fundamental properties. The notion generalizes the usual determinant and can be considered for any operator in a finite von Neumann algebra (M, τ) with a faithful normal trace.

Let $T \in M$ be normal and $|T| := (T^*T)^{1/2}$ its modulus. By the spectral theorem one can represent T as an integral

$$T = \int_{\operatorname{Sp}(T)} \lambda dE(\lambda),$$

where $E(\lambda)$ is a projection valued measure and $\operatorname{Sp}(T)$ is the spectrum of T. The measure $\mu_T := \tau \circ E$ becomes a probability measure on the complex plane and has the support in the spectrum $\operatorname{Sp}(T)$.

For any $T \in M$ the Fuglede-Kadison determinant (*FK-determinant*) is defined by

$$\Delta_{FK}(T) := \exp\left(\int_0^\infty \ln t d\mu_{|T|}\right).$$

If T is invertible, then

$$\Delta_{FK}(T) := \exp(\tau(\ln(|T|))),$$

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where $\ln(|T|)$ is defined by the use of functional calculus.

Let $B(H)$ be the space of all bounded linear operators on a Hilbert space H, and I stands for the identity operator on H. An operator A in $B(H)$ is said to be positive (in symbol: $A \geq 0$) if $\langle Ax, x\rangle \geq 0$ for all $x \in H$. In particular, $A > 0$ means that A is positive and invertible. For a pair A, B of selfadjoint operators the order relation $A \geq B$ means as usual that $A - B$ is positive.

In 1998, Fujii et al. [7], [8], introduced the *normalized determinant* $\Delta_x(A)$ for positive invertible operators A on a Hilbert space H and a fixed unit vector $x \in H$, namely $\|x\| = 1$, defined by

$$\Delta_x(A) := \exp\langle \ln Ax, x\rangle$$

and discussed it as a continuous geometric mean and observed some inequalities around the determinant from this point of view. For some recent results, see [9].

We need now some preparations for trace of operators in Hilbert spaces.

Let $(H, \langle\cdot,\cdot\rangle)$ be a complex Hilbert space and $\{e_i\}_{i\in I}$ an *orthonormal basis* of H. We say that $A \in \mathcal{B}(H)$ is a *Hilbert-Schmidt operator* if

$$\sum_{i\in I} \|Ae_i\|^2 < \infty. \tag{1.1}$$

It is well know that, if $\{e_i\}_{i\in I}$ and $\{f_j\}_{j\in J}$ are orthonormal bases for H and $A \in \mathcal{B}(H)$ then

$$\sum_{i\in I} \|Ae_i\|^2 = \sum_{j\in I} \|Af_j\|^2 = \sum_{j\in I} \|A^* f_j\|^2 \tag{1.2}$$

showing that the definition (1.1) is independent of the orthonormal basis and A is a Hilbert-Schmidt operator iff A^* is a Hilbert-Schmidt operator.

Let $\mathcal{B}_2(H)$ the set of *Hilbert-Schmidt operators* in $\mathcal{B}(H)$. For $A \in \mathcal{B}_2(H)$ we define

$$\|A\|_2 := \left(\sum_{i\in I} \|Ae_i\|^2\right)^{1/2} \tag{1.3}$$

for $\{e_i\}_{i\in I}$ an orthonormal basis of H.

Using the triangle inequality in $l^2(I)$, one checks that $\mathcal{B}_2(H)$ is a *vector space* and that $\|\cdot\|_2$ is a norm on $\mathcal{B}_2(H)$, which is usually called in the literature as the *Hilbert-Schmidt norm.*

Denote *the modulus* of an operator $A \in \mathcal{B}(H)$ by $|A| := (A^*A)^{1/2}$.

Because $\||A|x\| = \|Ax\|$ for all $x \in H$, A is Hilbert-Schmidt iff $|A|$ is Hilbert-Schmidt and $\|A\|_2 = \||A|\|_2$. From (1.2) we have that if $A \in \mathcal{B}_2(H)$, then $A^* \in \mathcal{B}_2(H)$ and $\|A\|_2 = \|A^*\|_2$.

The following theorem collects some of the most important properties of Hilbert-Schmidt operators:

Theorem 1. *We have:*

(i) $(\mathcal{B}_2(H), \|\cdot\|_2)$ *is a Hilbert space with inner product*

$$\langle A, B\rangle_2 := \sum_{i\in I} \langle Ae_i, Be_i\rangle = \sum_{i\in I} \langle B^*Ae_i, e_i\rangle \tag{1.4}$$

and the definition does not depend on the choice of the orthonormal basis $\{e_i\}_{i\in I}$;

(ii) We have the inequalities

$$\|A\| \le \|A\|_2 \tag{1.5}$$

for any $A \in \mathcal{B}_2(H)$ *and, if* $A \in \mathcal{B}_2(H)$ *and* $T \in \mathcal{B}(H)$, *then* $AT, TA \in \mathcal{B}_2(H)$ *with*

$$\|AT\|_2, \ \ \|TA\|_2 \le \|T\| \|A\|_2 \tag{1.6}$$

(iii) $\mathcal{B}_2(H)$ *is an operator ideal in* $\mathcal{B}(H)$, *i.e.*

$$\mathcal{B}(H)\mathcal{B}_2(H)\mathcal{B}(H) \subseteq \mathcal{B}_2(H).$$

If $\{e_i\}_{i\in I}$ an orthonormal basis of H, we say that $A \in \mathcal{B}(H)$ is *trace class* if

$$\|A\|_1 := \sum_{i\in I} \langle |A| e_i, e_i\rangle < \infty. \tag{1.7}$$

The definition of $\|A\|_1$ does not depend on the choice of the orthonormal basis $\{e_i\}_{i\in I}$. We denote by $\mathcal{B}_1(H)$ the set of trace class operators in $\mathcal{B}(H)$.

The following proposition holds:

Proposition 1. *If* $A \in \mathcal{B}(H)$, *then the following are equivalent:*
(i) $A \in \mathcal{B}_1(H)$;
(ii) $|A|^{1/2} \in \mathcal{B}_2(H)$.

The following properties are also well known:

Theorem 2. *With the above notations:*
(i) We have

$$\|A\|_1 = \|A^*\|_1 \ \text{ and } \ \|A\|_2 \le \|A\|_1 \tag{1.8}$$

for any $A \in \mathcal{B}_1(H)$;
(ii) $\mathcal{B}_1(H)$ *is an operator ideal in* $\mathcal{B}(H)$, *i.e.*

$$\mathcal{B}(H)\mathcal{B}_1(H)\mathcal{B}(H) \subseteq \mathcal{B}_1(H);$$

(iii) We have

$$\mathcal{B}_2(H)\mathcal{B}_2(H) = \mathcal{B}_1(H);$$

(iv) We have

$$\|A\|_1 = \sup\{\langle A, B\rangle_2 \mid B \in \mathcal{B}_2(H), \ \ \|B\|_2 \le 1\};$$

(v) $(\mathcal{B}_1(H), \|\cdot\|_1)$ *is a Banach space.*

We define the *trace* of a trace class operator $A \in \mathcal{B}_1(H)$ to be

$$\operatorname{tr}(A) := \sum_{i\in I} \langle Ae_i, e_i\rangle, \tag{1.9}$$

where $\{e_i\}_{i\in I}$ an orthonormal basis of H. Note that this coincides with the usual definition of the trace if H is finite-dimensional. We observe that the series (1.9) converges absolutely and it is independent from the choice of basis.

The following result collects some properties of the trace:

Theorem 3. *We have:*
(i) If $A \in \mathcal{B}_1(H)$ *then* $A^* \in \mathcal{B}_1(H)$ *and*

$$\operatorname{tr}(A^*) = \overline{\operatorname{tr}(A)}; \tag{1.10}$$

(ii) If $A \in \mathcal{B}_1(H)$ *and* $T \in \mathcal{B}(H)$, *then* $AT,\ TA \in \mathcal{B}_1(H)$,

$$\operatorname{tr}(AT) = \operatorname{tr}(TA) \text{ and } |\operatorname{tr}(AT)| \le \|A\|_1 \|T\|; \tag{1.11}$$

(iii) $\operatorname{tr}(\cdot)$ *is a bounded linear functional on* $\mathcal{B}_1(H)$ *with* $\|\operatorname{tr}\| = 1$;
(iv) If $A,\ B \in \mathcal{B}_2(H)$ *then* $AB,\ BA \in \mathcal{B}_1(H)$ *and* $\operatorname{tr}(AB) = \operatorname{tr}(BA)$.

Now, if we assume that $P \ge 0$ and $P \in \mathcal{B}_1(H)$, then for all $T \in \mathcal{B}(H)$, PT, $TP \in \mathcal{B}_1(H)$ and $\operatorname{tr}(PT) = \operatorname{tr}(TP)$. Also, since $P^{1/2} \in \mathcal{B}_2(H)$, $TP^{1/2} \in \mathcal{B}_2(H)$, hence $P^{1/2}TP^{1/2}$ and $TP^{1/2}P^{1/2} = TP \in \mathcal{B}_1(H)$ with $\operatorname{tr}\left(P^{1/2}TP^{1/2}\right) = \operatorname{tr}(TP)$. Therefore, if $P \ge 0$ and $P \in \mathcal{B}_1(H)$,

$$\operatorname{tr}(PT) = \operatorname{tr}(TP) = \operatorname{tr}\left(P^{1/2}TP^{1/2}\right)$$

for all $T \in \mathcal{B}(H)$.

If $T \ge 0$, then $P^{1/2}TP^{1/2} \ge 0$, which implies that $\operatorname{tr}(PT) \ge 0$ that shows that the functional $\mathcal{B}(H) \ni T \longmapsto \operatorname{tr}(PT)$ is linear and isotonic functional. Also, by (1.11), if $T_n \to T$ for $n \to \infty$ in $\mathcal{B}(H)$ then $\lim_{n\to\infty} \operatorname{tr}(PT_n) = \operatorname{tr}(PT)$, namely $\mathcal{B}(H) \ni T \longmapsto \operatorname{tr}(PT)$ is also continuous in the norm topology.

For a survey on recent trace inequalities see [4] and the references therein.

Now, for a given $P \ge 0$ with $P \in \mathcal{B}_1(H)$ and $\operatorname{tr}(P) = 1$, we define the P-*determinant* of the positive invertible operator A by

$$\Delta_P(A) := \exp \operatorname{tr}(P \ln A) = \exp \operatorname{tr}((\ln A) P) = \exp \operatorname{tr}\left(P^{1/2}(\ln A) P^{1/2}\right). \tag{1.12}$$

Assume that $P \ge 0$ with $P \in \mathcal{B}_1(H)$ and $\operatorname{tr}(P) = 1$. We observe that we have the following elementary properties:

(i) *continuity*: the map $A \to \Delta_P(A)$ is norm continuous;
(ii) *power equality:* $\Delta_P(A^t) = \Delta_P(A)^t$ for all $t > 0$;
(iii) *homogeneity*: $\Delta_P(tA) = t\Delta_x(A)$ and $\Delta_P(tI) = t$ for all $t > 0$;
(iv) *monotonicity*: $0 < A \le B$ implies $\Delta_P(A) \le \Delta_P(B)$.

In the recent paper [5] we obtained the following results:

Theorem 4. *Let* $P \ge 0$ *with* $P \in \mathcal{B}_1(H)$ *and* $\operatorname{tr}(P) = 1$, *then for all* $A,\ B > 0$ *and* $t \in [0,1]$,

$$\Delta_P((1-t)A + tB) \ge [\Delta_P(A)]^{1-t} [\Delta_P(B)]^t.$$

and

Theorem 5. *Let* $P \ge 0$ *with* $P \in \mathcal{B}_1(H)$ *and* $\operatorname{tr}(P) = 1$, *then for all* $A > 0$ *and* $a > 0$ *we have the double inequality*

$$a \exp\left[1 - a \operatorname{tr}\left(PA^{-1}\right)\right] \le \Delta_P(A) \le a \exp\left[a^{-1} \operatorname{tr}(PA) - 1\right].$$

In particular

$$1 \le \frac{\operatorname{tr}(PA)}{\Delta_P(A)} \le \exp\left[\operatorname{tr}(PA) \operatorname{tr}\left(PA^{-1}\right) - 1\right]$$

and

$$1 \le \frac{\Delta_P(A)}{\left[\operatorname{tr}\left(PA^{-1}\right)\right]^{-1}} \le \exp\left[\operatorname{tr}\left(PA^{-1}\right) \operatorname{tr}(PA) - 1\right].$$

Kittaneh and Manasrah [10], [11] provided a refinement and an additive reverse for Young inequality as follows:

$$r\left(\sqrt{a} - \sqrt{b}\right)^2 \le (1-\nu)a + \nu b - a^{1-\nu}b^{\nu} \le R\left(\sqrt{a} - \sqrt{b}\right)^2 \tag{1.13}$$

where $a,\ b>0,\ \nu\in[0,1],\ r=\min\{1-\nu,\nu\}$ and $R=\max\{1-\nu,\nu\}$. The case $\nu=\frac{1}{2}$ reduces (1.13) to an identity.

For some operator versions of (1.13) see [10] and [11].

We also have the following inequality that provides a refinement and a reverse for the celebrated Young's inequality

$$\frac{1}{2}\nu(1-\nu)\frac{(b-a)^2}{\max\{a,b\}}\le(1-\nu)a+\nu b-a^{1-\nu}b^{\nu}\le\frac{1}{2}\nu(1-\nu)\frac{(b-a)^2}{\min\{a,b\}} \tag{1.14}$$

for any $a,\ b>0$ and $\nu\in[0,1]$.

This result was obtained in 1978 by Cartwright and Field [1] who established a more general result for n variables and gave an application for a probability measure supported on a finite interval.

Motivated by the above results, in this paper we show among others that, if A is an operator satisfying the condition $0<mI\le A\le MI$, then

$$\begin{aligned}1&\le\exp\left[\frac{1}{2M^2}\operatorname{tr}[P(MI-A)(A-mI)]\right]\\&\le\frac{\Delta_P(A)}{m^{\frac{M-\operatorname{tr}(PA)}{M-m}}M^{\frac{\operatorname{tr}(PA)-m}{M-m}}}\\&\le\exp\left[\frac{1}{2m^2}\operatorname{tr}[P(MI-A)(A-mI)]\right]\\&\le\exp\left[\frac{1}{2m^2}(M-\operatorname{tr}(PA))(\operatorname{tr}(PA)-m)\right]\le\exp\left[\frac{1}{8}\left(\frac{M}{m}-1\right)^2\right].\end{aligned}$$

2. Main Results

The first result is as follows:

Theorem 6. *Let $P\ge0$ with $P\in\mathcal{B}_1(H)$ and $\operatorname{tr}(P)=1$. Assume that $0<mI\le A\le MI$, then*

$$\begin{aligned}1&\le\frac{\Delta_P(A)}{m^{\frac{M-\operatorname{tr}(PA)}{M-m}}M^{\frac{\operatorname{tr}(PA)-m}{M-m}}}\le\exp\left[\frac{1}{Mm}\operatorname{tr}[P(MI-A)(A-mI)]\right]\\&\le\exp\left[\frac{1}{Mm}(M-\operatorname{tr}(PA))(\operatorname{tr}(PA)-m)\right]\\&\le\exp\left[\frac{1}{4Mm}(M-m)^2\right].\end{aligned} \tag{2.1}$$

Proof. In [2] we obtained the following reverses of Young's inequality:

$$1\le\frac{(1-\nu)a+\nu b}{a^{1-\nu}b^{\nu}}\le\exp\left[4\nu(1-\nu)\left(K\left(\frac{a}{b}\right)-1\right)\right],$$

where $a,\ b>0,\ \nu\in[0,1]$.

This is equivalent, by taking the logarithm, with

$$0\le\ln((1-\nu)a+\nu b)-(1-\nu)\ln a-\nu\ln b\le\nu(1-\nu)\frac{(b-a)^2}{ba}$$

where $a,\ b>0,\ \nu\in[0,1]$.

If we take $a=m$, $b=M$, $t\in[m,M]$ and $\nu=\frac{t-m}{M-m}\in[0,1]$, then we get

$$\begin{aligned} 0 &\le \ln t-\frac{M-t}{M-m}\ln m-\frac{t-m}{M-m}\ln M\le \frac{(M-t)(t-m)}{(M-m)^2}\frac{(M-m)^2}{Mm} \\ &= \frac{(M-t)(t-m)}{Mm}. \end{aligned}$$

Using the continuous functional calculus for selfadjoint operators, we have

$$0\le \ln A-\frac{MI-A}{M-m}\ln m-\frac{AP^{1/2}-mI}{M-m}\ln M\le \frac{(MI-A)(A-mI)}{Mm}.$$

If we multiply both sides by $P^{1/2}$ we get

$$\begin{aligned} 0 &\le P^{1/2}(\ln A)P^{1/2}-\frac{MP-P^{1/2}AP^{1/2}}{M-m}\ln m-\frac{P^{1/2}AP^{1/2}-mP}{M-m}\ln M \\ &\le \frac{P^{1/2}(MI-A)(A-mI)P^{1/2}}{Mm}. \end{aligned}$$

If we take the trace and use the fact that $\operatorname{tr}(P)=1$, then we obtain

$$\begin{aligned} 0 &\le \operatorname{tr}(P\ln A)-\frac{M-\operatorname{tr}(PA)}{M-m}\ln m-\frac{\operatorname{tr}(PA)-m}{M-m}\ln M \\ &\le \frac{1}{Mm}\operatorname{tr}[P(MI-A)(A-mI)]. \end{aligned}$$

If we take the exponential, then we get

$$\begin{aligned} 1 &\le \frac{\exp[\operatorname{tr}(P\ln A)]}{\exp\left[\frac{M-\operatorname{tr}(PA)}{M-m}\ln m+\frac{\operatorname{tr}(PA)-m}{M-m}\ln M\right]} \\ &\le \exp\left[\frac{1}{Mm}\operatorname{tr}[P(MI-A)(A-mI)]\right]. \end{aligned} \tag{2.2}$$

Observe that

$$\begin{aligned} \exp\left[\frac{M-\operatorname{tr}(PA)}{M-m}\ln m+\frac{\operatorname{tr}(PA)-m}{M-m}\ln M\right] &= \exp\left[\ln\left(m^{\frac{M-\operatorname{tr}(PA)}{M-m}}M^{\frac{\operatorname{tr}(PA)-m}{M-m}}\right)\right] \\ &= m^{\frac{M-\operatorname{tr}(PA)}{M-m}}M^{\frac{\operatorname{tr}(PA)-m}{M-m}} \end{aligned}$$

and by (2.2) we obtain the first inequality in (2.1).

The function $g(t)=(M-t)(t-m)$ is concave on $[m,M]$ and by Jensen's inequality for trace

$$\operatorname{tr}(Pg(A))\le g(\operatorname{tr}(PA)),$$

for $P\ge 0$ with $P\in\mathcal{B}_1(H)$ and $\operatorname{tr}(P)=1$, we have

$$\operatorname{tr}[(MI-A)(A-mI)]\le((M-\operatorname{tr}(PA))(\operatorname{tr}(PA)-m))$$

which proves the third inequality in (2.1). □

Corollary 1. *With the assumptions of Theorem 6,*

$$1 \leq \frac{M^{\frac{m^{-1}-\operatorname{tr}\left(PA^{-1}\right)}{m^{-1}-M^{-1}}} m^{\frac{\operatorname{tr}\left(PA^{-1}\right)-M^{-1}}{m^{-1}-M^{-1}}}}{\Delta_P(A)} \tag{2.3}$$
$$\leq \exp\left[mM \operatorname{tr}\left[P\left(m^{-1}I - A^{-1}\right)\left(A^{-1} - M^{-1}I\right)\right]\right]$$
$$\leq \exp\left[mM\left(m^{-1} - \operatorname{tr}\left(PA^{-1}\right)\right)\left(\operatorname{tr}\left(PA^{-1}\right) - M^{-1}\right)\right]$$
$$\leq \exp\left[\frac{1}{4}mM\left(M-m\right)^2\right].$$

Proof. Observe that $0 < mI \leq A \leq MI$ implies that $0 < M^{-1}I \leq A^{-1} \leq m^{-1}I$. If we write the inequality (2.1) for A^{-1}, then we get

$$1 \leq \frac{\Delta_P(A^{-1})}{M^{-\frac{m^{-1}-\operatorname{tr}\left(PA^{-1}\right)}{m^{-1}-M^{-1}}} m^{-\frac{\operatorname{tr}\left(PA^{-1}\right)-M^{-1}}{m^{-1}-M^{-1}}}}$$
$$\leq \exp\left[\frac{1}{m^{-1}M^{-1}} \operatorname{tr}\left[P\left(m^{-1}I - A^{-1}\right)\left(A^{-1} - M^{-1}I\right)\right]\right]$$
$$\leq \exp\left[\frac{1}{m^{-1}M^{-1}}\left(m^{-1} - \operatorname{tr}\left(PA^{-1}\right)\right)\left(\operatorname{tr}\left(PA^{-1}\right) - M^{-1}\right)\right]$$
$$\leq \exp\left[\frac{1}{4m^{-1}M^{-1}}\left(m^{-1} - M^{-1}\right)^2\right],$$

which is equivalent to (2.3). □

In [3] we obtained the following refinement and reverse of Young's inequality:

$$\exp\left[\frac{1}{2}\nu\left(1-\nu\right)\left(1 - \frac{\min\{a,b\}}{\max\{a,b\}}\right)^2\right] \tag{2.4}$$
$$\leq \frac{\left(1-\nu\right)a + \nu b}{a^{1-\nu}b^{\nu}}$$
$$\leq \exp\left[\frac{1}{2}\nu\left(1-\nu\right)\left(\frac{\max\{a,b\}}{\min\{a,b\}} - 1\right)^2\right],$$

for any a, $b > 0$ and $\nu \in [0,1]$.

Theorem 7. *Let $P \geq 0$ with $P \in \mathcal{B}_1(H)$ and $\operatorname{tr}(P) = 1$. Assume that $0 < mI \leq A \leq MI$, then*

$$1 \leq \exp\left[\frac{1}{2M^2} \operatorname{tr}\left[P\left(MI - A\right)\left(A - mI\right)\right]\right] \tag{2.5}$$
$$\leq \frac{\Delta_P(A)}{m^{\frac{M-\operatorname{tr}(PA)}{M-m}} M^{\frac{\operatorname{tr}(PA)-m}{M-m}}}$$
$$\leq \exp\left[\frac{1}{2m^2} \operatorname{tr}\left[P\left(MI - A\right)\left(A - mI\right)\right]\right]$$
$$\leq \exp\left[\frac{1}{2m^2}\left(M - \operatorname{tr}\left(PA\right)\right)\left(\operatorname{tr}\left(PA\right) - m\right)\right]$$
$$\leq \exp\left[\frac{1}{8}\left(\frac{M}{m} - 1\right)^2\right].$$

Proof. From (2.4) we have

$$\begin{aligned}&\exp\left[\frac{1}{2}\nu(1-\nu)\left(1-\frac{m}{M}\right)^2\right]\\&\leq\frac{(1-\nu)m+\nu M}{m^{1-\nu}M^{\nu}}\leq\exp\left[\frac{1}{2}\nu(1-\nu)\left(\frac{M}{m}-1\right)^2\right],\end{aligned}$$

for $\nu\in[0,1]$.

By taking the logarithm, we obtain

$$\begin{aligned}&\frac{1}{2}\nu(1-\nu)\left(1-\frac{m}{M}\right)^2\\&\leq\ln((1-\nu)m+\nu M)-(1-\nu)\ln m-\nu\ln M\\&\leq\frac{1}{2}\nu(1-\nu)\left(\frac{M}{m}-1\right)^2,\end{aligned}\tag{2.6}$$

for $\nu\in[0,1]$.

If we take $a=m$, $b=M$, $t\in[m,M]$ and $\nu=\frac{t-m}{M-m}\in[0,1]$, then we get

$$\begin{aligned}\frac{(M-t)(t-m)}{2M^2}&\leq\ln t-\frac{M-t}{M-m}\ln m-\frac{t-m}{M-m}\ln M\\&\leq\frac{(M-t)(t-m)}{2m^2}\end{aligned}$$

for $t\in[m,M]$.

As above, we get the trace inequality

$$\begin{aligned}&\frac{1}{2M^2}\operatorname{tr}[P(MI-A)(A-mI)]\\&\leq\operatorname{tr}(P\ln A)-\frac{M-\operatorname{tr}(PA)}{M-m}\ln m-\frac{\operatorname{tr}(PA)-m}{M-m}\ln M\\&\leq\frac{1}{2m^2}\operatorname{tr}[P(MI-A)(A-mI)].\end{aligned}$$

If we take the exponential, then we derive

$$\begin{aligned}&\exp\left[\frac{1}{2M^2}\operatorname{tr}[P(MI-A)(A-mI)]\right]\\&\leq\frac{\exp[\operatorname{tr}(P\ln A)]}{\exp\left[\frac{M-\operatorname{tr}(PA)}{M-m}\ln m+\frac{\operatorname{tr}(PA)-m}{M-m}\ln M\right]}\\&\leq\exp\left[\frac{1}{2m^2}\operatorname{tr}[P(MI-A)(A-mI)]\right],\end{aligned}$$

which proves the first part of (2.5).

The second part is obvious. □

Corollary 2. *With the assumptions of Theorem 6,*

$$\begin{aligned}
1 &\le \exp\left[\frac{1}{2}m^2 \operatorname{tr}\left[P\left(m^{-1}I - A^{-1}\right)\left(A^{-1} - M^{-1}I\right)\right]\right] \\
&\le \frac{M^{\frac{m^{-1}-\operatorname{tr}\left(PA^{-1}\right)}{m^{-1}-M^{-1}}} m^{\frac{\operatorname{tr}\left(PA^{-1}\right)-M^{-1}}{m^{-1}-M^{-1}}}}{\Delta_P(A)} \\
&\le \exp\left[\frac{1}{2}M^2 \operatorname{tr}\left[P\left(m^{-1}I - A^{-1}\right)\left(A^{-1} - M^{-1}I\right)\right]\right] \\
&\le \exp\left[\frac{1}{2}M^2\left(m^{-1} - \operatorname{tr}\left(PA^{-1}\right)\right)\left(\operatorname{tr}\left(PA^{-1}\right) - M^{-1}\right)\right] \\
&\le \exp\left[\frac{1}{8}\left(\frac{M}{m} - 1\right)^2\right].
\end{aligned} \tag{2.7}$$

3. Related Results

We also have:

Theorem 8. *Let $P \ge 0$ with $P \in \mathcal{B}_1(H)$ and $\operatorname{tr}(P) = 1$. Assume that $I < mI \le A \le MI$, then*

$$\begin{aligned}
0 &\le \ln \Delta_P(A) - (\ln m)^{\frac{\ln M - \operatorname{tr}(P\ln A)}{\ln M - \ln m}} (\ln M)^{\frac{\operatorname{tr}(P\ln A) - \ln m}{\ln M - \ln m}} \\
&\le \frac{(\ln M - \operatorname{tr}(P\ln A))(\operatorname{tr}(P\ln A) - \ln m)}{\ln M - \ln m} \ln\left(\frac{\ln M}{\ln m}\right) \\
&\le \frac{1}{4}(\ln M - \ln m)\ln\left(\frac{\ln M}{\ln m}\right).
\end{aligned} \tag{3.1}$$

Proof. In the recent paper [2] we obtained the following reverses of Young's inequality as well:

$$0 \le (1-\nu)a + \nu b - a^{1-\nu}b^{\nu} \le \nu(1-\nu)(a-b)(\ln a - \ln b) \tag{3.2}$$

where $a, b > 0$, $\nu \in [0,1]$.

If we take the exponential in (3.2), then we get

$$\begin{aligned}
1 &\le \frac{\exp[(1-\nu)a + \nu b]}{\exp(a^{1-\nu}b^{\nu})} \le \exp[\nu(1-\nu)(a-b)(\ln a - \ln b)] \\
&= \exp\left[\ln\left(\frac{b}{a}\right)^{\nu(1-\nu)(b-a)}\right] = \left(\frac{b}{a}\right)^{\nu(1-\nu)(b-a)}.
\end{aligned} \tag{3.3}$$

If we put $(1-\nu)a + \nu b = s > 0$, then $\nu = \frac{s-a}{b-a}$, $1-\nu = \frac{b-s}{b-a}$ and by (3.3) we obtain

$$1 \le \frac{\exp s}{\exp\left(a^{\frac{b-s}{b-a}} b^{\frac{s-a}{b-a}}\right)} \le \left(\frac{b}{a}\right)^{\frac{(s-a)(b-s)}{b-a}} \le \left(\frac{b}{a}\right)^{\frac{1}{4}(b-a)}. \tag{3.4}$$

Now, we take $a=\ln m$, $s=\operatorname{tr}(P\ln A)$ and $b=\ln M$, in (3.4) to get

$$1\leq\frac{\exp\operatorname{tr}(P\ln A)}{\exp\left((\ln m)^{\frac{\ln M-\operatorname{tr}(P\ln A)}{\ln M-\ln m}}(\ln M)^{\frac{\operatorname{tr}(P\ln A)-\ln m}{\ln M-\ln m}}\right)}$$
$$\leq\left(\frac{\ln M}{\ln m}\right)^{\frac{(\ln M-\operatorname{tr}(P\ln A))(\operatorname{tr}(P\ln A)-\ln m)}{\ln M-\ln m}}\leq\left(\frac{\ln M}{\ln m}\right)^{\frac{1}{4}(\ln M-\ln m)}.$$

By taking the logarithm we then obtain (3.1). □

We also have:

Theorem 9. *With the assumption of Theorem 8, then*

$$\begin{aligned}(3.5)\qquad 0&\leq\left(\frac{1}{2}-\frac{1}{\ln M-\ln m}\left|\operatorname{tr}(P\ln A)-\frac{\ln m+\ln M}{2}\right|\right)\\&\quad\times\left(\sqrt{\ln M}-\sqrt{\ln m}\right)^2\\&\leq\ln\Delta_P(A)-(\ln m)^{\frac{\ln M-\operatorname{tr}(P\ln A)}{\ln M-\ln m}}(\ln M)^{\frac{\operatorname{tr}(P\ln A)-\ln m}{\ln M-\ln m}}\\&\leq\left(\frac{1}{2}+\frac{1}{\ln M-\ln m}\left|\operatorname{tr}(P\ln A)-\frac{\ln m+\ln M}{2}\right|\right)\\&\quad\times\left(\sqrt{\ln M}-\sqrt{\ln m}\right)^2\\&\leq\left(\sqrt{\ln M}-\sqrt{\ln m}\right)^2.\end{aligned}$$

Proof. If we take the exponential in (1.13) we get

$$\begin{aligned}(3.6)\qquad 1&\leq\exp\left[\min\{1-\nu,\nu\}\left(\sqrt{a}-\sqrt{b}\right)^2\right]\\&\leq\frac{\exp[(1-\nu)a+\nu b]}{\exp\left(a^{1-\nu}b^{\nu}\right)}\\&\leq\exp\left[\max\{1-\nu,\nu\}\left(\sqrt{a}-\sqrt{b}\right)^2\right]\end{aligned}$$

for $a,b>0$, $\nu\in[0,1]$.

If we put $(1-\nu)a+\nu b=s>0$, then $\nu=\frac{s-a}{b-a}$,

$$\min\{1-\nu,\nu\}=\frac{1}{2}-\frac{1}{b-a}\left|s-\frac{a+b}{2}\right|,$$
$$\max\{1-\nu,\nu\}=\frac{1}{2}+\frac{1}{b-a}\left|s-\frac{a+b}{2}\right|,$$

and by (3.6) we get

$$\begin{aligned}(3.7)\qquad 1&\leq\exp\left[\left(\frac{1}{2}-\frac{1}{b-a}\left|s-\frac{a+b}{2}\right|\right)\left(\sqrt{a}-\sqrt{b}\right)^2\right]\\&\leq\frac{\exp s}{\exp\left(a^{\frac{b-s}{b-a}}b^{\frac{s-a}{b-a}}\right)}\\&\leq\exp\left[\left(\frac{1}{2}+\frac{1}{b-a}\left|s-\frac{a+b}{2}\right|\right)\left(\sqrt{a}-\sqrt{b}\right)^2\right]\end{aligned}$$

for $s\in[a,b]$.

Now, we take $a=\ln m$, $s=\operatorname{tr}(P\ln A)$ and $b=\ln M$ in (3.7) to get

$$\begin{aligned}1&\leq\exp\left[\left(\frac{1}{2}-\frac{1}{\ln M-\ln m}\left|\operatorname{tr}(P\ln A)-\frac{\ln m+\ln M}{2}\right|\right)\left(\sqrt{\ln m}-\sqrt{\ln M}\right)^2\right]\\&\leq\frac{\exp\operatorname{tr}(P\ln A)}{\exp\left((\ln m)^{\frac{\ln M-\operatorname{tr}(P\ln A)}{\ln M-\ln m}}(\ln M)^{\frac{\operatorname{tr}(P\ln A)-\ln m}{\ln M-\ln m}}\right)}\\&\leq\exp\left[\left(\frac{1}{2}+\frac{1}{\ln M-\ln m}\left|\operatorname{tr}(P\ln A)-\frac{\ln m+\ln M}{2}\right|\right)\left(\sqrt{\ln m}-\sqrt{\ln M}\right)^2\right].\end{aligned}$$

Taking the logarithm, we obtain

$$\begin{aligned}0&\leq\left(\frac{1}{2}-\frac{1}{\ln M-\ln m}\left|\operatorname{tr}(P\ln A)-\frac{\ln m+\ln M}{2}\right|\right)\left(\sqrt{\ln M}-\sqrt{\ln m}\right)^2\\&\leq\ln\Delta_P(A)-(\ln m)^{\frac{\ln M-\operatorname{tr}(P\ln A)}{\ln M-\ln m}}(\ln M)^{\frac{\operatorname{tr}(P\ln A)-\ln m}{\ln M-\ln m}}\\&\leq\left(\frac{1}{2}+\frac{1}{\ln M-\ln m}\left|\operatorname{tr}(P\ln A)-\frac{\ln m+\ln M}{2}\right|\right)\left(\sqrt{\ln M}-\sqrt{\ln m}\right)^2\\&\leq\left(\sqrt{\ln M}-\sqrt{\ln m}\right)^2,\end{aligned}$$

which proves the desired result. □

We also have:

Theorem 10. *With the assumptions of Theorem 9,*

$$\begin{aligned}0&\leq\frac{1}{2}\frac{(\operatorname{tr}(P\ln A)-\ln m)(\ln M-\operatorname{tr}(P\ln A))}{\ln M}\\&\leq\ln\Delta_P(A)-(\ln m)^{\frac{\ln M-\operatorname{tr}(P\ln A)}{\ln M-\ln m}}(\ln M)^{\frac{\operatorname{tr}(P\ln A)-\ln m}{\ln M-\ln m}}\\&\leq\frac{1}{2}\frac{(\operatorname{tr}(P\ln A)-\ln m)(\ln M-\operatorname{tr}(P\ln A))}{\ln m}\\&\leq\frac{1}{8\ln m}(\ln M-\ln m)^2.\end{aligned}\tag{3.8}$$

Proof. If we take the exponential in (1.14), then we get

$$\begin{aligned}1&\leq\exp\left[\frac{1}{2}\nu(1-\nu)\frac{(b-a)^2}{\max\{a,b\}}\right]\\&\leq\frac{\exp[(1-\nu)a+\nu b]}{\exp(a^{1-\nu}b^{\nu})}\\&\leq\exp\left[\frac{1}{2}\nu(1-\nu)\frac{(b-a)^2}{\min\{a,b\}}\right]\end{aligned}\tag{3.9}$$

for $a,\ b>0$, $\nu\in[0,1]$.

If we put $(1-\nu)a+\nu b=s>0$, then $\nu=\frac{s-a}{b-a}$, $1-\nu=\frac{b-s}{b-a}$ and by (3.9) we derive

$$\exp\left[\frac{1}{2}\frac{(s-a)(b-s)}{\max\{a,b\}}\right] \le \frac{\exp s}{\exp\left(a^{\frac{b-s}{b-a}}b^{\frac{s-a}{b-a}}\right)} \le \exp\left[\frac{1}{2}\frac{(s-a)(b-s)}{\min\{a,b\}}\right]. \tag{3.10}$$

Now, we put $a=\ln m$, $s=\operatorname{tr}(P\ln A)$ and $b=\ln M$ in (3.10) to get

$$\begin{aligned}1 &\le \exp\left[\frac{1}{2}\frac{(\operatorname{tr}(P\ln A)-\ln m)(\ln M-\operatorname{tr}(P\ln A))}{\ln M}\right]\\ &\le \frac{\exp\operatorname{tr}(P\ln A)}{\exp\left((\ln m)^{\frac{\ln M-\operatorname{tr}(P\ln A)}{\ln M-\ln m}}(\ln M)^{\frac{\operatorname{tr}(P\ln A)-\ln m}{\ln M-\ln m}}\right)}\\ &\le \exp\left[\frac{1}{2}\frac{(\operatorname{tr}(P\ln A)-\ln m)(\ln M-\operatorname{tr}(P\ln A))}{\ln m}\right]\end{aligned}$$

and by taking the logarithm we obtain the first part of (3.8).

The second part is obvious. □

In [3] we also obtained the following result

$$\begin{aligned}\frac{1}{2}\nu(1-\nu)(\ln a-\ln b)^2\min\{a,b\} &\le (1-\nu)a+\nu b-a^{1-\nu}b^{\nu}\\ &\le \frac{1}{2}\nu(1-\nu)(\ln a-\ln b)^2\max\{a,b\}\end{aligned} \tag{3.11}$$

for any $a,\ b>0$ and $\nu\in[0,1]$.

Theorem 11. *With the assumptions of Theorem 9,*

$$\begin{aligned}0 &\le \frac{1}{2}(\operatorname{tr}(P\ln A)-\ln m)(\ln M-\operatorname{tr}(P\ln A))\\ &\quad\times\left[\frac{\ln(\ln M)-\ln(\ln m)}{\ln M-\ln m}\right]^2\ln(\ln m)\\ &\le \ln\Delta_P(A)-(\ln m)^{\frac{\ln M-\operatorname{tr}(P\ln A)}{\ln M-\ln m}}(\ln M)^{\frac{\operatorname{tr}(P\ln A)-\ln m}{\ln M-\ln m}}\\ &\le \frac{1}{2}(\operatorname{tr}(P\ln A)-\ln m)(\ln M-\operatorname{tr}(P\ln A))\\ &\quad\times\left[\frac{\ln(\ln M)-\ln(\ln m)}{\ln M-\ln m}\right]^2\ln(\ln M)\\ &\le \frac{1}{8}[\ln(\ln M)-\ln(\ln m)]^2\ln(\ln M).\end{aligned} \tag{3.12}$$

Proof. If we take the exponential in (3.11), then we get

$$\begin{aligned}1 &\le \exp\left[\frac{1}{2}\nu(1-\nu)(\ln a-\ln b)^2\min\{a,b\}\right]\\ &\le \frac{\exp[(1-\nu)a+\nu b]}{\exp(a^{1-\nu}b^{\nu})}\\ &\le \exp\left[\frac{1}{2}\nu(1-\nu)(\ln a-\ln b)^2\max\{a,b\}\right]\end{aligned}$$

for any a, $b > 0$ and $\nu \in [0, 1]$.

By utilizing a similar argument to the one from Theorem 10 we deduce the desired result (3.12).

The details are omitted. $\square$

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