SOME BOUNDS FOR TRACE CLASS P-DETERMINANT OF POSITIVE OPERATORS IN HILBERT SPACES VIA TOMINAGA'S RESULTS

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Abstract. Let H be a complex Hilbert space. For a given operator $P \geq 0$ with $P \in \mathcal{B}_1(H)$, the trace class associated to $\mathcal{B}(H)$ and $\operatorname{tr}(P) = 1$, we define the P-*determinant* of the positive invertible operator A by

$$\Delta_P(A) := \exp \operatorname{tr}(P \ln A).$$

In this paper we show, among others that, if A is an operator satisfying the condition $0 < mI \leq A \leq MI$, then

$$0 \leq \Delta_P(A) - m^{\frac{M-\operatorname{tr}(PA)}{M-m}} M^{\frac{\operatorname{tr}(PA)-mP}{M-m}} \leq L(M,m) \log S\left(\frac{M}{m}\right),$$

where L is the *logarithmic mean* and S is the *Specht's ratio*.

1. Introduction

In 1952, in the paper [3], B. Fuglede and R. V. Kadison introduced the determinant of a (invertible) operator and established its fundamental properties. The notion generalizes the usual determinant and can be considered for any operator in a finite von Neumann algebra (M, τ) with a faithful normal trace.

Let $T \in M$ be normal and $|T| := (T^*T)^{1/2}$ its modulus. By the spectral theorem one can represent T as an integral

$$T = \int_{\operatorname{Sp}(T)} \lambda dE(\lambda),$$

where $E(\lambda)$ is a projection valued measure and $\operatorname{Sp}(T)$ is the spectrum of T. The measure $\mu_T := \tau \circ E$ becomes a probability measure on the complex plane and has the support in the spectrum $\operatorname{Sp}(T)$.

For any $T \in M$ the Fuglede-Kadison determinant (*FK-determinant*) is defined by

$$\Delta_{FK}(T) := \exp\left(\int_0^\infty \ln t d\mu_{|T|}\right).$$

If T is invertible, then

$$\Delta_{FK}(T) := \exp(\tau(\ln(|T|))),$$

where $\ln(|T|)$ is defined by the use of functional calculus.

Let $B(H)$ be the space of all bounded linear operators on a Hilbert space H, and I stands for the identity operator on H. An operator A in $B(H)$ is said to be positive (in symbol: $A \geq 0$) if $\langle Ax, x\rangle \geq 0$ for all $x \in H$. In particular, $A > 0$

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means that A is positive and invertible. For a pair A, B of selfadjoint operators the order relation $A \geq B$ means as usual that $A - B$ is positive.

In 1998, Fujii et al. [4], [5], introduced the *normalized determinant* $\Delta_x(A)$ for positive invertible operators A on a Hilbert space H and a fixed unit vector $x \in H$, namely $\|x\| = 1$, defined by

$$\Delta_x(A) := \exp \langle \ln Ax, x \rangle$$

and discussed it as a continuous geometric mean and observed some inequalities around the determinant from this point of view. For some recent results, see [8].

We need now some preparations for trace of operators in Hilbert spaces.

Let $(H, \langle \cdot, \cdot \rangle)$ be a complex Hilbert space and $\{e_i\}_{i \in I}$ an *orthonormal basis* of H. We say that $A \in \mathcal{B}(H)$ is a *Hilbert-Schmidt operator* if

$$\sum_{i \in I} \|Ae_i\|^2 < \infty. \tag{1.1}$$

It is well know that, if $\{e_i\}_{i \in I}$ and $\{f_j\}_{j \in J}$ are orthonormal bases for H and $A \in \mathcal{B}(H)$ then

$$\sum_{i \in I} \|Ae_i\|^2 = \sum_{j \in I} \|Af_j\|^2 = \sum_{j \in I} \|A^* f_j\|^2 \tag{1.2}$$

showing that the definition (1.1) is independent of the orthonormal basis and A is a Hilbert-Schmidt operator iff A^* is a Hilbert-Schmidt operator.

Let $\mathcal{B}_2(H)$ the set of *Hilbert-Schmidt operators* in $\mathcal{B}(H)$. For $A \in \mathcal{B}_2(H)$ we define

$$\|A\|_2 := \left(\sum_{i \in I} \|Ae_i\|^2 \right)^{1/2} \tag{1.3}$$

for $\{e_i\}_{i \in I}$ an orthonormal basis of H.

Using the triangle inequality in $l^2(I)$, one checks that $\mathcal{B}_2(H)$ is a *vector space* and that $\|\cdot\|_2$ is a norm on $\mathcal{B}_2(H)$, which is usually called in the literature as the *Hilbert-Schmidt norm*.

Denote *the modulus* of an operator $A \in \mathcal{B}(H)$ by $|A| := (A^*A)^{1/2}$.

Because $\||A|x\| = \|Ax\|$ for all $x \in H$, A is Hilbert-Schmidt iff $|A|$ is Hilbert-Schmidt and $\|A\|_2 = \||A|\|_2$. From (1.2) we have that if $A \in \mathcal{B}_2(H)$, then $A^* \in \mathcal{B}_2(H)$ and $\|A\|_2 = \|A^*\|_2$.

The following theorem collects some of the most important properties of Hilbert-Schmidt operators:

Theorem 1. *We have:*

(i) $(\mathcal{B}_2(H), \|\cdot\|_2)$ *is a Hilbert space with inner product*

$$\langle A, B \rangle_2 := \sum_{i \in I} \langle Ae_i, Be_i \rangle = \sum_{i \in I} \langle B^* Ae_i, e_i \rangle \tag{1.4}$$

and the definition does not depend on the choice of the orthonormal basis $\{e_i\}_{i \in I}$;

(ii) We have the inequalities

$$\|A\| \leq \|A\|_2 \tag{1.5}$$

for any $A \in \mathcal{B}_2(H)$ *and, if* $A \in \mathcal{B}_2(H)$ *and* $T \in \mathcal{B}(H)$, *then* AT, $TA \in \mathcal{B}_2(H)$ *with*

$$\|AT\|_2, \ \|TA\|_2 \leq \|T\| \|A\|_2 \tag{1.6}$$

(iii) $\mathcal{B}_2(H)$ *is an operator ideal in* $\mathcal{B}(H)$, *i.e.*

$$\mathcal{B}(H)\mathcal{B}_2(H)\mathcal{B}(H)\subseteq\mathcal{B}_2(H).$$

If $\{e_i\}_{i\in I}$ an orthonormal basis of H, we say that $A\in\mathcal{B}(H)$ is *trace class* if

$$\|A\|_1:=\sum_{i\in I}\langle|A|e_i,e_i\rangle<\infty. \tag{1.7}$$

The definition of $\|A\|_1$ does not depend on the choice of the orthonormal basis $\{e_i\}_{i\in I}$. We denote by $\mathcal{B}_1(H)$ the set of trace class operators in $\mathcal{B}(H)$.

The following proposition holds:

Proposition 1. *If* $A\in\mathcal{B}(H)$, *then the following are equivalent:*

(i) $A\in\mathcal{B}_1(H)$;

(ii) $|A|^{1/2}\in\mathcal{B}_2(H)$.

The following properties are also well known:

Theorem 2. *With the above notations:*

(i) We have

$$\|A\|_1=\|A^*\|_1 \text{ and } \|A\|_2\le\|A\|_1 \tag{1.8}$$

for any $A\in\mathcal{B}_1(H)$;

(ii) $\mathcal{B}_1(H)$ *is an operator ideal in* $\mathcal{B}(H)$, *i.e.*

$$\mathcal{B}(H)\mathcal{B}_1(H)\mathcal{B}(H)\subseteq\mathcal{B}_1(H);$$

(iii) We have

$$\mathcal{B}_2(H)\mathcal{B}_2(H)=\mathcal{B}_1(H);$$

(iv) We have

$$\|A\|_1=\sup\left\{\langle A,B\rangle_2 \mid B\in\mathcal{B}_2(H),\ \|B\|_2\le1\right\};$$

(v) $(\mathcal{B}_1(H),\|\cdot\|_1)$ *is a Banach space.*

We define the *trace* of a trace class operator $A\in\mathcal{B}_1(H)$ to be

$$\operatorname{tr}(A):=\sum_{i\in I}\langle Ae_i,e_i\rangle, \tag{1.9}$$

where $\{e_i\}_{i\in I}$ an orthonormal basis of H. Note that this coincides with the usual definition of the trace if H is finite-dimensional. We observe that the series (1.9) converges absolutely and it is independent from the choice of basis.

The following result collects some properties of the trace:

Theorem 3. *We have:*

(i) If $A\in\mathcal{B}_1(H)$ *then* $A^*\in\mathcal{B}_1(H)$ *and*

$$\operatorname{tr}(A^*)=\overline{\operatorname{tr}(A)}; \tag{1.10}$$

(ii) If $A\in\mathcal{B}_1(H)$ *and* $T\in\mathcal{B}(H)$, *then* $AT,\ TA\in\mathcal{B}_1(H)$,

$$\operatorname{tr}(AT)=\operatorname{tr}(TA) \text{ and } |\operatorname{tr}(AT)|\le\|A\|_1\|T\|; \tag{1.11}$$

(iii) $\operatorname{tr}(\cdot)$ *is a bounded linear functional on* $\mathcal{B}_1(H)$ *with* $\|\operatorname{tr}\|=1$;

(iv) If $A,\ B\in\mathcal{B}_2(H)$ *then* $AB,\ BA\in\mathcal{B}_1(H)$ *and* $\operatorname{tr}(AB)=\operatorname{tr}(BA)$.

Now, if we assume that $P \geq 0$ and $P \in \mathcal{B}_1(H)$, then for all $T \in \mathcal{B}(H)$, PT, $TP \in \mathcal{B}_1(H)$ and $\operatorname{tr}(PT) = \operatorname{tr}(TP)$. Also, since $P^{1/2} \in \mathcal{B}_2(H)$, $TP^{1/2} \in \mathcal{B}_2(H)$, hence $P^{1/2}TP^{1/2}$ and $TP^{1/2}P^{1/2} = TP \in \mathcal{B}_1(H)$ with $\operatorname{tr}\left(P^{1/2}TP^{1/2}\right) = \operatorname{tr}(TP)$. Therefore, if $P \geq 0$ and $P \in \mathcal{B}_1(H)$,

$$\operatorname{tr}(PT) = \operatorname{tr}(TP) = \operatorname{tr}\left(P^{1/2}TP^{1/2}\right)$$

for all $T \in \mathcal{B}(H)$.

If $T \geq 0$, then $P^{1/2}TP^{1/2} \geq 0$, which implies that $\operatorname{tr}(PT) \geq 0$ that shows that the functional $\mathcal{B}(H) \ni T \longmapsto \operatorname{tr}(PT)$ is linear and isotonic functional. Also, by (1.11), if $T_n \to T$ for $n \to \infty$ in $\mathcal{B}(H)$ then $\lim_{n\to\infty} \operatorname{tr}(PT_n) = \operatorname{tr}(PT)$, namely $\mathcal{B}(H) \ni T \longmapsto \operatorname{tr}(PT)$ is also continuous in the norm topology.

For a survey on recent trace inequalities see [1] and the references therein.

Now, for a given $P \geq 0$ with $P \in \mathcal{B}_1(H)$ and $\operatorname{tr}(P) = 1$, we define the P-*determinant* of the positive invertible operator A by

$$\Delta_P(A) := \exp \operatorname{tr}(P \ln A) = \exp \operatorname{tr}((\ln A) P) = \exp \operatorname{tr}\left(P^{1/2}(\ln A) P^{1/2}\right). \tag{1.12}$$

Assume that $P \geq 0$ with $P \in \mathcal{B}_1(H)$ and $\operatorname{tr}(P) = 1$. We observe that we have the following elementary properties:

(i) *continuity*: the map $A \to \Delta_P(A)$ is norm continuous;
(ii) *power equality:* $\Delta_P(A^t) = \Delta_P(A)^t$ for all $t > 0$;
(iii) *homogeneity*: $\Delta_P(tA) = t\Delta_x(A)$ and $\Delta_P(tI) = t$ for all $t > 0$;
(iv) *monotonicity*: $0 < A \leq B$ implies $\Delta_P(A) \leq \Delta_P(B)$.

In the recent paper [2] we obtained the following results:

Theorem 4. *Let $P \geq 0$ with $P \in \mathcal{B}_1(H)$ and $\operatorname{tr}(P) = 1$, then for all A, $B > 0$ and $t \in [0,1]$,*

$$\Delta_P((1-t)A + tB) \geq [\Delta_P(A)]^{1-t}[\Delta_P(B)]^t.$$

and

Theorem 5. *Let $P \geq 0$ with $P \in \mathcal{B}_1(H)$ and $\operatorname{tr}(P) = 1$, then for all $A > 0$ and $a > 0$ we have the double inequality*

$$a \exp\left[1 - a \operatorname{tr}\left(PA^{-1}\right)\right] \leq \Delta_P(A) \leq a \exp\left[a^{-1} \operatorname{tr}(PA) - 1\right].$$

In particular

$$1 \leq \frac{\operatorname{tr}(PA)}{\Delta_P(A)} \leq \exp\left[\operatorname{tr}(PA) \operatorname{tr}\left(PA^{-1}\right) - 1\right] \tag{1.13}$$

and

$$1 \leq \frac{\Delta_P(A)}{\left[\operatorname{tr}(PA^{-1})\right]^{-1}} \leq \exp\left[\operatorname{tr}\left(PA^{-1}\right) \operatorname{tr}(PA) - 1\right].$$

We recall that *Specht's ratio* is defined by [9]

$$S(h) := \begin{cases} \dfrac{h^{\frac{1}{h-1}}}{e \ln\left(h^{\frac{1}{h-1}}\right)} \text{ if } h \in (0,1) \cup (1,\infty) \\ \\ 1 \text{ if } h = 1. \end{cases} \tag{1.14}$$

It is well known that $\lim_{h\to 1} S(h) = 1$, $S(h) = S\left(\frac{1}{h}\right) > 1$ for $h > 0$, $h \neq 1$. The function is decreasing on $(0,1)$ and increasing on $(1,\infty)$.

The following inequality provides a refinement and a multiplicative reverse for Young's inequality

$$\left(a^{1-\nu}b^{\nu}\leq\right)S\left(\left(\frac{a}{b}\right)^{r}\right)a^{1-\nu}b^{\nu}\leq\left(1-\nu\right)a+\nu b\leq S\left(\frac{a}{b}\right)a^{1-\nu}b^{\nu}, \tag{1.15}$$

where a, $b>0$, $\nu\in[0,1]$, $r=\min\{1-\nu,\nu\}$.

The second inequality in (1.15) is due to Tominaga [10] while the first one is due to Furuichi [6].

In [10] Tominaga also obtained the following additive reverse inequality

$$0\leq\left(1-\nu\right)a+\nu b-a^{1-\nu}b^{\nu}\leq L\left(a,b\right)\log S\left(\frac{a}{b}\right) \tag{1.16}$$

where the *logarithmic mean* of two positive numbers a, b is defined by

$$L\left(a,b\right):=\begin{cases}\frac{b-a}{\ln b-\ln a}\text{ if }b\neq a,\\ \\ a\text{ if }b=a.\end{cases}$$

Motivated by the above results, in this paper we show, among others that, if A is an operator satisfying the condition $0<mI\leq A\leq MI$, then

$$0\leq\Delta_{P}(A)-m^{\frac{M-\operatorname{tr}(PA)}{M-m}}M^{\frac{\operatorname{tr}(PA)-mP}{M-m}}\leq L\left(M,m\right)\log S\left(\frac{M}{m}\right),$$

where L is the *logarithmic mean* and S is the *Specht's ratio.*

2. Main Results

Our first main result is as follows:

Theorem 6. *Let $P\geq0$ with $P\in\mathcal{B}_{1}\left(H\right)$ and $\operatorname{tr}\left(P\right)=1$. If $0<mI\leq A\leq MI$ for positive numbers m, M, then*

$$\begin{aligned}1&\leq\exp\left[\operatorname{tr}\left(P\ln S\left(\left(\frac{M}{m}\right)^{\frac{1}{2}I-\frac{1}{M-m}\left|A-\frac{1}{2}(m+M)I\right|}\right)\right)\right]\\&\leq\frac{\Delta_{P}(A)}{m^{\frac{M-\operatorname{tr}(PA)}{M-m}}M^{\frac{\operatorname{tr}(PA)-m}{M-m}}}\leq S\left(\frac{M}{m}\right).\end{aligned} \tag{2.1}$$

Proof. Assume that $t\in[m,M]$ and consider $\nu=\frac{t-m}{M-m}\in[0,1]$. Then

$$\begin{aligned}\min\{1-\nu,\nu\}&=\frac{1}{2}-\left|\nu-\frac{1}{2}\right|=\frac{1}{2}-\left|\frac{t-m}{M-m}-\frac{1}{2}\right|\\&=\frac{1}{2}-\frac{1}{M-m}\left|t-\frac{1}{2}\left(m+M\right)\right|,\end{aligned}$$

$$\left(1-\nu\right)m+\nu M=\frac{M-t}{M-m}m+\frac{t-m}{M-m}M=t$$

and

$$m^{1-\nu}M^{\nu}=m^{\frac{M-t}{M-m}}M^{\frac{t-m}{M-m}}.$$

By using the inequality (1.15) we deduce

$$m^{\frac{M-t}{M-m}}M^{\frac{t-m}{M-m}} \le S\left(\left(\frac{M}{m}\right)^{\frac{1}{2}-\frac{1}{M-m}\left|t-\frac{1}{2}(m+M)\right|}\right)m^{\frac{M-t}{M-m}}M^{\frac{t-m}{M-m}} \le t \le S\left(\frac{M}{m}\right)m^{\frac{M-t}{M-m}}M^{\frac{t-m}{M-m}} \tag{2.2}$$

for $t \in [m, M]$.

By taking the log in (2.2) we get

$$\begin{aligned} &\frac{M-t}{M-m}\ln m + \frac{t-m}{M-m}\ln M \\ &\le \ln S\left(\left(\frac{M}{m}\right)^{\frac{1}{2}-\frac{1}{M-m}\left|t-\frac{1}{2}(m+M)\right|}\right) + \frac{M-t}{M-m}\ln m + \frac{t-m}{M-m}\ln M \\ &\le \ln t \le \ln S\left(\frac{M}{m}\right) + \frac{M-t}{M-m}\ln m + \frac{t-m}{M-m}\ln M, \end{aligned} \tag{2.3}$$

for $t \in [m, M]$.

If $0 < mI \le A \le MI$, then by using the continuous functional calculus for selfadjoint operators we get from (2.3) that

$$\begin{aligned} &\ln m\frac{MI-A}{M-m} + \ln M\frac{A-mI}{M-m} \\ &\le \ln S\left(\left(\frac{M}{m}\right)^{\frac{1}{2}I-\frac{1}{M-m}\left|A-\frac{1}{2}(m+M)I\right|}\right) + \ln m\frac{MI-A}{M-m} + \ln M\frac{A-mI}{M-m} \\ &\le \ln A \le \ln S\left(\frac{M}{m}\right)I + \ln m\frac{MI-A}{M-m} + \ln M\frac{A-mI}{M-m}. \end{aligned}$$

Now, if multiply this inequality both sides by $P^{1/2}$ we get

$$\begin{aligned} &\ln m\frac{MP-P^{1/2}AP^{1/2}}{M-m} + \ln M\frac{P^{1/2}AP^{1/2}-mP}{M-m} \\ &\le P^{1/2}\ln S\left(\left(\frac{M}{m}\right)^{\frac{1}{2}I-\frac{1}{M-m}\left|A-\frac{1}{2}(m+M)I\right|}\right)P^{1/2} \\ &+ \ln m\frac{MP-P^{1/2}AP^{1/2}}{M-m} + \ln M\frac{P^{1/2}AP^{1/2}-mP}{M-m} \\ &\le P^{1/2}(\ln A)P^{1/2} \\ &\le \ln S\left(\frac{M}{m}\right)P + \ln m\frac{MP-P^{1/2}AP^{1/2}}{M-m} + \ln M\frac{P^{1/2}AP^{1/2}-mIP}{M-m}. \end{aligned}$$

If we take the trace and use the fact that $\operatorname{tr}(P)=1$, then we obtain

$$\begin{aligned}
&\ln m\frac{M-\operatorname{tr}(PA)}{M-m}+\ln M\frac{\operatorname{tr}(PA)-m}{M-m}\\
&\le \operatorname{tr}\left[P\ln S\left(\left(\frac{M}{m}\right)^{\frac{1}{2}I-\frac{1}{M-m}\left|A-\frac{1}{2}(m+M)I\right|}\right)\right]\\
&+\ln m\frac{M-\operatorname{tr}(PA)}{M-m}+\ln M\frac{\operatorname{tr}(PA)-m}{M-m}\\
&\le \operatorname{tr}[P(\ln A)]\le \ln S\left(\frac{M}{m}\right)+\ln m\frac{M-\operatorname{tr}(PA)}{M-m}+\frac{\operatorname{tr}(PA)-m}{M-m}\ln M.
\end{aligned}$$

This inequality can also be written as

$$\begin{aligned}
&\ln\left(m^{\frac{M-\operatorname{tr}(PA)}{M-m}}M^{\frac{\operatorname{tr}(PA)-m}{M-m}}\right)\\
&\le \ln\exp\left(\operatorname{tr}\left[P\ln S\left(\left(\frac{M}{m}\right)^{\frac{1}{2}I-\frac{1}{M-m}\left|A-\frac{1}{2}(m+M)I\right|}\right)\right]\right)\\
&+\ln\left(m^{\frac{M-\operatorname{tr}(PA)}{M-m}}M^{\frac{\operatorname{tr}(PA)-m}{M-m}}\right)\\
&\le \operatorname{tr}[P(\ln A)]\le \ln S\left(\frac{M}{m}\right)+\ln\left(m^{\frac{M-\operatorname{tr}(PA)}{M-m}}M^{\frac{\operatorname{tr}(PA)-m}{M-m}}\right).
\end{aligned}\tag{2.4}$$

If we take the exponential in (2.4), then we get

$$\begin{aligned}
&m^{\frac{M-\operatorname{tr}(PA)}{M-m}}M^{\frac{\operatorname{tr}(PA)-m}{M-m}}\\
&\le\left(\exp\left(\operatorname{tr}\left[P\ln S\left(\left(\frac{M}{m}\right)^{\frac{1}{2}I-\frac{1}{M-m}\left|A-\frac{1}{2}(m+M)I\right|}\right)\right]\right)\right)m^{\frac{M-\operatorname{tr}(PA)}{M-m}}M^{\frac{\operatorname{tr}(PA)-m}{M-m}}\\
&\le\le \operatorname{tr}[P(\ln A)]\le S\left(\frac{M}{m}\right)m^{\frac{M-\operatorname{tr}(PA)}{M-m}}M^{\frac{\operatorname{tr}(PA)-m}{M-m}}
\end{aligned}$$

and the inequality (2.1) is proved. □

Corollary 1. *With the assumption of Theorem 6, we get*

$$\begin{aligned}
1&\le\exp\left[\operatorname{tr}\left(P\ln S\left(\left(\frac{M}{m}\right)^{\frac{1}{2}I-\frac{1}{m^{-1}-M^{-1}}\left|A^{-1}-\frac{1}{2}\left(M^{-1}+m^{-1}\right)I\right|}\right)\right)\right]\\
&\le\frac{M^{\frac{m^{-1}-\operatorname{tr}\left(PA^{-1}\right)}{m^{-1}-M^{-1}}}m^{\frac{\operatorname{tr}\left(PA^{-1}\right)-M^{-1}}{m^{-1}-M^{-1}}}}{\Delta_P(A)}\le S\left(\frac{M}{m}\right).
\end{aligned}\tag{2.5}$$

Proof. If we write the inequality for A^{-1} that satisfies the condition $0<M^{-1}I\le A^{-1}\le m^{-1}I$, then

$$\begin{aligned}
1&\le\exp\left[\operatorname{tr}\left(P\ln S\left(\left(\frac{M}{m}\right)^{\frac{1}{2}I-\frac{1}{m^{-1}-M^{-1}}\left|A^{-1}-\frac{1}{2}\left(M^{-1}+m^{-1}\right)I\right|}\right)\right)\right]\\
&\le\frac{\Delta_P(A^{-1})}{M^{-\frac{m^{-1}-\operatorname{tr}\left(PA^{-1}\right)}{m^{-1}-M^{-1}}}m^{-\frac{\operatorname{tr}\left(PA^{-1}\right)-M^{-1}}{m^{-1}-M^{-1}}}}\le S\left(\frac{m^{-1}}{M^{-1}}\right),
\end{aligned}$$

namely

$$1 \le \exp\left[\operatorname{tr}\left(P \ln S\left(\left(\frac{M}{m}\right)^{\frac{1}{2}I - \frac{1}{m^{-1}-M^{-1}}\left|A^{-1}-\frac{1}{2}\left(M^{-1}+m^{-1}\right)I\right|}\right)\right)\right]$$
$$\le \frac{\Delta_P(A^{-1})}{M^{-\frac{m^{-1}-\operatorname{tr}(PA^{-1})}{m^{-1}-M^{-1}}} m^{-\frac{\operatorname{tr}(PA^{-1})-M^{-1}}{m^{-1}-M^{-1}}}} \le S\left(\frac{M}{m}\right),$$

or

$$1 \le \exp\left[\operatorname{tr}\left(P \ln S\left(\left(\frac{M}{m}\right)^{\frac{1}{2}I - \frac{1}{m^{-1}-M^{-1}}\left|A^{-1}-\frac{1}{2}\left(M^{-1}+m^{-1}\right)I\right|}\right)\right)\right]$$
$$\le \frac{[\Delta_P(A)]^{-1}}{\left(M^{\frac{m^{-1}-\operatorname{tr}(PA^{-1})}{m^{-1}-M^{-1}}} m^{\frac{\operatorname{tr}(PA^{-1})-M^{-1}}{m^{-1}-M^{-1}}}\right)^{-1}} \le S\left(\frac{M}{m}\right),$$

which is equivalent to the desired result (2.5). □

Corollary 2. *If* $0 < mI \le A$, $B \le MI$ *for positive numbers* m, M, *then*

$$\begin{aligned} &\Theta(A, B, m, M, P) \\ &\le \frac{\int_0^1 \Delta_P((1-t)A + tB)dt}{\frac{m^{\frac{M-1}{M-m}} M^{\frac{1-m}{M-m}}}{\ln\left(\frac{M}{m}\right)}} \\ &\le S\left(\frac{M}{m}\right)\Theta(A, B, m, M, P), \end{aligned} \tag{2.6}$$

where

$$\Theta(A, B, m, M, P) := \begin{cases} \dfrac{\left(\frac{M}{m}\right)^{\frac{\operatorname{tr}[P(B-A)]}{M-m}} - 1}{\frac{\operatorname{tr}[P(B-A)]}{M-m}} & \text{if } \operatorname{tr}[P(B-A)] \ne 0, \\ 1 & \text{if } \operatorname{tr}[P(B-A)] = 0. \end{cases}$$

Proof. From (2.4) we get

$$\begin{aligned} &m^{\frac{M-\operatorname{tr}(P[(1-t)A+tB])}{M-m}} M^{\frac{\operatorname{tr}(P[(1-t)A+tB])-m}{M-m}} \\ &\le \Delta_P((1-t)A + tB) \\ &\le S\left(\frac{M}{m}\right) m^{\frac{M-\operatorname{tr}(P[(1-t)A+tB])}{M-m}} M^{\frac{\operatorname{tr}(P[(1-t)A+tB])-m}{M-m}} \end{aligned}$$

for $t \in [0, 1]$.

If we take the integral over $t \in [0, 1]$, then we get

$$\begin{aligned} &\int_0^1 m^{\frac{M-\operatorname{tr}(P[(1-t)A+tB])}{M-m}} M^{\frac{\operatorname{tr}(P[(1-t)A+tB])-m}{M-m}} dt \\ &\le \int_0^1 \Delta_P((1-t)A + tB)dt \\ &\le S\left(\frac{M}{m}\right)\int_0^1 m^{\frac{M-\operatorname{tr}(P[(1-t)A+tB])}{M-m}} M^{\frac{\operatorname{tr}(P[(1-t)A+tB])-m}{M-m}} dt. \end{aligned} \tag{2.7}$$

Observe that

$$\begin{aligned}
&\int_0^1 m^{\frac{M-\operatorname{tr}(P[(1-t)A+tB])}{M-m}} M^{\frac{\operatorname{tr}(P[(1-t)A+tB])-m}{M-m}} dt \\
&= m^{\frac{M}{M-m}} M^{\frac{-m}{M-m}} \int_0^1 \left(\frac{M}{m}\right)^{\frac{\operatorname{tr}(P[(1-t)A+tB])}{M-m}} dt \\
&= m^{\frac{M}{M-m}} M^{\frac{-m}{M-m}} \left(\frac{M}{m}\right)^{\frac{1}{M-m}} \int_0^1 \left(\frac{M}{m}\right)^{t\frac{\operatorname{tr}[P(B-A)]}{M-m}} dt \\
&= m^{\frac{M-1}{M-m}} M^{\frac{1-m}{M-m}} \int_0^1 \left(\frac{M}{m}\right)^{t\frac{\operatorname{tr}[P(B-A)]}{M-m}} dt.
\end{aligned}$$

Since for $a > 0$, $a \neq 1$ and $b \in \mathbb{R}$ we have

$$\int_0^1 a^{bx} dx = \frac{a^b - 1}{b \ln a},$$

then for $\operatorname{tr}[P(B-A)] \neq 0$

$$\int_0^1 \left(\frac{M}{m}\right)^{t\frac{\operatorname{tr}[P(B-A)]}{M-m}} dt = \frac{\left(\frac{M}{m}\right)^{\frac{\operatorname{tr}[P(B-A)]}{M-m}} - 1}{\frac{\operatorname{tr}[P(B-A)]}{M-m} \ln\left(\frac{M}{m}\right)}$$

and by (2.7) we derive (2.6). □

We also have

Theorem 7. *With the assumption of Theorem 6,*

$$\begin{aligned}
(2.8)\qquad m^{\frac{M-\operatorname{tr}(PA)}{M-m}} M^{\frac{\operatorname{tr}(PA)-mP}{M-m}} &\le \Delta_P(A) \\
&\le \operatorname{tr}\left(P m^{\frac{MI-A}{M-m}} M^{\frac{A-mI}{M-m}}\right) + L(M,m) \log S\left(\frac{M}{m}\right) \\
&\le m^{\frac{M-\operatorname{tr}(PA)}{M-m}} M^{\frac{\operatorname{tr}(PA)-mP}{M-m}} + L(M,m) \log S\left(\frac{M}{m}\right).
\end{aligned}$$

We also have the simpler inequality

$$0 \le \Delta_P(A) - m^{\frac{M-\operatorname{tr}(PA)}{M-m}} M^{\frac{\operatorname{tr}(PA)-mP}{M-m}} \le L(M,m) \log S\left(\frac{M}{m}\right).$$

Proof. From (1.16) we get

$$0 \le t - m^{\frac{M-t}{M-m}} M^{\frac{t-m}{M-m}} \le L(M,m) \log S\left(\frac{M}{m}\right)$$

for all $t \in [m, M]$, namely

$$m^{\frac{M-t}{M-m}} M^{\frac{t-m}{M-m}} \le t \le m^{\frac{M-t}{M-m}} M^{\frac{t-m}{M-m}} + L(M,m) \log S\left(\frac{M}{m}\right)$$

for all $t \in [m, M]$.

By taking the logarithm, we derive

$$\begin{aligned}
\frac{M-t}{M-m} \ln m + \frac{t-m}{M-m} \ln M &\le \ln t \\
&\le \ln\left(m^{\frac{M-t}{M-m}} M^{\frac{t-m}{M-m}} + L(M,m) \log S\left(\frac{M}{m}\right)\right)
\end{aligned}$$

for all $t \in [m, M]$, which implies the operator inequalities

$$\ln m \frac{MI - A}{M - m} + \ln M \frac{A - mI}{M - m} \le \ln A$$
$$\le \ln \left(m^{\frac{MI-A}{M-m}} M^{\frac{A-mI}{M-m}} + L(M,m) \log S\left(\frac{M}{m}\right) I \right).$$

Now if we multiply both sides by $P^{1/2}$, then we get

$$\ln m \frac{MP - P^{1/2}AP^{1/2}}{M - m} + \ln M \frac{P^{1/2}AP^{1/2} - mP}{M - m}$$
$$\le P^{1/2} (\ln A) P^{1/2}$$
$$\le P^{1/2} \ln \left(m^{\frac{MI-A}{M-m}} M^{\frac{A-mI}{M-m}} + L(M,m) \log S\left(\frac{M}{m}\right) I \right) P^{1/2}.$$

If we take the trace and use the fact that $\operatorname{tr}(P) = 1$,

$$\ln m \frac{M - \operatorname{tr}(PA)}{M - m} + \ln M \frac{\operatorname{tr}(PA) - mP}{M - m} \tag{2.9}$$
$$\le \operatorname{tr}[P(\ln A)]$$
$$\le \operatorname{tr}\left[P \ln \left(m^{\frac{MI-A}{M-m}} M^{\frac{A-mI}{M-m}} + L(M,m) \log S\left(\frac{M}{m}\right) I \right) \right].$$

By the Jensen's trace inequality for concave function ln, we derive

$$\operatorname{tr}\left[P \ln \left(m^{\frac{MI-A}{M-m}} M^{\frac{A-mI}{M-m}} + L(M,m) \log S\left(\frac{M}{m}\right) I \right) \right] \tag{2.10}$$
$$\le \ln \left(\operatorname{tr}\left[P \left(m^{\frac{MI-A}{M-m}} M^{\frac{A-mI}{M-m}} + L(M,m) \log S\left(\frac{M}{m}\right) I \right) \right] \right)$$
$$= \ln \left(\operatorname{tr}\left(P m^{\frac{MI-A}{M-m}} M^{\frac{A-mI}{M-m}} \right) + L(M,m) \log S\left(\frac{M}{m}\right) \right).$$

By (2.9) and (2.10) we derive

$$\ln \left(m^{\frac{M - \operatorname{tr}(PA)}{M-m}} M^{\frac{\operatorname{tr}(PA) - mP}{M-m}} \right)$$
$$\le \operatorname{tr}[P(\ln A)]$$
$$\le \ln \left(\operatorname{tr}\left(P m^{\frac{MI-A}{M-m}} M^{\frac{A-mI}{M-m}} \right) + L(M,m) \log S\left(\frac{M}{m}\right) \right)$$

and by taking the exponential, we derive the first two inequalities in (2.8).

Observe that

$$g(t) := m^{\frac{M-t}{M-m}} M^{\frac{t-m}{M-m}} = m^{\frac{M}{M-m}} M^{-\frac{m}{M-m}} \left(\frac{M}{m}\right)^{\frac{t}{M-m}},$$

which shows that g is convex on $[m, M]$.

By using the Jensen's trace inequality for convex function g, we also have

$$\operatorname{tr}\left(P m^{\frac{MI-A}{M-m}} M^{\frac{A-mI}{M-m}} \right) \le m^{\frac{M - \operatorname{tr}(PA)}{M-m}} M^{\frac{\operatorname{tr}(PA) - mP}{M-m}}$$

and the last part of (2.8) is also proved. $\square$

3. Related Results

We also have:

Theorem 8. *With the assumption of Theorem 6, we have that*

$$1 \leq S\left(\left(\frac{M}{m}\right)^{\frac{1}{2}-\frac{1}{\ln M-\ln m}\left|\operatorname{tr}(P\ln A)-\frac{\ln M+\ln m}{2}\right|}\right) \leq \frac{\frac{\ln M-\operatorname{tr}(P\ln A)}{\ln M-\ln m}m+\frac{\operatorname{tr}(P\ln A)-\ln m}{\ln M-\ln m}M}{\Delta_P(A)} \leq S\left(\frac{M}{m}\right). \tag{3.1}$$

Proof. Assume that $m^{1-\nu}M^{\nu}=\exp s$, then $s=(1-\nu)\ln m+\nu\ln M\in[\ln m,\ln M]$, which gives that

$$\nu=\frac{s-\ln m}{\ln M-\ln m}.$$

Also

$$\min\{1-\nu,\nu\}=\frac{1}{2}-\left|\frac{s-\ln m}{\ln M-\ln m}-\frac{1}{2}\right| =\frac{1}{2}-\frac{1}{\ln M-\ln m}\left|s-\frac{\ln M+\ln m}{2}\right|.$$

From (2.1) we get

$$\exp s\leq S\left(\left(\frac{M}{m}\right)^{\frac{1}{2}-\frac{1}{\ln M-\ln m}\left|s-\frac{\ln M+\ln m}{2}\right|}\right)\exp s \leq \frac{\ln M-s}{\ln M-\ln m}m+\frac{s-\ln m}{\ln M-\ln m}M \leq S\left(\frac{M}{m}\right)\exp s,$$

namely

$$1\leq S\left(\left(\frac{M}{m}\right)^{\frac{1}{2}-\frac{1}{\ln M-\ln m}\left|s-\frac{\ln M+\ln m}{2}\right|}\right) \leq \frac{\frac{\ln M-s}{\ln M-\ln m}m+\frac{s-\ln m}{\ln M-\ln m}M}{\exp s}\leq S\left(\frac{M}{m}\right)$$

for $s\in[\ln m,\ln M]$.

If $0<m\leq A\leq M$, then $\ln m\leq\operatorname{tr}(P\ln A)\leq\ln M$ and for $s=\operatorname{tr}(P\ln A)$ we deduce

$$1 \leq S\left(\left(\frac{M}{m}\right)^{\frac{1}{2}-\frac{1}{\ln M-\ln m}\left|\operatorname{tr}(P\ln A)-\frac{\ln M+\ln m}{2}\right|}\right) \leq \frac{\frac{\ln M-\operatorname{tr}(P\ln A)}{\ln M-\ln m}m+\frac{\operatorname{tr}(P\ln A)-\ln m}{\ln M-\ln m}M}{\exp\operatorname{tr}(P\ln A)} \leq S\left(\frac{M}{m}\right),$$

which is equivalent to (3.1). □

Corollary 3. *With the assumption of Theorem 6, we get*

$$1 \le S\left(\left(\frac{M}{m}\right)^{\frac{1}{2}-\frac{1}{\ln M-\ln m}\left|\operatorname{tr}(P\ln A)-\frac{\ln m+\ln M}{2}\right|}\right) \tag{3.2}$$
$$\le \frac{\Delta_P(A)}{\left(\frac{\operatorname{tr}(P\ln A)-\ln m}{\ln M-\ln m}M^{-1}+\frac{\ln M-\operatorname{tr}(P\ln A)}{\ln M-\ln m}m^{-1}\right)^{-1}} \le S\left(\frac{M}{m}\right).$$

Proof. If we write the inequality (3.1) for A^{-1} that satisfies the condition $0 < M^{-1}I \le A^{-1} \le m^{-1}I$, then we obtain

$$1 \le S\left(\left(\frac{m^{-1}}{M^{-1}}\right)^{\frac{1}{2}-\frac{1}{\ln m^{-1}-\ln M^{-1}}\left|\operatorname{tr}\left(P\ln A^{-1}\right)-\frac{\ln m^{-1}+\ln M^{-1}}{2}\right|}\right)$$
$$\le \frac{\frac{\ln m^{-1}-\operatorname{tr}\left(P\ln A^{-1}\right)}{\ln m^{-1}-\ln M^{-1}}M^{-1}+\frac{\operatorname{tr}\left(P\ln A^{-1}\right)-\ln M^{-1}}{\ln m^{-1}-\ln M^{-1}}m^{-1}}{\Delta_P(A^{-1})} \le S\left(\frac{m^{-1}}{M^{-1}}\right),$$

namely

$$1 \le S\left(\left(\frac{M}{m}\right)^{\frac{1}{2}-\frac{1}{\ln M-\ln m}\left|\operatorname{tr}(P\ln A)-\frac{\ln m+\ln M}{2}\right|}\right)$$
$$\le \frac{\frac{\operatorname{tr}(P\ln A)-\ln m}{\ln M-\ln m}M^{-1}+\frac{\ln M-\operatorname{tr}(P\ln A)}{\ln M-\ln m}m^{-1}}{\Delta_P(A^{-1})} \le S\left(\frac{M}{m}\right).$$

This proves (3.2). □

Finally, we also have:

Theorem 9. *With the assumption of Theorem 6, we get*

$$1 \le \frac{\exp\left(\frac{\ln M-\operatorname{tr}(P\ln A)}{\ln M-\ln m}m+\frac{\operatorname{tr}(P\ln A)-\ln m}{\ln M-\ln m}M\right)}{\Delta_P(A)} \le \left[S\left(\frac{M}{m}\right)\right]^{L(M,m)}. \tag{3.3}$$

Proof. Assume that $m^{1-\nu}M^{\nu} = \exp s$, then $s = (1-\nu)\ln m + \nu \ln M \in [\ln m, \ln M]$, which gives that

$$\nu = \frac{s-\ln m}{\ln M - \ln m}.$$

By the additive Tominaga's inequality,

$$0 \le \frac{\ln M - s}{\ln M - \ln m}m + \frac{s-\ln m}{\ln M-\ln m}M - s \le L(M,m)\log S\left(\frac{M}{m}\right)$$

for $s \in [\ln m, \ln M]$.

If $0 < m \le A \le M$, then $\ln m \le \operatorname{tr}(P\ln A) \le \ln M$ and for $s = \operatorname{tr}(P\ln A)$ we deduce

$$0 \le \frac{\ln M - \operatorname{tr}(P\ln A)}{\ln M-\ln m}m + \frac{\operatorname{tr}(P\ln A)-\ln m}{\ln M-\ln m}M - \operatorname{tr}(P\ln A)$$
$$\le L(M,m)\log S\left(\frac{M}{m}\right).$$

If we take the exponential in this inequality, then we get

$$1 \leq \frac{\exp\left(\frac{\ln M - \operatorname{tr}(P\ln A)}{\ln M - \ln m} m + \frac{\operatorname{tr}(P\ln A) - \ln m}{\ln M - \ln m} M\right)}{\exp\left[\operatorname{tr}\left(P\ln A\right)\right]}$$
$$\leq \exp\left[L\left(M,m\right)\log S\left(\frac{M}{m}\right)\right] = \left[S\left(\frac{M}{m}\right)\right]^{L(M,m)}$$

and the inequality (3.3) is thus proved. □

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