INEQUALITIES FOR TRACE CLASS P-DETERMINANT OF POSITIVE OPERATORS IN HILBERT SPACES VIA OSTROWSKI TYPE RESULTS

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Abstract. Let H be a complex Hilbert space. For a given operator $P \geq 0$ with $P \in \mathcal{B}_1(H)$, the trace class associated to $\mathcal{B}(H)$ and $\operatorname{tr}(P) = 1$, we define the P-*determinant* of the positive invertible operator A by

$$\Delta_P(A) := \exp \operatorname{tr}(P \ln A).$$

In this paper we show among others that, if A is an operator satisfying the condition $0 < mI \leq A \leq MI$, then

$$\begin{aligned} \frac{m}{M} &\leq \left(\frac{m}{M}\right)^{\left[\frac{1}{2}+\frac{1}{M-m}\operatorname{tr}\left(P\left|A-\frac{m+M}{2}I\right|\right)\right]} \\ &\leq \frac{\Delta_P(A)}{I_d(m,M)} \\ &\leq \left(\frac{M}{m}\right)^{\left[\frac{1}{2}+\frac{1}{M-m}\operatorname{tr}\left(P\left|A-\frac{m+M}{2}I\right|\right)\right]} \leq \frac{M}{m}, \end{aligned}$$

where I_d is the *identric mean.*

1. Introduction

In 1952, in the paper [6], B. Fuglede and R. V. Kadison introduced the determinant of a (invertible) operator and established its fundamental properties. The notion generalizes the usual determinant and can be considered for any operator in a finite von Neumann algebra (M, τ) with a faithful normal trace.

Let $T \in M$ be normal and $|T| := (T^*T)^{1/2}$ its modulus. By the spectral theorem one can represent T as an integral

$$T = \int_{\operatorname{Sp}(T)} \lambda dE(\lambda),$$

where $E(\lambda)$ is a projection valued measure and $\operatorname{Sp}(T)$ is the spectrum of T. The measure $\mu_T := \tau \circ E$ becomes a probability measure on the complex plane and has the support in the spectrum $\operatorname{Sp}(T)$.

For any $T \in M$ the Fuglede-Kadison determinant (*FK-determinant*) is defined by

$$\Delta_{FK}(T) := \exp\left(\int_0^\infty \ln t d\mu_{|T|}\right).$$

If T is invertible, then

$$\Delta_{FK}(T) := \exp(\tau(\ln(|T|))),$$

where $\ln(|T|)$ is defined by the use of functional calculus.

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Let $B(H)$ be the space of all bounded linear operators on a Hilbert space H, and I stands for the identity operator on H. An operator A in $B(H)$ is said to be positive (in symbol: $A \geq 0$) if $\langle Ax, x\rangle \geq 0$ for all $x \in H$. In particular, $A > 0$ means that A is positive and invertible. For a pair A, B of selfadjoint operators the order relation $A \geq B$ means as usual that $A - B$ is positive.

In 1998, Fujii et al. [7], [8], introduced the *normalized determinant* $\Delta_x(A)$ for positive invertible operators A on a Hilbert space H and a fixed unit vector $x \in H$, namely $\|x\| = 1$, defined by

$$\Delta_x(A) := \exp \langle \ln Ax, x\rangle$$

and discussed it as a continuous geometric mean and observed some inequalities around the determinant from this point of view. For some recent results, see [9].

We need now some preparations for trace of operators in Hilbert spaces.

Let $(H, \langle \cdot, \cdot \rangle)$ be a complex Hilbert space and $\{e_i\}_{i\in I}$ an *orthonormal basis* of H. We say that $A \in \mathcal{B}(H)$ is a *Hilbert-Schmidt operator* if

$$\sum_{i\in I} \|Ae_i\|^2 < \infty. \tag{1.1}$$

It is well know that, if $\{e_i\}_{i\in I}$ and $\{f_j\}_{j\in J}$ are orthonormal bases for H and $A \in \mathcal{B}(H)$ then

$$\sum_{i\in I} \|Ae_i\|^2 = \sum_{j\in I} \|Af_j\|^2 = \sum_{j\in I} \|A^* f_j\|^2 \tag{1.2}$$

showing that the definition (1.1) is independent of the orthonormal basis and A is a Hilbert-Schmidt operator iff A^* is a Hilbert-Schmidt operator.

Let $\mathcal{B}_2(H)$ the set of *Hilbert-Schmidt operators* in $\mathcal{B}(H)$. For $A \in \mathcal{B}_2(H)$ we define

$$\|A\|_2 := \left(\sum_{i\in I} \|Ae_i\|^2\right)^{1/2} \tag{1.3}$$

for $\{e_i\}_{i\in I}$ an orthonormal basis of H.

Using the triangle inequality in $l^2(I)$, one checks that $\mathcal{B}_2(H)$ is a *vector space* and that $\|\cdot\|_2$ is a norm on $\mathcal{B}_2(H)$, which is usually called in the literature as the *Hilbert-Schmidt norm.*

Denote *the modulus* of an operator $A \in \mathcal{B}(H)$ by $|A| := (A^*A)^{1/2}$.

Because $\||A|x\| = \|Ax\|$ for all $x \in H$, A is Hilbert-Schmidt iff $|A|$ is Hilbert-Schmidt and $\|A\|_2 = \||A|\|_2$. From (1.2) we have that if $A \in \mathcal{B}_2(H)$, then $A^* \in \mathcal{B}_2(H)$ and $\|A\|_2 = \|A^*\|_2$.

The following theorem collects some of the most important properties of Hilbert-Schmidt operators:

Theorem 1. *We have:*

(i) $(\mathcal{B}_2(H), \|\cdot\|_2)$ *is a Hilbert space with inner product*

$$\langle A, B\rangle_2 := \sum_{i\in I} \langle Ae_i, Be_i\rangle = \sum_{i\in I} \langle B^*Ae_i, e_i\rangle \tag{1.4}$$

and the definition does not depend on the choice of the orthonormal basis $\{e_i\}_{i\in I}$*;*

(ii) We have the inequalities

$$\|A\| \leq \|A\|_2 \tag{1.5}$$

for any $A\in\mathcal{B}_2(H)$ *and, if* $A\in\mathcal{B}_2(H)$ *and* $T\in\mathcal{B}(H)$, *then* AT, $TA\in\mathcal{B}_2(H)$ *with*

$$\|AT\|_2,\ \|TA\|_2\le\|T\|\,\|A\|_2 \tag{1.6}$$

(iii) $\mathcal{B}_2(H)$ *is an operator ideal in* $\mathcal{B}(H)$, *i.e.*

$$\mathcal{B}(H)\,\mathcal{B}_2(H)\,\mathcal{B}(H)\subseteq\mathcal{B}_2(H).$$

If $\{e_i\}_{i\in I}$ an orthonormal basis of H, we say that $A\in\mathcal{B}(H)$ is *trace class* if

$$\|A\|_1:=\sum_{i\in I}\langle|A|\,e_i,e_i\rangle<\infty. \tag{1.7}$$

The definition of $\|A\|_1$ does not depend on the choice of the orthonormal basis $\{e_i\}_{i\in I}$. We denote by $\mathcal{B}_1(H)$ the set of trace class operators in $\mathcal{B}(H)$.

The following proposition holds:

Proposition 1. *If* $A\in\mathcal{B}(H)$, *then the following are equivalent:*

(i) $A\in\mathcal{B}_1(H)$;

(ii) $|A|^{1/2}\in\mathcal{B}_2(H)$.

The following properties are also well known:

Theorem 2. *With the above notations:*

(i) We have

$$\|A\|_1=\|A^*\|_1 \text{ and } \|A\|_2\le\|A\|_1 \tag{1.8}$$

for any $A\in\mathcal{B}_1(H)$;

(ii) $\mathcal{B}_1(H)$ *is an operator ideal in* $\mathcal{B}(H)$, *i.e.*

$$\mathcal{B}(H)\,\mathcal{B}_1(H)\,\mathcal{B}(H)\subseteq\mathcal{B}_1(H);$$

(iii) We have

$$\mathcal{B}_2(H)\,\mathcal{B}_2(H)=\mathcal{B}_1(H);$$

(iv) We have

$$\|A\|_1=\sup\{\langle A,B\rangle_2\ \mid\ B\in\mathcal{B}_2(H),\ \|B\|_2\le1\};$$

(v) $(\mathcal{B}_1(H),\|\cdot\|_1)$ *is a Banach space.*

We define the *trace* of a trace class operator $A\in\mathcal{B}_1(H)$ to be

$$\operatorname{tr}(A):=\sum_{i\in I}\langle Ae_i,e_i\rangle, \tag{1.9}$$

where $\{e_i\}_{i\in I}$ an orthonormal basis of H. Note that this coincides with the usual definition of the trace if H is finite-dimensional. We observe that the series (1.9) converges absolutely and it is independent from the choice of basis.

The following result collects some properties of the trace:

Theorem 3. *We have:*

(i) If $A\in\mathcal{B}_1(H)$ *then* $A^*\in\mathcal{B}_1(H)$ *and*

$$\operatorname{tr}(A^*)=\overline{\operatorname{tr}(A)}; \tag{1.10}$$

(ii) If $A\in\mathcal{B}_1(H)$ *and* $T\in\mathcal{B}(H)$, *then* AT, $TA\in\mathcal{B}_1(H)$,

$$\operatorname{tr}(AT)=\operatorname{tr}(TA) \text{ and } |\operatorname{tr}(AT)|\le\|A\|_1\|T\|; \tag{1.11}$$

(iii) $\operatorname{tr}(\cdot)$ *is a bounded linear functional on* $\mathcal{B}_1(H)$ *with* $\|\operatorname{tr}\|=1$;

(iv) If A, $B\in\mathcal{B}_2(H)$ *then* AB, $BA\in\mathcal{B}_1(H)$ *and* $\operatorname{tr}(AB)=\operatorname{tr}(BA)$.

Now, if we assume that $P \geq 0$ and $P \in \mathcal{B}_1(H)$, then for all $T \in \mathcal{B}(H)$, PT, $TP \in \mathcal{B}_1(H)$ and $\operatorname{tr}(PT) = \operatorname{tr}(TP)$. Also, since $P^{1/2} \in \mathcal{B}_2(H)$, $TP^{1/2} \in \mathcal{B}_2(H)$, hence $P^{1/2}TP^{1/2}$ and $TP^{1/2}P^{1/2} = TP \in \mathcal{B}_1(H)$ with $\operatorname{tr}\left(P^{1/2}TP^{1/2}\right) = \operatorname{tr}(TP)$. Therefore, if $P \geq 0$ and $P \in \mathcal{B}_1(H)$,

$$\operatorname{tr}(PT) = \operatorname{tr}(TP) = \operatorname{tr}\left(P^{1/2}TP^{1/2}\right)$$

for all $T \in \mathcal{B}(H)$.

If $T \geq 0$, then $P^{1/2}TP^{1/2} \geq 0$, which implies that $\operatorname{tr}(PT) \geq 0$ that shows that the functional $\mathcal{B}(H) \ni T \longmapsto \operatorname{tr}(PT)$ is linear and isotonic functional. Also, by (1.11), if $T_n \to T$ for $n \to \infty$ in $\mathcal{B}(H)$ then $\lim_{n\to\infty} \operatorname{tr}(PT_n) = \operatorname{tr}(PT)$, namely $\mathcal{B}(H) \ni T \longmapsto \operatorname{tr}(PT)$ is also continuous in the norm topology.

For a survey on recent trace inequalities see [1] and the references therein.

Now, for a given $P \geq 0$ with $P \in \mathcal{B}_1(H)$ and $\operatorname{tr}(P) = 1$, we define the *P-determinant* of the positive invertible operator A by

$$\Delta_P(A) := \exp\operatorname{tr}(P\ln A) = \exp\operatorname{tr}((\ln A)P) = \exp\operatorname{tr}\left(P^{1/2}(\ln A)P^{1/2}\right). \tag{1.12}$$

Assume that $P \geq 0$ with $P \in \mathcal{B}_1(H)$ and $\operatorname{tr}(P) = 1$. We observe that we have the following elementary properties:

(i) *continuity*: the map $A \to \Delta_P(A)$ is norm continuous;
(ii) *power equality:* $\Delta_P(A^t) = \Delta_P(A)^t$ for all $t > 0$;
(iii) *homogeneity*: $\Delta_P(tA) = t\Delta_x(A)$ and $\Delta_P(tI) = t$ for all $t > 0$;
(iv) *monotonicity*: $0 < A \leq B$ implies $\Delta_P(A) \leq \Delta_P(B)$.

In the recent paper [3] we obtained the following results:

Theorem 4. *Let $P \geq 0$ with $P \in \mathcal{B}_1(H)$ and $\operatorname{tr}(P) = 1$, then for all A, $B > 0$ and $t \in [0,1]$,*

$$\Delta_P((1-t)A + tB) \geq [\Delta_P(A)]^{1-t}[\Delta_P(B)]^t.$$

and

Theorem 5. *Let $P \geq 0$ with $P \in \mathcal{B}_1(H)$ and $\operatorname{tr}(P) = 1$, then for all $A > 0$ and $a > 0$ we have the double inequality*

$$a\exp\left[1 - a\operatorname{tr}\left(PA^{-1}\right)\right] \leq \Delta_P(A) \leq a\exp\left[a^{-1}\operatorname{tr}(PA) - 1\right].$$

In particular

$$1 \leq \frac{\operatorname{tr}(PA)}{\Delta_P(A)} \leq \exp\left[\operatorname{tr}(PA)\operatorname{tr}\left(PA^{-1}\right) - 1\right]$$

and

$$1 \leq \frac{\Delta_P(A)}{\left[\operatorname{tr}\left(PA^{-1}\right)\right]^{-1}} \leq \exp\left[\operatorname{tr}\left(PA^{-1}\right)\operatorname{tr}(PA) - 1\right].$$

Motivated by the above results, in this paper we show among others that, if A is an operator satisfying the condition $0 < mI \leq A \leq MI$, then

$$\begin{aligned}\frac{m}{M} &\leq \left(\frac{m}{M}\right)^{\left[\frac{1}{2}+\frac{1}{M-m}\operatorname{tr}\left(P\left|A-\frac{m+M}{2}I\right|\right)\right]}\\ &\leq \frac{\Delta_P(A)}{I_d(m,M)}\\ &\leq \left(\frac{M}{m}\right)^{\left[\frac{1}{2}+\frac{1}{M-m}\operatorname{tr}\left(P\left|A-\frac{m+M}{2}I\right|\right)\right]} \leq \frac{M}{m},\end{aligned}$$

where I_d is the *identric mean.*

2. Main Results

Recall the *identric mean*

$$I_d(a,b) := \begin{cases} \frac{1}{e}\left(\frac{b^b}{a^a}\right)^{\frac{1}{b-a}} & \text{if} \quad b \neq a \\ a & \text{if} \quad b = a \end{cases} ; \ a, b > 0.$$

It is easy to observe the connection between the integral mean of the logarithmic function and the logarithm of the identric mean,

$$\frac{1}{b-a}\int_a^b \ln t dt = \ln I_d(a,b)$$

for $a \neq b$ positive numbers.

Theorem 6. *Let $P \geq 0$ with $P \in \mathcal{B}_1(H)$ and $\operatorname{tr}(P) = 1$. Assume that the operator A satisfies the condition $0 < mI \leq A \leq MI$, where m, M are positive numbers, then*

$$\begin{aligned} &\exp\left[-\frac{1}{2}\left(\frac{M}{m}-1\right)\right] \\ &\leq \exp\left(-\left(\frac{M}{m}-1\right)\left[\frac{1}{4}+\frac{1}{(M-m)^2}\operatorname{tr}\left[P\left(A-\frac{m+M}{2}I\right)^2\right]\right]\right) \\ &\leq \frac{\Delta_P(A)}{I_d(m,M)} \\ &\leq \exp\left(\left(\frac{M}{m}-1\right)\left[\frac{1}{4}+\frac{1}{(M-m)^2}\operatorname{tr}\left[P\left(A-\frac{m+M}{2}I\right)^2\right]\right]\right) \\ &\leq \exp\left[\frac{1}{2}\left(\frac{M}{m}-1\right)\right]. \end{aligned} \tag{2.1}$$

Also, we have

$$\begin{aligned} &\exp\left[-\frac{1}{2}\left(\frac{M}{m}-1\right)\right] \\ &\leq \exp\left(-\left(\frac{M}{m}-1\right)\right. \\ &\times\left.\left[\frac{1}{4}+\frac{1}{(m^{-1}-M^{-1})^2}\operatorname{tr}\left[P\left(A^{-1}-\frac{m^{-1}+M^{-1}}{2}I\right)^2\right]\right]\right) \end{aligned} \tag{2.2}$$

$$
\begin{aligned}
&\leq \frac{\left[I_d\left(m^{-1}, M^{-1}\right)\right]^{-1}}{\Delta_P(A)} \\
&\leq \exp\left(-\left(\frac{M}{m}-1\right)\right. \\
&\times\left.\left[\frac{1}{4}+\frac{1}{\left(m^{-1}-M^{-1}\right)^2}\operatorname{tr}\left[P\left(A^{-1}-\frac{m^{-1}+M^{-1}}{2}I\right)^2\right]\right]\right) \\
&\leq \exp\left[\frac{1}{2}\left(\frac{M}{m}-1\right)\right].
\end{aligned}
$$

Proof. We use Ostrowski's inequality [10]:

Let $f:[a,b]\to\mathbb{R}$ be continuous on $[a,b]$ and differentiable on (a,b) such that $f':(a,b)\to\mathbb{R}$ is bounded on (a,b), i.e., $\|f'\|_\infty := \sup_{s\in(a,b)}|f'(s)| < \infty$, then

$$
\left|f(t)-\frac{1}{b-a}\int_a^b f(s)\,ds\right| \leq \left[\frac{1}{4}+\left(\frac{t-\frac{a+b}{2}}{b-a}\right)^2\right]\|f'\|_\infty (b-a), \tag{2.3}
$$

for all $t\in[a,b]$ and the constant $\frac{1}{4}$ is the best possible.

If we take $f(t)=\ln t$, $t\in[a,b]\subset(0,\infty)$ in (2.3) and observe that

$$
\|f'\|_\infty = \sup_{t\in[a,b]} t^{-1} = \frac{1}{a},
$$

then we get

$$
|\ln t - \ln I_d(a,b)| \leq \left[\frac{1}{4}+\left(\frac{t-\frac{a+b}{2}}{b-a}\right)^2\right]\left(\frac{b}{a}-1\right),
$$

for all $t\in[a,b]$.

This inequality is equivalent to

$$
\begin{aligned}
&-\left[\frac{1}{4}+\left(\frac{t-\frac{a+b}{2}}{b-a}\right)^2\right]\left(\frac{b}{a}-1\right) \\
&\leq \ln t - \ln I_d(a,b) \leq \left[\frac{1}{4}+\left(\frac{t-\frac{a+b}{2}}{b-a}\right)^2\right]\left(\frac{b}{a}-1\right),
\end{aligned} \tag{2.4}
$$

for all $t\in[a,b]$.

By utilizing the continuous functional calculus for selfadjoint operators, we get from (2.4) that

$$
\begin{aligned}
&-\left(\frac{M}{m}-1\right)\left[\frac{1}{4}I+\frac{1}{(M-m)^2}\left(A-\frac{m+M}{2}I\right)^2\right] \\
&\leq \ln A - \ln I_d(m,M)\,I \\
&\leq \left(\frac{M}{m}-1\right)\left[\frac{1}{4}I+\frac{1}{(M-m)^2}\left(A-\frac{m+M}{2}I\right)^2\right].
\end{aligned}
$$

If we multiply both sides with $P^{1/2}$, then we get

$$
\begin{aligned}
&-\left(\frac{M}{m}-1\right)\left[\frac{1}{4}P+\frac{1}{(M-m)^2}P^{1/2}\left(A-\frac{m+M}{2}I\right)^2 P^{1/2}\right]\\
&\leq P^{1/2}(\ln A)P^{1/2}-\ln I_d(m,M)P\\
&\leq \left(\frac{M}{m}-1\right)\left[\frac{1}{4}P+\frac{1}{(M-m)^2}P^{1/2}\left(A-\frac{m+M}{2}I\right)^2 P^{1/2}\right].
\end{aligned}
$$

Now, if we take the trace and use the fact that $\operatorname{tr} P=1$, then we obtain

$$
\begin{aligned}
&-\left(\frac{M}{m}-1\right)\left[\frac{1}{4}+\frac{1}{(M-m)^2}\operatorname{tr}\left[P\left(A-\frac{m+M}{2}I\right)^2\right]\right]\\
&\leq \operatorname{tr}\left[P^{1/2}(\ln A)P^{1/2}\right]-\ln I_d(m,M)\\
&\leq \left(\frac{M}{m}-1\right)\left[\frac{1}{4}+\frac{1}{(M-m)^2}\operatorname{tr}\left[P\left(A-\frac{m+M}{2}I\right)^2\right]\right].
\end{aligned}
\tag{2.5}
$$

By taking the exponential in (2.5) we derive

$$
\begin{aligned}
&\exp\left(-\left(\frac{M}{m}-1\right)\left[\frac{1}{4}+\frac{1}{(M-m)^2}\operatorname{tr}\left[P\left(A-\frac{m+M}{2}I\right)^2\right]\right]\right)\\
&\leq \frac{\operatorname{tr}\left[P^{1/2}(\ln A)P^{1/2}\right]}{I_d(m,M)}\\
&\leq \exp\left(\frac{M}{m}-1\right)\left[\frac{1}{4}+\frac{1}{(M-m)^2}\operatorname{tr}\left[P\left(A-\frac{m+M}{2}I\right)^2\right]\right].
\end{aligned}
$$

Since

$$
\operatorname{tr}\left[P\left(A-\frac{m+M}{2}I\right)^2\right]\leq \frac{1}{4}(M-m)^2,
$$

hence

$$
\frac{1}{4}+\frac{1}{(M-m)^2}\operatorname{tr}\left[P\left(A-\frac{m+M}{2}I\right)^2\right]\leq \frac{1}{2}
$$

and

$$
-\frac{1}{2}\leq -\left(\frac{1}{4}+\frac{1}{(M-m)^2}\operatorname{tr}\left[P\left(A-\frac{m+M}{2}I\right)^2\right]\right).
$$

These prove the desired result (2.1).

If $0<mI\leq A\leq MI$, then $0<M^{-1}I\leq A^{-1}\leq m^{-1}I$ and if we write the inequality (2.1) for A^{-1}, we derive (2.2). □

Theorem 7. *With the assumptions of Theorem 6, we have the inequalities*

$$\begin{aligned}\frac{m}{M} &\leq \left(\frac{m}{M}\right)^{\left[\frac{1}{2}+\frac{1}{M-m}\operatorname{tr}\left(P\left|A-\frac{m+M}{2}I\right|\right)\right]} \\ &\leq \frac{\Delta_P(A)}{I_d(m,M)} \\ &\leq \left(\frac{M}{m}\right)^{\left[\frac{1}{2}+\frac{1}{M-m}\operatorname{tr}\left(P\left|A-\frac{m+M}{2}I\right|\right)\right]} \leq \frac{M}{m}\end{aligned} \tag{2.6}$$

and

$$\begin{aligned}\frac{m}{M} &\leq \left(\frac{m}{M}\right)^{\left[\frac{1}{2}+\frac{1}{m^{-1}-M^{-1}}\operatorname{tr}\left(P\left|A-\frac{M^{-1}+m^{-1}}{2}I\right|\right)\right]} \\ &\leq \frac{\left[I_d\left(m^{-1},M^{-1}\right)\right]^{-1}}{\Delta_P(A)} \\ &\leq \left(\frac{M}{m}\right)^{\left[\frac{1}{2}+\frac{1}{m^{-1}-M^{-1}}\operatorname{tr}\left(P\left|A-\frac{M^{-1}+m^{-1}}{2}I\right|\right)\right]} \leq \frac{M}{m}.\end{aligned} \tag{2.7}$$

Proof. In 1997, Dragomir and Wang proved the following Ostrowski type inequality [4]:

Let $f:[a,b]\to\mathbb{R}$ be an absolutely continuous function on $[a,b]$, then

$$\left|f(t)-\frac{1}{b-a}\int_a^b f(s)\,ds\right| \leq \left[\frac{1}{2}+\frac{\left|t-\frac{a+b}{2}\right|}{b-a}\right]\|f'\|_{[a,b],1}, \tag{2.8}$$

for all $t\in[a,b]$, where $\|\cdot\|_1$ is the Lebesgue norm on $L_1[a,b]$, i.e., we recall it

$$\|g\|_{[a,b],1} := \int_a^b |g(t)|\,dt.$$

The constant $\frac{1}{2}$ is best possible.

If we take $f(t)=\ln t$, $t\in[a,b]\subset(0,\infty)$ in (2.8) and observe that

$$\|f'\|_{[a,b],1} = \ln b - \ln a,$$

then by (2.8) we get

$$|\ln t - \ln I_d(a,b)| \leq \left[\frac{1}{2}+\frac{1}{b-a}\left|t-\frac{a+b}{2}\right|\right](\ln b-\ln a),$$

for all $t\in[a,b]$.

This inequality is equivalent to

$$\begin{aligned}&-\left[\frac{1}{2}+\frac{1}{b-a}\left|t-\frac{a+b}{2}\right|\right](\ln b-\ln a) \\ &\leq \ln t-\ln I_d(a,b) \\ &\leq \left[\frac{1}{2}+\frac{1}{b-a}\left|t-\frac{a+b}{2}\right|\right](\ln b-\ln a),\end{aligned} \tag{2.9}$$

for all $t\in[a,b]$.

By utilizing the continuous functional calculus for selfadjoint operators, we get from (2.4) that

$$
\begin{aligned}
& -(\ln M-\ln m)\left[\frac{1}{2}I+\frac{1}{M-m}\left|A-\frac{m+M}{2}I\right|\right] \\
& \leq \ln A-\ln I_d(m,M)\,I \\
& \leq (\ln M-\ln m)\left[\frac{1}{2}I+\frac{1}{M-m}\left|A-\frac{m+M}{2}I\right|\right],
\end{aligned}
$$

which, as above, implies the trace inequalities

$$
\begin{aligned}
& -(\ln M-\ln m)\left[\frac{1}{2}+\frac{1}{M-m}\operatorname{tr}\left(P\left|A-\frac{m+M}{2}I\right|\right)\right] \\
& \leq \operatorname{tr}(P\ln A)-\ln I_d(m,M) \\
& \leq (\ln M-\ln m)\left[\frac{1}{2}+\frac{1}{M-m}\operatorname{tr}\left(P\left|A-\frac{m+M}{2}I\right|\right)\right].
\end{aligned}
$$

By taking the exponential, we derive

$$
\begin{aligned}
& \exp\left(-(\ln M-\ln m)\left[\frac{1}{2}+\frac{1}{M-m}\operatorname{tr}\left(P\left|A-\frac{m+M}{2}I\right|\right)\right]\right) \\
& \leq \frac{\exp\operatorname{tr}(P\ln A)}{I_d(m,M)} \\
& \leq \exp\left((\ln M-\ln m)\left[\frac{1}{2}+\frac{1}{M-m}\operatorname{tr}\left(P\left|A-\frac{m+M}{2}I\right|\right)\right]\right).
\end{aligned}
$$

Also, we notice that

$$
\operatorname{tr}\left(P\left|A-\frac{m+M}{2}I\right|\right)\leq\frac{1}{2}(M-m)
$$

for $P\geq 0$ with $P\in\mathcal{B}_1(H)$ and $\operatorname{tr}(P)=1$. These prove the desired result (2.6).

The inequality (2.7) follows by (2.6) applied for A^{-1}. □

Theorem 8. *With the assumptions of Theorem 6, we have the inequalities*

$$
\begin{aligned}
& \exp\left(-\frac{(M-m)^{1/q}\left(M^{p-1}-m^{p-1}\right)^{1/p}}{(q+1)^{1/q}(p-1)^{1/p}m^{1/q}M^{1/q}}\right) \\
& \leq \exp\left(-\frac{(M-m)^{1/q}\left(M^{p-1}-m^{p-1}\right)^{1/p}}{(q+1)^{1/q}(p-1)^{1/p}m^{1/q}M^{1/q}}\right. \\
& \qquad\left.\times\left(\operatorname{tr}P\left[\left(\frac{A-mI}{M-m}\right)^{q+1}+\left(\frac{MI-A}{M-m}\right)^{q+1}\right]\right)^{1/q}\right) \\
& \leq \exp\left(-\frac{(M-m)^{1/q}\left(M^{p-1}-m^{p-1}\right)^{1/p}}{(q+1)^{1/q}(p-1)^{1/p}m^{1/q}M^{1/q}}\right. \\
& \qquad\left.\times\operatorname{tr}\left(P\left[\left(\frac{A-mI}{M-m}\right)^{q+1}+\left(\frac{MI-A}{M-m}\right)^{q+1}\right]^{1/q}\right)\right)
\end{aligned}
\tag{2.10}
$$

$$
\begin{aligned}
&\leq \frac{\Delta_P(A)}{I_d(m,M)} \\
&\leq \exp\left(\frac{(M-m)^{1/q}\left(M^{p-1}-m^{p-1}\right)^{1/p}}{(q+1)^{1/q}(p-1)^{1/p}m^{1/q}M^{1/q}}\right. \\
&\left.\times \operatorname{tr}\left(P\left[\left(\frac{A-mI}{M-m}\right)^{q+1}+\left(\frac{MI-A}{M-m}\right)^{q+1}\right]^{1/q}\right)\right) \\
&\leq \exp\left(\frac{(M-m)^{1/q}\left(M^{p-1}-m^{p-1}\right)^{1/p}}{(q+1)^{1/q}(p-1)^{1/p}m^{1/q}M^{1/q}}\right. \\
&\left.\times\left(\operatorname{tr}\left[P\left(\frac{A-mI}{M-m}\right)^{q+1}+\left(\frac{MI-A}{M-m}\right)^{q+1}\right]\right)^{1/q}\right) \\
&\leq \exp\left(\frac{(M-m)^{1/q}\left(M^{p-1}-m^{p-1}\right)^{1/p}}{(q+1)^{1/q}(p-1)^{1/p}m^{1/q}M^{1/q}}\right)
\end{aligned}
$$

where $p>1$, $\frac{1}{p}+\frac{1}{q}=1$.

Proof. In 1998, Dragomir and Wang proved the following Ostrowski type inequality [5]:

Let $f:[a,b]\to\mathbb{R}$ be an absolutely continuous function on $[a,b]$. If $f'\in L_p[a,b]$, then we have the inequality

$$
\begin{aligned}
&\left|f(t)-\frac{1}{b-a}\int_a^b f(s)\,dt\right| \\
&\leq \frac{1}{(q+1)^{1/q}}\left[\left(\frac{t-a}{b-a}\right)^{q+1}+\left(\frac{b-t}{b-a}\right)^{q+1}\right]^{1/q}(b-a)^{1/q}\|f'\|_{[a,b],p}, \qquad (2.11)
\end{aligned}
$$

for all $t\in[a,b]$, where $p>1$, $\frac{1}{p}+\frac{1}{q}=1$ and $\|\cdot\|_{[a,b],p}$ is the p-Lebesgue norm on $L_p[a,b]$, i.e., we recall it

$$
\|g\|_{[a,b],p}:=\left(\int_a^b |g(t)|^p\,dt\right)^{1/p}.
$$

If we take $f(t)=\ln t$, $t\in[a,b]\subset(0,\infty)$ in (2.11) and observe that

$$
\begin{aligned}
\|f'\|_{[a,b],p}&:=\left(\int_a^b t^{-p}dt\right)^{1/p}=\left(\frac{b^{-p+1}-a^{-p+1}}{1-p}\right)^{1/p} \\
&=\left(\frac{\frac{1}{b^{p-1}}-\frac{1}{a^{p-1}}}{1-p}\right)^{1/p}=\left[\frac{b^{p-1}-a^{p-1}}{(p-1)a^{p-1}b^{p-1}}\right]^{1/p} \\
&=\frac{\left(b^{p-1}-a^{p-1}\right)^{1/p}}{(p-1)^{1/p}a^{1-1/p}b^{1-1/p}}=\frac{\left(b^{p-1}-a^{p-1}\right)^{1/p}}{(p-1)^{1/p}a^{1/q}b^{1/q}},
\end{aligned}
$$

then we get

$$
\begin{aligned}
(2.12)\qquad & \left|\ln t-\ln I_d\left(a,b\right)\right| \\
&\le \frac{1}{\left(q+1\right)^{1/q}\left(p-1\right)^{1/p}}\left[\left(\frac{t-a}{b-a}\right)^{q+1}+\left(\frac{b-t}{b-a}\right)^{q+1}\right]^{1/q} \\
&\times \frac{\left(b-a\right)^{1/q}\left(b^{p-1}-a^{p-1}\right)^{1/p}}{a^{1/q}b^{1/q}} \\
&\le \frac{1}{\left(q+1\right)^{1/q}\left(p-1\right)^{1/p}}\frac{\left(b-a\right)^{1/q}\left(b^{p-1}-a^{p-1}\right)^{1/p}}{a^{1/q}b^{1/q}}
\end{aligned}
$$

for $t\in[a,b]$, since

$$
\left(\frac{t-a}{b-a}\right)^{q+1}+\left(\frac{b-t}{b-a}\right)^{q+1}\le 1
$$

for $t\in[a,b]$.

This implies as above

$$
\begin{aligned}
&-\frac{1}{\left(q+1\right)^{1/q}\left(p-1\right)^{1/p}}\left[\left(\frac{A-mI}{M-m}\right)^{q+1}+\left(\frac{MI-A}{M-m}\right)^{q+1}\right]^{1/q} \\
&\times \frac{\left(M-m\right)^{1/q}\left(M^{p-1}-m^{p-1}\right)^{1/p}}{m^{1/q}M^{1/q}} \\
&\le \ln A-\ln I_d\left(a,b\right)I \\
&\le \frac{1}{\left(q+1\right)^{1/q}\left(p-1\right)^{1/p}}\left[\left(\frac{A-mI}{M-m}\right)^{q+1}+\left(\frac{MI-A}{M-m}\right)^{q+1}\right]^{1/q} \\
&\times \frac{\left(M-m\right)^{1/q}\left(M^{p-1}-m^{p-1}\right)^{1/p}}{m^{1/q}M^{1/q}}.
\end{aligned}
$$

Following a similar argument as above, we get the trace inequalities

$$
\begin{aligned}
&-\frac{\left(M-m\right)^{1/q}\left(M^{p-1}-m^{p-1}\right)^{1/p}}{\left(q+1\right)^{1/q}\left(p-1\right)^{1/p}m^{1/q}M^{1/q}} \\
&\times \operatorname{tr}\left(P\left[\left(\frac{A-mI}{M-m}\right)^{q+1}+\left(\frac{MI-A}{M-m}\right)^{q+1}\right]^{1/q}\right) \\
&\le \operatorname{tr}\left(P\ln A\right)-\ln I_d\left(a,b\right) \\
&\le \frac{\left(M-m\right)^{1/q}\left(M^{p-1}-m^{p-1}\right)^{1/p}}{\left(q+1\right)^{1/q}\left(p-1\right)^{1/p}m^{1/q}M^{1/q}} \\
&\times \operatorname{tr}\left(P\left[\left(\frac{A-mI}{M-m}\right)^{q+1}+\left(\frac{MI-A}{M-m}\right)^{q+1}\right]^{1/q}\right)
\end{aligned}
$$

By Jensen's trace inequality for concave functions, we also have that

$$\operatorname{tr}\left(P\left[\left(\frac{A-mI}{M-m}\right)^{q+1}+\left(\frac{MI-A}{M-m}\right)^{q+1}\right]^{1/q}\right)$$
$$\leq\left(\operatorname{tr}\left[P\left(\left(\frac{A-mI}{M-m}\right)^{q+1}+\left(\frac{MI-A}{M-m}\right)^{q+1}\right)\right]\right)^{1/q}.$$

The last part follows by the fact that

$$\left(\operatorname{tr}\left[P\left(\left(\frac{A-mI}{M-m}\right)^{q+1}+\left(\frac{MI-A}{M-m}\right)^{q+1}\right)\right]\right)^{1/q}\leq 1$$

for $P\geq 0$ with $P\in\mathcal{B}_1(H)$ and $\operatorname{tr}(P)=1$.

Now, by taking the exponential and making use of a similar argument as above, we derive the desired result (2.10). □

Corollary 1. *With the assumption of Theorem 6 we have*

$$\exp\left(-\frac{M-m}{\sqrt{3mM}}\right) \tag{2.13}$$
$$\leq\exp\left(-\frac{M-m}{\sqrt{3mM}}\left[\operatorname{tr}\left(P\left[\left(\frac{A-mI}{M-m}\right)^{3}+\left(\frac{MI-A}{M-m}\right)^{3}\right]\right)\right]^{1/2}\right)$$
$$\leq\exp\left(-\frac{M-m}{\sqrt{3mM}}\operatorname{tr}\left(P\left[\left(\frac{A-mI}{M-m}\right)^{3}+\left(\frac{MI-A}{M-m}\right)^{3}\right]^{1/2}\right)\right)$$
$$\leq\frac{\Delta_P(A)}{I_d(m,M)}$$
$$\leq\exp\left(\frac{M-m}{\sqrt{3mM}}\operatorname{tr}\left(P\left[\left(\frac{A-mI}{M-m}\right)^{3}+\left(\frac{MI-A}{M-m}\right)^{3}\right]^{1/2}\right)\right)$$
$$\leq\exp\left(\frac{M-m}{\sqrt{3mM}}\left[\operatorname{tr}\left(P\left[\left(\frac{A-mI}{M-m}\right)^{3}+\left(\frac{MI-A}{M-m}\right)^{3}\right]\right)\right]^{1/2}\right)$$
$$\leq\exp\left(\frac{M-m}{\sqrt{3mM}}\right).$$

Remark 1. *If we apply the inequality (2.10) for A^{-1}, then we get*

$$\exp\left(-\frac{\left(m^{-1}-M^{-1}\right)^{1/q}\left(m^{1-p}-M^{1-p}\right)^{1/p}}{(q+1)^{1/q}(p-1)^{1/p}M^{-1/q}m^{-1/q}}\right) \tag{2.14}$$
$$\leq\exp\left(-\frac{\left(m^{-1}-M^{-1}\right)^{1/q}\left(m^{1-p}-M^{1-p}\right)^{1/p}}{(q+1)^{1/q}(p-1)^{1/p}M^{-1/q}m^{-1/q}}\right.$$
$$\left.\times\left[\operatorname{tr}\left(P\left[\left(\frac{A^{-1}-M^{-1}I}{m^{-1}-M^{-1}}\right)^{q+1}+\left(\frac{m^{-1}I-A^{-1}}{m^{-1}-M^{-1}}\right)^{q+1}\right]\right)\right]^{1/q}\right)$$

$$\leq \exp\left(-\frac{\left(m^{-1}-M^{-1}\right)^{1/q}\left(m^{1-p}-M^{1-p}\right)^{1/p}}{(q+1)^{1/q}(p-1)^{1/p}M^{-1/q}m^{-1/q}}\right.$$
$$\left.\times\operatorname{tr}\left(P\left[\left(\frac{A^{-1}-M^{-1}I}{m^{-1}-M^{-1}}\right)^{q+1}+\left(\frac{m^{-1}I-A^{-1}}{m^{-1}-M^{-1}}\right)^{q+1}\right]^{1/q}\right)\right)$$

$$\leq \frac{\left[I_d\left(m^{-1},M^{-1}\right)\right]^{-1}}{\Delta_P(A)}$$
$$\leq \exp\left(\frac{\left(m^{-1}-M^{-1}\right)^{1/q}\left(m^{1-p}-M^{1-p}\right)^{1/p}}{(q+1)^{1/q}(p-1)^{1/p}M^{-1/q}m^{-1/q}}\right.$$
$$\left.\times\operatorname{tr}\left(P\left[\left(\frac{A^{-1}-M^{-1}I}{m^{-1}-M^{-1}}\right)^{q+1}+\left(\frac{m^{-1}I-A^{-1}}{m^{-1}-M^{-1}}\right)^{q+1}\right]^{1/q}\right)\right)$$
$$\leq \exp\left(\frac{\left(m^{-1}-M^{-1}\right)^{1/q}\left(m^{1-p}-M^{1-p}\right)^{1/p}}{(q+1)^{1/q}(p-1)^{1/p}M^{-1/q}m^{-1/q}}\right.$$
$$\left.\times\left[\operatorname{tr}\left(P\left[\left(\frac{A^{-1}-M^{-1}I}{m^{-1}-M^{-1}}\right)^{q+1}+\left(\frac{m^{-1}I-A^{-1}}{m^{-1}-M^{-1}}\right)^{q+1}\right]\right)\right]^{1/q}\right)$$
$$\leq \exp\left(\frac{\left(m^{-1}-M^{-1}\right)^{1/q}\left(m^{1-p}-M^{1-p}\right)^{1/p}}{(q+1)^{1/q}(p-1)^{1/p}M^{-1/q}m^{-1/q}}\right).$$

3. Related Results

The following results of Ostrowski type holds, see [2]:

Lemma 1. *Let $f:[a,b]\subset\mathbb{R}\to\mathbb{R}$ be a convex function on $[a,b]$. Then for any $t\in[a,b]$ one has the inequality*

$$\begin{aligned}(3.1)\quad &\frac{1}{2}\left[(b-t)^2 f'_+(t)-(t-a)^2 f'_-(t)\right]\\ &\leq \int_a^b f(s)\,ds-(b-a)f(t)\\ &\leq \frac{1}{2}\left[(b-t)^2 f'_-(b)-(t-a)^2 f'_+(a)\right].\end{aligned}$$

The constant $\frac{1}{2}$ is sharp in both inequalities. The second inequality also holds for $t=a$ or $t=b$.

If the function is differentiable in $t\in(a,b)$ then the first inequality in (3.1) becomes

$$(3.2)\quad \left(\frac{a+b}{2}-t\right)f'(t)\leq\frac{1}{b-a}\int_a^b f(s)\,ds-f(t).$$

We also have:

Theorem 9. *Assume that the operator A satisfies the condition $0 < mI \leq A \leq MI$, where m, M are positive numbers, then*

$$\begin{aligned}
(3.3)\qquad & \exp\left(1 - \frac{m+M}{2}\operatorname{tr}\left(PA^{-1}\right)\right) \\
& \leq \frac{\Delta_P(A)}{I_d(m,M)} \\
& \leq \exp\left(\frac{1}{m}\operatorname{tr}\left(P(A-mI)^2\right) - \frac{1}{M}\operatorname{tr}\left(P(MI-A)^2\right)\right)
\end{aligned}$$

and

$$\begin{aligned}
(3.4)\qquad & \exp\left(1 - \frac{m^{-1}+M^{-1}}{2}\operatorname{tr}(PA)\right) \\
& \leq \frac{\left[I_d\left(m^{-1}, M^{-1}\right)\right]^{-1}}{\Delta_P(A)} \\
& \leq \exp\left(M\operatorname{tr}\left[P\left(A^{-1} - M^{-1}I\right)^2\right] - m\operatorname{tr}\left[P\left(m^{-1}I - A^{-1}\right)^2\right]\right).
\end{aligned}$$

Proof. Writing (3.1) and (3.2) for the convex function $f(t) = -\ln t$, then we get

$$1 - \frac{a+b}{2}t^{-1} \leq \ln t - \ln I_d(a,b) \leq \frac{(t-a)^2}{a} - \frac{(b-t)^2}{b},$$

for all $t \in [a,b] \subset (0,\infty)$.

If we use the functional calculus, we derive the operator inequality

$$1 - \frac{m+M}{2}A^{-1} \leq \ln A - \ln I_d(m,M) \leq \frac{(A-mI)^2}{m} - \frac{(MI-A)^2}{M},$$

which gives

$$\begin{aligned}
& 1 - \frac{m+M}{2}\operatorname{tr}\left(PA^{-1}\right) \\
& \leq \operatorname{tr}(P\ln A) - \ln I_d(m,M) \\
& \leq \frac{1}{m}\operatorname{tr}\left(P(A-mI)^2\right) - \frac{1}{M}\operatorname{tr}\left(P(MI-A)^2\right).
\end{aligned}$$

If we take the exponential, then we get

$$\begin{aligned}
& \exp\left(1 - \frac{m+M}{2}\operatorname{tr}\left(PA^{-1}\right)\right) \\
& \leq \frac{\exp\operatorname{tr}(P\ln A)}{I_d(m,M)} \\
& \leq \exp\left(\frac{1}{m}\operatorname{tr}\left(P(A-mI)^2\right) - \frac{1}{M}\operatorname{tr}\left(P(MI-A)^2\right)\right).
\end{aligned}$$

The inequality (3.4) follows by (3.1) applied for A^{-1}. □

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