SEVERAL BOUNDS FOR THE TRACE CLASS P-DETERMINANT OF POSITIVE OPERATORS IN HILBERT SPACES

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Abstract. Let H be a complex Hilbert space. For a given operator $P\geq 0$ with $P\in\mathcal{B}_1(H)$, the trace class associated to $\mathcal{B}(H)$ and $\operatorname{tr}(P)=1$, we define the P-*determinant* of the positive invertible operator A by

$$\Delta_P(A):=\exp\operatorname{tr}(P\ln A).$$

In this paper we show among others that, if $P_i\geq 0$ with $P_i\in\mathcal{B}_1(H)$ and $\operatorname{tr}(P_i)=1$ for $i\in\{1,...,n\}$, $p_i\geq 0$ with $\sum_{i=1}^n p_i=1$ and $A_i>0$ for $i\in\{1,...,n\}$, then for all $a>0$

$$a\exp\left(1-a\sum_{i=1}^n p_i\operatorname{tr}\left(P_iA_i^{-1}\right)\right)\leq\prod_{i=1}^n\left[\Delta_{P_i}(A_i)\right]^{p_i}$$
$$\leq a\exp\left(a^{-1}\sum_{i=1}^n p_i\operatorname{tr}(P_iA_i)-1\right).$$

1. Introduction

In 1952, in the paper [5], B. Fuglede and R. V. Kadison introduced the determinant of a (invertible) operator and established its fundamental properties. The notion generalizes the usual determinant and can be considered for any operator in a finite von Neumann algebra (M,τ) with a faithful normal trace.

Let $T\in M$ be normal and $|T|:=(T^*T)^{1/2}$ its modulus. By the spectral theorem one can represent T as an integral

$$T=\int_{\operatorname{Sp}(T)}\lambda dE(\lambda),$$

where $E(\lambda)$ is a projection valued measure and $\operatorname{Sp}(T)$ is the spectrum of T. The measure $\mu_T:=\tau\circ E$ becomes a probability measure on the complex plane and has the support in the spectrum $\operatorname{Sp}(T)$.

For any $T\in M$ the Fuglede-Kadison determinant (*FK-determinant*) is defined by

$$\Delta_{FK}(T):=\exp\left(\int_0^\infty\ln t\,d\mu_{|T|}\right).$$

If T is invertible, then

$$\Delta_{FK}(T):=\exp\left(\tau\left(\ln\left(|T|\right)\right)\right),$$

where $\ln(|T|)$ is defined by the use of functional calculus.

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Let $B(H)$ be the space of all bounded linear operators on a Hilbert space H, and I stands for the identity operator on H. An operator A in $B(H)$ is said to be positive (in symbol: $A \geq 0$) if $\langle Ax, x\rangle \geq 0$ for all $x \in H$. In particular, $A > 0$ means that A is positive and invertible. For a pair A, B of selfadjoint operators the order relation $A \geq B$ means as usual that $A - B$ is positive.

In 1998, Fujii et al. [6], [7], introduced the *normalized determinant* $\Delta_x(A)$ for positive invertible operators A on a Hilbert space H and a fixed unit vector $x \in H$, namely $\|x\| = 1$, defined by

$$\Delta_x(A) := \exp \langle \ln Ax, x\rangle$$

and discussed it as a continuous geometric mean and observed some inequalities around the determinant from this point of view. For some recent results, see [10].

We need now some preparations for trace of operators in Hilbert spaces.

Let $(H, \langle \cdot, \cdot \rangle)$ be a complex Hilbert space and $\{e_i\}_{i\in I}$ an *orthonormal basis* of H. We say that $A \in \mathcal{B}(H)$ is a *Hilbert-Schmidt operator* if

$$\sum_{i\in I} \|Ae_i\|^2 < \infty. \tag{1.1}$$

It is well know that, if $\{e_i\}_{i\in I}$ and $\{f_j\}_{j\in J}$ are orthonormal bases for H and $A \in \mathcal{B}(H)$ then

$$\sum_{i\in I} \|Ae_i\|^2 = \sum_{j\in I} \|Af_j\|^2 = \sum_{j\in I} \|A^* f_j\|^2 \tag{1.2}$$

showing that the definition (1.1) is independent of the orthonormal basis and A is a Hilbert-Schmidt operator iff A^* is a Hilbert-Schmidt operator.

Let $\mathcal{B}_2(H)$ the set of *Hilbert-Schmidt operators* in $\mathcal{B}(H)$. For $A \in \mathcal{B}_2(H)$ we define

$$\|A\|_2 := \left(\sum_{i\in I} \|Ae_i\|^2\right)^{1/2} \tag{1.3}$$

for $\{e_i\}_{i\in I}$ an orthonormal basis of H.

Using the triangle inequality in $l^2(I)$, one checks that $\mathcal{B}_2(H)$ is a *vector space* and that $\|\cdot\|_2$ is a norm on $\mathcal{B}_2(H)$, which is usually called in the literature as the *Hilbert-Schmidt norm.*

Denote *the modulus* of an operator $A \in \mathcal{B}(H)$ by $|A| := (A^*A)^{1/2}$.

Because $\||A|x\| = \|Ax\|$ for all $x \in H$, A is Hilbert-Schmidt iff $|A|$ is Hilbert-Schmidt and $\|A\|_2 = \||A|\|_2$. From (1.2) we have that if $A \in \mathcal{B}_2(H)$, then $A^* \in \mathcal{B}_2(H)$ and $\|A\|_2 = \|A^*\|_2$.

The following theorem collects some of the most important properties of Hilbert-Schmidt operators:

Theorem 1. *We have:*

(i) $(\mathcal{B}_2(H), \|\cdot\|_2)$ *is a Hilbert space with inner product*

$$\langle A, B\rangle_2 := \sum_{i\in I} \langle Ae_i, Be_i\rangle = \sum_{i\in I} \langle B^*Ae_i, e_i\rangle \tag{1.4}$$

and the definition does not depend on the choice of the orthonormal basis $\{e_i\}_{i\in I}$;

(ii) We have the inequalities

$$\|A\| \leq \|A\|_2 \tag{1.5}$$

for any $A \in \mathcal{B}_2(H)$ *and, if* $A \in \mathcal{B}_2(H)$ *and* $T \in \mathcal{B}(H)$, *then* AT, $TA \in \mathcal{B}_2(H)$ *with*

$$\|AT\|_2, \ \|TA\|_2 \leq \|T\| \|A\|_2 \tag{1.6}$$

(iii) $\mathcal{B}_2(H)$ *is an operator ideal in* $\mathcal{B}(H)$, *i.e.*

$$\mathcal{B}(H) \mathcal{B}_2(H) \mathcal{B}(H) \subseteq \mathcal{B}_2(H).$$

If $\{e_i\}_{i\in I}$ an orthonormal basis of H, we say that $A \in \mathcal{B}(H)$ is *trace class* if

$$\|A\|_1 := \sum_{i\in I} \langle |A| e_i, e_i\rangle < \infty. \tag{1.7}$$

The definition of $\|A\|_1$ does not depend on the choice of the orthonormal basis $\{e_i\}_{i\in I}$. We denote by $\mathcal{B}_1(H)$ the set of trace class operators in $\mathcal{B}(H)$.

The following proposition holds:

Proposition 1. *If* $A \in \mathcal{B}(H)$, *then the following are equivalent:*

(i) $A \in \mathcal{B}_1(H)$;

(ii) $|A|^{1/2} \in \mathcal{B}_2(H)$.

The following properties are also well known:

Theorem 2. *With the above notations:*

(i) We have

$$\|A\|_1 = \|A^*\|_1 \text{ and } \|A\|_2 \leq \|A\|_1 \tag{1.8}$$

for any $A \in \mathcal{B}_1(H)$;

(ii) $\mathcal{B}_1(H)$ *is an operator ideal in* $\mathcal{B}(H)$, *i.e.*

$$\mathcal{B}(H) \mathcal{B}_1(H) \mathcal{B}(H) \subseteq \mathcal{B}_1(H);$$

(iii) We have

$$\mathcal{B}_2(H) \mathcal{B}_2(H) = \mathcal{B}_1(H);$$

(iv) We have

$$\|A\|_1 = \sup\{\langle A, B\rangle_2 \mid B \in \mathcal{B}_2(H), \ \|B\|_2 \leq 1\};$$

(v) $(\mathcal{B}_1(H), \|\cdot\|_1)$ *is a Banach space.*

We define the *trace* of a trace class operator $A \in \mathcal{B}_1(H)$ to be

$$\operatorname{tr}(A) := \sum_{i\in I} \langle A e_i, e_i\rangle, \tag{1.9}$$

where $\{e_i\}_{i\in I}$ an orthonormal basis of H. Note that this coincides with the usual definition of the trace if H is finite-dimensional. We observe that the series (1.9) converges absolutely and it is independent from the choice of basis.

The following result collects some properties of the trace:

Theorem 3. *We have:*

(i) If $A \in \mathcal{B}_1(H)$ *then* $A^* \in \mathcal{B}_1(H)$ *and*

$$\operatorname{tr}(A^*) = \overline{\operatorname{tr}(A)}; \tag{1.10}$$

(ii) If $A \in \mathcal{B}_1(H)$ *and* $T \in \mathcal{B}(H)$, *then* AT, $TA \in \mathcal{B}_1(H)$,

$$\operatorname{tr}(AT) = \operatorname{tr}(TA) \text{ and } |\operatorname{tr}(AT)| \leq \|A\|_1 \|T\|; \tag{1.11}$$

(iii) $\operatorname{tr}(\cdot)$ *is a bounded linear functional on* $\mathcal{B}_1(H)$ *with* $\|\operatorname{tr}\| = 1$;

(iv) If A, $B \in \mathcal{B}_2(H)$ *then* AB, $BA \in \mathcal{B}_1(H)$ *and* $\operatorname{tr}(AB) = \operatorname{tr}(BA)$.

Now, if we assume that $P \geq 0$ and $P \in \mathcal{B}_1(H)$, then for all $T \in \mathcal{B}(H)$, PT, $TP \in \mathcal{B}_1(H)$ and $\operatorname{tr}(PT) = \operatorname{tr}(TP)$. Also, since $P^{1/2} \in \mathcal{B}_2(H)$, $TP^{1/2} \in \mathcal{B}_2(H)$, hence $P^{1/2}TP^{1/2}$ and $TP^{1/2}P^{1/2} = TP \in \mathcal{B}_1(H)$ with $\operatorname{tr}\left(P^{1/2}TP^{1/2}\right) = \operatorname{tr}(TP)$. Therefore, if $P \geq 0$ and $P \in \mathcal{B}_1(H)$,

$$\operatorname{tr}(PT) = \operatorname{tr}(TP) = \operatorname{tr}\left(P^{1/2}TP^{1/2}\right)$$

for all $T \in \mathcal{B}(H)$.

If $T \geq 0$, then $P^{1/2}TP^{1/2} \geq 0$, which implies that $\operatorname{tr}(PT) \geq 0$ that shows that the functional $\mathcal{B}(H) \ni T \longmapsto \operatorname{tr}(PT)$ is linear and isotonic functional. Also, by (1.11), if $T_n \to T$ for $n \to \infty$ in $\mathcal{B}(H)$ then $\lim_{n\to\infty} \operatorname{tr}(PT_n) = \operatorname{tr}(PT)$, namely $\mathcal{B}(H) \ni T \longmapsto \operatorname{tr}(PT)$ is also continuous in the norm topology.

For a survey on recent trace inequalities see [3] and the references therein.

Now, for a given $P \geq 0$ with $P \in \mathcal{B}_1(H)$ and $\operatorname{tr}(P) = 1$, we define the P-*determinant* of the positive invertible operator A by

$$\Delta_P(A) := \exp \operatorname{tr}(P \ln A) = \exp \operatorname{tr}((\ln A) P) = \exp \operatorname{tr}\left(P^{1/2}(\ln A) P^{1/2}\right). \tag{1.12}$$

Assume that $P \geq 0$ with $P \in \mathcal{B}_1(H)$ and $\operatorname{tr}(P) = 1$. We observe that we have the following elementary properties:

(i) *continuity*: the map $A \to \Delta_P(A)$ is norm continuous;
(ii) *power equality:* $\Delta_P(A^t) = \Delta_P(A)^t$ for all $t > 0$;
(iii) *homogeneity*: $\Delta_P(tA) = t\Delta_x(A)$ and $\Delta_P(tI) = t$ for all $t > 0$;
(iv) *monotonicity*: $0 < A \leq B$ implies $\Delta_P(A) \leq \Delta_P(B)$.

In the recent paper [4] we obtained the following results:

Theorem 4. *Let $P \geq 0$ with $P \in \mathcal{B}_1(H)$ and $\operatorname{tr}(P) = 1$, then for all A, $B > 0$ and $t \in [0,1]$,*

$$\Delta_P((1-t)A + tB) \geq [\Delta_P(A)]^{1-t}[\Delta_P(B)]^t.$$

and

Theorem 5. *Let $P \geq 0$ with $P \in \mathcal{B}_1(H)$ and $\operatorname{tr}(P) = 1$, then for all $A > 0$ and $a > 0$ we have the double inequality*

$$a \exp\left[1 - a \operatorname{tr}\left(PA^{-1}\right)\right] \leq \Delta_P(A) \leq a \exp\left[a^{-1} \operatorname{tr}(PA) - 1\right]. \tag{1.13}$$

In particular

$$1 \leq \frac{\operatorname{tr}(PA)}{\Delta_P(A)} \leq \exp\left[\operatorname{tr}(PA)\operatorname{tr}\left(PA^{-1}\right) - 1\right] \tag{1.14}$$

and

$$1 \leq \frac{\Delta_P(A)}{\left[\operatorname{tr}\left(PA^{-1}\right)\right]^{-1}} \leq \exp\left[\operatorname{tr}\left(PA^{-1}\right)\operatorname{tr}(PA) - 1\right]. \tag{1.15}$$

The first inequalities in (1.14) and 1.15) are best possible from (1.13).

Motivated by the above results, in this paper we show among others that, if $P_i \geq 0$ with $P_i \in \mathcal{B}_1(H)$ and $\operatorname{tr}(P_i) = 1$ for $i \in \{1, ..., n\}$, $p_i \geq 0$ with $\sum_{i=1}^n p_i = 1$

and $A_i > 0$ for $i \in \{1, ..., n\}$, then for all $a > 0$ we have the lower and upper bounds

$$a \exp\left(1 - a \sum_{i=1}^{n} p_i \operatorname{tr}\left(P_i A_i^{-1}\right)\right) \leq \prod_{i=1}^{n} \left[\Delta_{Pi}(A_i)\right]^{p_i} \leq a \exp\left(a^{-1} \sum_{i=1}^{n} p_i \operatorname{tr}(P_i A_i) - 1\right).$$

2. Main Results

We start to the following double trace inequality that is of interest in itself as well:

Lemma 1. *Assume that f is differentiable convex on the interior $\mathring{I}$ of the interval I and the derivative f' is continuous on $\mathring{I}$. Let $Q_i \geq 0$ with $Q_i \in \mathcal{B}_1(H)$ for $i \in \{1, ..., n\}$ and $\sum_{i=1}^{n} \operatorname{tr}(Q_i) > 0$, then for all B_i with the spectra $\operatorname{Sp}(B_i) \subset \mathring{I}$ for $i \in \{1, ..., n\}$ and $a \in \mathring{I}$ we have the double inequality*

$$\begin{aligned} &\frac{\sum_{i=1}^{n} \operatorname{tr}(Q_i f'(B_i) B_i)}{\sum_{i=1}^{n} \operatorname{tr}(Q_i)} - a \frac{\sum_{i=1}^{n} \operatorname{tr}(Q_i f'(B_i))}{\sum_{i=1}^{n} \operatorname{tr}(Q_i)} + f(a) \\ &\geq \frac{\sum_{i=1}^{n} \operatorname{tr}(Q_i f(B_i))}{\sum_{i=1}^{n} \operatorname{tr}(Q_i)} \\ &\geq f'(a)\left(\frac{\sum_{i=1}^{n} \operatorname{tr}(Q_i B_i)}{\sum_{i=1}^{n} \operatorname{tr}(Q_i)} - a\right) + f(a). \end{aligned} \tag{2.1}$$

Proof. We use the gradient inequality

$$f'(t)(t-a) + f(a) \geq f(t) \geq f'(a)(t-a) + f(a)$$

that holds for all $t, a \in \mathring{I}$.

Using the continuous functional calculus for the selfadjoint operators with spectra in $\mathring{I}$, we get

$$f'(B_i)(B_i - aI) + f(a) I \geq f(B_i) \geq f'(a)(B_i - a) + f(a)$$

for all $i \in \{1, ..., n\}$.

If we multiply this inequality both sides with $Q_i^{1/2}$, then we get

$$\begin{aligned} &Q_i^{1/2} f'(B_i) B_i Q_i^{1/2} - a Q_i^{1/2} f'(B_i) Q_i^{1/2} + f(a) Q_i \\ &\geq Q_i^{1/2} f(B_i) Q_i^{1/2} \\ &\geq f'(a) Q_i^{1/2} B_i Q_i^{1/2} - a f'(a) Q_i + f(a) Q_i \end{aligned} \tag{2.2}$$

for all $i \in \{1, ..., n\}$.

If we take the trace in (2.2), then we get

$$\begin{aligned} &\operatorname{tr}\left(Q_i^{1/2} f'(B_i) B_i Q_i^{1/2}\right) - a \operatorname{tr}\left(Q_i^{1/2} f'(B_i) Q_i^{1/2}\right) + f(a) \operatorname{tr}(Q_i) \\ &\geq \operatorname{tr}\left(Q_i^{1/2} f(B_i) Q_i^{1/2}\right) \\ &\geq f'(a) \operatorname{tr}\left(Q_i^{1/2} B_i Q_i^{1/2}\right) - a f'(a) \operatorname{tr}(Q_i) + f(a) \operatorname{tr}(Q_i), \end{aligned}$$

namely, by the properties of the trace

$$\begin{aligned}&\operatorname{tr}\left(Q_i f'\left(B_i\right) B_i\right)-a \operatorname{tr}\left(Q_i f'\left(B_i\right)\right)+f\left(a\right) \operatorname{tr}\left(Q_i\right)\\&\geq \operatorname{tr}\left(Q_i f\left(B_i\right)\right) \geq f'\left(a\right) \operatorname{tr}\left(Q_i B_i\right)-a f'\left(a\right) \operatorname{tr}\left(Q_i\right)+f\left(a\right) \operatorname{tr}\left(Q_i\right),\end{aligned}$$

for all $i \in \{1, ..., n\}$.

If we sum over i from 1 to n, then we get

$$\begin{aligned}&\sum_{i=1}^{n} \operatorname{tr}\left(Q_i f'\left(B_i\right) B_i\right)-a \sum_{i=1}^{n} \operatorname{tr}\left(Q_i f'\left(B_i\right)\right)+f\left(a\right) \sum_{i=1}^{n} \operatorname{tr}\left(Q_i\right)\\&\geq \sum_{i=1}^{n} \operatorname{tr}\left(Q_i f\left(B_i\right)\right)\\&\geq f'\left(a\right) \sum_{i=1}^{n} \operatorname{tr}\left(Q_i B_i\right)-a f'\left(a\right) \sum_{i=1}^{n} \operatorname{tr}\left(Q_i\right)+f\left(a\right) \sum_{i=1}^{n} \operatorname{tr}\left(Q_i\right)\end{aligned}$$

and by dividing with $\sum_{i=1}^{n} \operatorname{tr}\left(Q_i\right)>0$, we get (2.1). □

Corollary 1. *With the assumptions of Lemma 1 we have*

$$\begin{aligned}&\frac{\sum_{i=1}^{n} \operatorname{tr}\left(Q_i f'\left(B_i\right) B_i\right)}{\sum_{i=1}^{n} \operatorname{tr}\left(Q_i\right)}-\frac{\sum_{i=1}^{n} \operatorname{tr}\left(Q_i B_i\right)}{\sum_{i=1}^{n} \operatorname{tr}\left(Q_i\right)} \frac{\sum_{i=1}^{n} \operatorname{tr}\left(Q_i f'\left(B_i\right)\right)}{\sum_{i=1}^{n} \operatorname{tr}\left(Q_i\right)}\\&\geq \frac{\sum_{i=1}^{n} \operatorname{tr}\left(Q_i f\left(B_i\right)\right)}{\sum_{i=1}^{n} \operatorname{tr}\left(Q_i\right)}-f\left(\frac{\sum_{i=1}^{n} \operatorname{tr}\left(Q_i B_i\right)}{\sum_{i=1}^{n} \operatorname{tr}\left(Q_i\right)}\right) \geq 0.\end{aligned} \tag{2.3}$$

Proof. Since $\operatorname{Sp}\left(B_i\right) \subset \mathring{I}$ for $i \in \{1, ..., n\}$, then there exists $m<M$ such that $\operatorname{Sp}\left(B_i\right) \subseteq [m, M] \subset \mathring{I}$ for $i \in \{1, ..., n\}$. Therefore

$$mI \leq B_i \leq MI, \text{ for } i \in \{1, ..., n\}.$$

If we multiply this inequality both sides with $Q_i^{1/2}$, then we get

$$mQ_i \leq Q_i^{1/2} B_i Q_i^{1/2} \leq MQ_i, \text{ for } i \in \{1, ..., n\}.$$

By taking the trace and summing, we get

$$m \leq \frac{\sum_{i=1}^{n} \operatorname{tr}\left(Q_i B_i\right)}{\sum_{i=1}^{n} \operatorname{tr}\left(Q_i\right)} \leq M.$$

Then by taking

$$a=\frac{\sum_{i=1}^{n} \operatorname{tr}\left(Q_i B_i\right)}{\sum_{i=1}^{n} \operatorname{tr}\left(Q_i\right)}$$

in (2.1) we get (2.3). □

Remark 1. *The case of one operator is as follows:*

$$\frac{\operatorname{tr}\left(Q f'\left(B\right) B\right)}{\operatorname{tr}\left(Q\right)}-\frac{\operatorname{tr}\left(Q B\right)}{\operatorname{tr}\left(Q\right)} \frac{\operatorname{tr}\left(Q f'\left(B\right)\right)}{\operatorname{tr}\left(Q\right)} \geq \frac{\operatorname{tr}\left(Q f\left(B\right)\right)}{\operatorname{tr}\left(Q\right)}-f\left(\frac{\operatorname{tr}\left(Q B\right)}{\operatorname{tr}\left(Q\right)}\right) \geq 0, \tag{2.4}$$

where $Q \geq 0$ with $Q \in \mathcal{B}_1\left(H\right)$ and $\operatorname{tr}\left(Q\right)>0$ and B is selfadjoint with $\operatorname{Sp}\left(B\right) \subset \mathring{I}$. This result was obtained in a different way in [1].

Corollary 2. *With the assumptions of Lemma 1 and if* $\sum_{i=1}^{n}\operatorname{tr}\left(Q_i f'\left(B_i\right)\right)\neq 0$ *with*

$$\frac{\sum_{i=1}^{n}\operatorname{tr}\left(Q_i f'\left(B_i\right)B_i\right)}{\sum_{i=1}^{n}\operatorname{tr}\left(Q_i f'\left(B_i\right)\right)}\in \mathring{I} \text{ for } i\in\{1,...,n\},$$

then

$$\begin{aligned}(2.5)\qquad 0 &\leq f\left(\frac{\sum_{i=1}^{n}\operatorname{tr}\left(Q_i f'\left(B_i\right)B_i\right)}{\sum_{i=1}^{n}\operatorname{tr}\left(Q_i f'\left(B_i\right)\right)}\right)-\frac{\sum_{i=1}^{n}\operatorname{tr}\left(Q_i f\left(B_i\right)\right)}{\sum_{i=1}^{n}\operatorname{tr}\left(Q_i\right)}\\ &\leq f'\left(\frac{\sum_{i=1}^{n}\operatorname{tr}\left(Q_i f'\left(B_i\right)B_i\right)}{\sum_{i=1}^{n}\operatorname{tr}\left(Q_i f'\left(B_i\right)\right)}\right)\\ &\quad\times\left(\frac{\sum_{i=1}^{n}\operatorname{tr}\left(Q_i f'\left(B_i\right)B_i\right)}{\sum_{i=1}^{n}\operatorname{tr}\left(Q_i f'\left(B_i\right)\right)}-\frac{\sum_{i=1}^{n}\operatorname{tr}\left(Q_iB_i\right)}{\sum_{i=1}^{n}\operatorname{tr}\left(Q_i\right)}\right).\end{aligned}$$

Proof. From (2.1) we derive for

$$a=\frac{\sum_{i=1}^{n}\operatorname{tr}\left(Q_i f'\left(B_i\right)B_i\right)}{\sum_{i=1}^{n}\operatorname{tr}\left(Q_i f'\left(B_i\right)\right)}$$

that

$$\begin{aligned}&f\left(\frac{\sum_{i=1}^{n}\operatorname{tr}\left(Q_i f'\left(B_i\right)B_i\right)}{\sum_{i=1}^{n}\operatorname{tr}\left(Q_i f'\left(B_i\right)\right)}\right)\\ &\geq\frac{\sum_{i=1}^{n}\operatorname{tr}\left(Q_i f\left(B_i\right)\right)}{\sum_{i=1}^{n}\operatorname{tr}\left(Q_i\right)}\\ &\geq f'\left(\frac{\sum_{i=1}^{n}\operatorname{tr}\left(Q_i f'\left(B_i\right)B_i\right)}{\sum_{i=1}^{n}\operatorname{tr}\left(Q_i f'\left(B_i\right)\right)}\right)\\ &\times\left(\frac{\sum_{i=1}^{n}\operatorname{tr}\left(Q_iB_i\right)}{\sum_{i=1}^{n}\operatorname{tr}\left(Q_i\right)}-\frac{\sum_{i=1}^{n}\operatorname{tr}\left(Q_i f'\left(B_i\right)B_i\right)}{\sum_{i=1}^{n}\operatorname{tr}\left(Q_i f'\left(B_i\right)\right)}\right)\\ &+f\left(\frac{\sum_{i=1}^{n}\operatorname{tr}\left(Q_i f'\left(B_i\right)B_i\right)}{\sum_{i=1}^{n}\operatorname{tr}\left(Q_i f'\left(B_i\right)\right)}\right),\end{aligned}$$

namely

$$\begin{aligned}0 &\geq\frac{\sum_{i=1}^{n}\operatorname{tr}\left(Q_i f\left(B_i\right)\right)}{\sum_{i=1}^{n}\operatorname{tr}\left(Q_i\right)}-f\left(\frac{\sum_{i=1}^{n}\operatorname{tr}\left(Q_i f'\left(B_i\right)B_i\right)}{\sum_{i=1}^{n}\operatorname{tr}\left(Q_i f'\left(B_i\right)\right)}\right)\\ &\geq f'\left(\frac{\sum_{i=1}^{n}\operatorname{tr}\left(Q_i f'\left(B_i\right)B_i\right)}{\sum_{i=1}^{n}\operatorname{tr}\left(Q_i f'\left(B_i\right)\right)}\right)\\ &\quad\times\left(\frac{\sum_{i=1}^{n}\operatorname{tr}\left(Q_iB_i\right)}{\sum_{i=1}^{n}\operatorname{tr}\left(Q_i\right)}-\frac{\sum_{i=1}^{n}\operatorname{tr}\left(Q_i f'\left(B_i\right)B_i\right)}{\sum_{i=1}^{n}\operatorname{tr}\left(Q_i f'\left(B_i\right)\right)}\right),\end{aligned}$$

which is equivalent to (2.5). □

Remark 2. *The case of one operator is as follows: if* $\operatorname{tr}\left(Qf'\left(B\right)\right)\neq 0$ *with* $\frac{\operatorname{tr}\left(Qf'(B)B\right)}{\operatorname{tr}\left(Qf'(B)\right)}\in\mathring{I}$, *then*

$$\begin{aligned}(2.6)\qquad 0 &\leq f\left(\frac{\operatorname{tr}\left(Qf'\left(B\right)B\right)}{\operatorname{tr}\left(Qf'\left(B\right)\right)}\right)-\frac{\operatorname{tr}\left(Qf\left(B\right)\right)}{\operatorname{tr}\left(Q\right)}\\ &\leq f'\left(\frac{\operatorname{tr}\left(Qf'\left(B\right)B\right)}{\operatorname{tr}\left(Qf'\left(B\right)\right)}\right)\left(\frac{\operatorname{tr}\left(Qf'\left(B\right)B\right)}{\operatorname{tr}\left(Qf'\left(B\right)\right)}-\frac{\operatorname{tr}\left(QB\right)}{\operatorname{tr}\left(Q\right)}\right),\end{aligned}$$

which was obtained in a different way in [2].

We have the following main result:

Theorem 6. *if* $P_i \geq 0$ *with* $P_i \in \mathcal{B}_1(H)$ *and* $\operatorname{tr}(P_i) = 1$ *for* $i \in \{1, ..., n\}$, $p_i \geq 0$ *with* $\sum_{i=1}^{n} p_i = 1$ *and* $A_i > 0$ *for* $i \in \{1, ..., n\}$, *then for all* $a > 0$ *we have the lower and upper bounds*

$$a \exp\left(1 - a\sum_{i=1}^{n} p_i \operatorname{tr}\left(P_i A_i^{-1}\right)\right) \leq \prod_{i=1}^{n} \left[\Delta_{P i}(A_i)\right]^{p_i} \leq a \exp\left(a^{-1}\sum_{i=1}^{n} p_i \operatorname{tr}(P_i A_i) - 1\right). \tag{2.7}$$

Proof. If we write the inequality (2.1) for the convex function $f(t) = -\ln t$, $t > 0$, $Q_i = p_i P_i$ and $B_i = A_i$ for $i \in \{1, ..., n\}$, we get

$$\begin{aligned}
& -\frac{\sum_{i=1}^{n} p_i \operatorname{tr}(P_i)}{\sum_{i=1}^{n} p_i \operatorname{tr}(P_i)} + a\frac{\sum_{i=1}^{n} p_i \operatorname{tr}\left(A_i^{-1}\right)}{\sum_{i=1}^{n} p_i \operatorname{tr}(P_i)} - \ln a \\
& \geq -\frac{\sum_{i=1}^{n} p_i \operatorname{tr}(P_i \ln(A_i))}{\sum_{i=1}^{n} p_i \operatorname{tr}(P_i)} \\
& \geq -a^{-1}\left(\frac{\sum_{i=1}^{n} p_i \operatorname{tr}(P_i A_i)}{\sum_{i=1}^{n} p_i \operatorname{tr}(P_i)} - a\right) - \ln a,
\end{aligned}$$

namely

$$1 + \ln a - a\sum_{i=1}^{n} p_i \operatorname{tr}\left(P_i A_i^{-1}\right) \leq \sum_{i=1}^{n} p_i \operatorname{tr}(P_i \ln A_i) \leq a^{-1}\sum_{i=1}^{n} p_i \operatorname{tr}(P_i A_i) + \ln a - 1, \tag{2.8}$$

for all $a > 0$.

If we take the exponential in (2.8), then we get

$$\begin{aligned}
& \exp\left(1 + \ln a - a\sum_{i=1}^{n} p_i \operatorname{tr}\left(P_i A_i^{-1}\right)\right) \\
& \leq \exp\left(\sum_{i=1}^{n} p_i \operatorname{tr}(P_i \ln A_i)\right) \\
& \leq \exp\left(a^{-1}\sum_{i=1}^{n} p_i \operatorname{tr}(P_i A_i) + \ln a - 1\right),
\end{aligned}$$

namely

$$\begin{aligned}
a \exp\left(1 - a\sum_{i=1}^{n} p_i \operatorname{tr}\left(P_i A_i^{-1}\right)\right) & \leq \prod_{i=1}^{n} \left(\exp\left[\operatorname{tr}(P_i \ln A_i)\right]\right)^{p_i} \\
& \leq a \exp\left(a^{-1}\sum_{i=1}^{n} p_i \operatorname{tr}(P_i A_i) - 1\right)
\end{aligned}$$

and the inequality (2.7) is proved. $\square$

Corollary 3. *With the assumptions in Theorem 6 we have*

$$\exp\left(1-\sum_{i=1}^{n}p_i\operatorname{tr}(P_iA_i)\sum_{i=1}^{n}p_i\operatorname{tr}\left(P_iA_i^{-1}\right)\right)\le\frac{\prod_{i=1}^{n}\left[\Delta_{Pi}(A_i)\right]^{p_i}}{\sum_{i=1}^{n}p_i\operatorname{tr}(P_iA_i)}\le 1. \tag{2.9}$$

The second inequality in (2.9) is best possible from the second inequality in (2.7).

Proof. The inequality (2.9) follows by (2.7) on taking $a=\sum_{i=1}^{n}p_i\operatorname{tr}(P_iA_i)$.

Now, consider the function $f(t)=t\exp\left[t^{-1}\sum_{i=1}^{n}p_i\operatorname{tr}(P_iA_i)-1\right]$, $t>0$, then

$$\begin{aligned}f'(t)&=\exp\left[t^{-1}\sum_{i=1}^{n}p_i\operatorname{tr}(P_iA_i)-1\right]+t\exp\left[t^{-1}\sum_{i=1}^{n}p_i\operatorname{tr}(P_iA_i)-1\right]\\&\quad\times\left(-\frac{\sum_{i=1}^{n}p_i\operatorname{tr}(P_iA_i)}{t^2}\right)\\&=\exp\left[t^{-1}\sum_{i=1}^{n}p_i\operatorname{tr}(P_iA_i)-1\right]\left(1-\frac{\sum_{i=1}^{n}p_i\operatorname{tr}(P_iA_i)}{t}\right).\end{aligned}$$

We have that $f'(t_0)=0$ for $t_0=\sum_{i=1}^{n}p_i\operatorname{tr}(P_iA_i)$ which shows that f is strictly decreasing on $(0,\sum_{i=1}^{n}p_i\operatorname{tr}(P_iA_i))$ and strictly increasing on $(\sum_{i=1}^{n}p_i\operatorname{tr}(P_iA_i),\infty)$. Therefore

$$\inf_{t\in(0,\infty)}f(t)=f\left(\sum_{i=1}^{n}p_i\operatorname{tr}(P_iA_i)\right)=\sum_{i=1}^{n}p_i\operatorname{tr}(P_iA_i),$$

and the corollary is thus proved. □

Corollary 4. *With the assumptions in Theorem 6 we have*

$$\begin{aligned}1&\le\frac{\prod_{i=1}^{n}\left[\Delta_{Pi}(A_i)\right]^{p_i}}{\left(\sum_{i=1}^{n}p_i\operatorname{tr}\left(P_iA_i^{-1}\right)\right)^{-1}}\\&\le\exp\left(\sum_{i=1}^{n}p_i\operatorname{tr}\left(P_iA_i^{-1}\right)\sum_{i=1}^{n}p_i\operatorname{tr}(P_iA_i)-1\right).\end{aligned}\tag{2.10}$$

The first inequality in (2.10) is best possible from the first inequality in (2.7).

Proof. The inequality (2.10) follows by (2.7) for $a=\left[\operatorname{tr}\left(PA^{-1}\right)\right]^{-1}$.

we take the function $g(t)=t\exp\left[1-t\sum_{i=1}^{n}p_i\operatorname{tr}\left(A_i^{-1}\right)\right]$, $t>0$, then

$$\begin{aligned}g'(t)&=\exp\left[1-t\sum_{i=1}^{n}p_i\operatorname{tr}\left(P_iA_i^{-1}\right)\right]\\&\quad-t\sum_{i=1}^{n}p_i\operatorname{tr}\left(A_i^{-1}\right)\exp\left[1-t\sum_{i=1}^{n}p_i\operatorname{tr}\left(P_iA_i^{-1}\right)\right]\\&=\exp\left[1-t\sum_{i=1}^{n}p_i\operatorname{tr}\left(P_iA_i^{-1}\right)\right]\left(1-t\sum_{i=1}^{n}p_i\operatorname{tr}\left(P_iA_i^{-1}\right)\right),\end{aligned}$$

which shows that g is strictly increasing on $\left(0,\left[\sum_{i=1}^{n}p_{i}\operatorname{tr}\left(P_{i}A_{i}^{-1}\right)\right]^{-1}\right)$ and strictly decreasing on $\left(\left[\sum_{i=1}^{n}p_{i}\operatorname{tr}\left(P_{i}A_{i}^{-1}\right)\right]^{-1},\infty\right)$, therefore

$$\sup_{t\in(0,\infty)}g\left(t\right)=g\left(\left[\sum_{i=1}^{n}p_{i}\operatorname{tr}\left(P_{i}A_{i}^{-1}\right)\right]^{-1}\right)=\left[\sum_{i=1}^{n}p_{i}\operatorname{tr}\left(P_{i}A_{i}^{-1}\right)\right]^{-1},$$

which shows that the first inequality in (2.10) is best possible from the first inequality in (2.7). □

Remark 3. *The case of one operator gives Theorem 5 from the introduction.*

3. Related Results

When more information for the operators A_i, $i\in\{1,...,n\}$ is available, then we have some simple bounds as follows:

Theorem 7. *Assume that $P_i\geq 0$ with $P_i\in\mathcal{B}_1(H)$ and $\operatorname{tr}(P_i)=1$ for $i\in\{1,...,n\}$. If $p_i\geq 0$ with $\sum_{i=1}^{n}p_i=1$ and A_i with the property that $0<mI\leq A_i\leq MI$, $i\in\{1,...,n\}$ for $i\in\{1,...,n\}$, then*

$$1\leq\frac{\sum_{i=1}^{n}p_{i}\operatorname{tr}\left(P_{i}A_{i}\right)}{\prod_{i=1}^{n}\left[\Delta_{P i}\left(A_{i}\right)\right]^{p_{i}}}\leq\exp\left(\sum_{i=1}^{n}p_{i}\operatorname{tr}\left(P_{i}A_{i}^{-1}\right)\sum_{i=1}^{n}p_{i}\operatorname{tr}\left(P_{i}A_{i}\right)-1\right)\leq\exp\left[\frac{\left(M-m\right)^{2}}{4mM}\right]\tag{3.1}$$

and

$$1\leq\frac{\prod_{i=1}^{n}\left[\Delta_{P i}\left(A_{i}\right)\right]^{p_{i}}}{\left(\sum_{i=1}^{n}p_{i}\operatorname{tr}\left(P_{i}A_{i}^{-1}\right)\right)^{-1}}\leq\exp\left(\sum_{i=1}^{n}p_{i}\operatorname{tr}\left(P_{i}A_{i}^{-1}\right)\sum_{i=1}^{n}p_{i}\operatorname{tr}\left(P_{i}A_{i}\right)-1\right)\leq\exp\left[\frac{\left(M-m\right)^{2}}{4mM}\right].\tag{3.2}$$

Proof. If $t\in[m,M]\subset(0,\infty)$, then $(M-t)\left(m^{-1}-t^{-1}\right)\geq 0$. Since $0<mI\leq A_i\leq MI$, $i\in\{1,...,n\}$ hence by using the functional calculus for selfadjoint operators we get

$$\left(MI-A_{i}\right)\left(m^{-1}I-A_{i}^{-1}\right)\geq 0$$

for all $i\in\{1,...,n\}$, which is equivalent to

$$\left(M+m\right)I\geq MmA_{i}^{-1}+A_{i}\tag{3.3}$$

for all $i\in\{1,...,n\}$.

If we multiply (3.3) both sides by $P_i^{1/2}$ we get

$$\left(M+m\right)P_{i}\geq MmP_{i}^{1/2}A_{i}^{-1}P_{i}^{1/2}+P_{i}^{1/2}A_{i}P_{i}^{1/2}$$

for all $i\in\{1,...,n\}$.

If we take the trace and use its properties, we get

$$M+m\geq Mm\operatorname{tr}\left(P_{i}A_{i}^{-1}\right)+\operatorname{tr}\left(P_{i}A_{i}\right)$$

for all $i \in \{1, ..., n\}$.

If we multiply by $p_i \geq 0$ and summing over i from 1 to n, we get

$$M + m \geq Mm \sum_{i=1}^{n} p_i \operatorname{tr}\left(P_i A_i^{-1}\right) + \sum_{i=1}^{n} p_i \operatorname{tr}\left(P_i A_i\right). \tag{3.4}$$

By making use of the arithmetic mean-geometric mean inequality we also have

$$\begin{aligned} &Mm \sum_{i=1}^{n} p_i \operatorname{tr}\left(P_i A_i^{-1}\right) + \sum_{i=1}^{n} p_i \operatorname{tr}\left(P_i A_i\right) \\ &\geq 2\sqrt{Mm \sum_{i=1}^{n} p_i \operatorname{tr}\left(P_i A_i^{-1}\right) \sum_{i=1}^{n} p_i \operatorname{tr}\left(P_i A_i\right)}, \end{aligned}$$

which implies that

$$M + m \geq 2\sqrt{Mm \sum_{i=1}^{n} p_i \operatorname{tr}\left(P_i A_i^{-1}\right) \sum_{i=1}^{n} p_i \operatorname{tr}\left(P_i A_i\right)}.$$

This is equivalent to

$$\sum_{i=1}^{n} p_i \operatorname{tr}\left(P_i A_i^{-1}\right) \sum_{i=1}^{n} p_i \operatorname{tr}\left(P_i A_i\right) \leq \frac{(M+m)^2}{4mM}$$

or to,

$$\sum_{i=1}^{n} p_i \operatorname{tr}\left(P_i A_i^{-1}\right) \sum_{i=1}^{n} p_i \operatorname{tr}\left(P_i A_i\right) - 1 \leq \frac{(M-m)^2}{4mM}.$$

Now, by utilizing Corollaries 3 and 4 we deduce the desired results. □

Theorem 8. *With the assumptions in Theorem 7 we have*

$$\begin{aligned} 1 &\leq \frac{\sum_{i=1}^{n} p_i \operatorname{tr}\left(P_i A_i\right)}{\prod_{i=1}^{n} \left[\Delta_{P_i}\left(A_i\right)\right]^{p_i}} \leq \exp\left(\sum_{i=1}^{n} p_i \operatorname{tr}\left(P_i A_i^{-1}\right) \sum_{i=1}^{n} p_i \operatorname{tr}\left(P_i A_i\right) - 1\right) \\ &\leq \exp\left[\frac{\left(\sqrt{M}-\sqrt{m}\right)^2}{mM} \sum_{i=1}^{n} p_i \operatorname{tr}\left(P_i A_i\right)\right] \leq \exp\left[\frac{\left(\sqrt{M}-\sqrt{m}\right)^2}{m}\right] \end{aligned} \tag{3.5}$$

and

$$\begin{aligned} 1 &\leq \frac{\prod_{i=1}^{n} \left[\Delta_{P_i}\left(A_i\right)\right]^{p_i}}{\left(\sum_{i=1}^{n} p_i \operatorname{tr}\left(P_i A_i^{-1}\right)\right)^{-1}} \leq \exp\left(\sum_{i=1}^{n} p_i \operatorname{tr}\left(P_i A_i^{-1}\right) \sum_{i=1}^{n} p_i \operatorname{tr}\left(P_i A_i\right) - 1\right) \\ &\leq \exp\left[\frac{\left(\sqrt{M}-\sqrt{m}\right)^2}{mM} \sum_{i=1}^{n} p_i \operatorname{tr}\left(P_i A_i\right)\right] \leq \exp\left[\frac{\left(\sqrt{M}-\sqrt{m}\right)^2}{m}\right]. \end{aligned} \tag{3.6}$$

Proof. From (3.4) we get

$$\sum_{i=1}^{n} p_i \operatorname{tr}\left(P_i A_i^{-1}\right) \leq \frac{1}{m} + \frac{1}{M} - \frac{1}{mM} \sum_{i=1}^{n} p_i \operatorname{tr}\left(P_i A_i\right),$$

which implies that

$$\begin{aligned}
&\sum_{i=1}^{n} p_i \operatorname{tr}\left(P_i A_i^{-1}\right) - \left(\sum_{i=1}^{n} p_i \operatorname{tr}\left(P_i A_i\right)\right)^{-1} \\
&\leq \frac{1}{m} + \frac{1}{M} - \frac{1}{mM} \sum_{i=1}^{n} p_i \operatorname{tr}\left(P_i A_i\right) - \left(\sum_{i=1}^{n} p_i \operatorname{tr}\left(P_i A_i\right)\right)^{-1} \\
&= \left(\frac{1}{\sqrt{m}} - \frac{1}{\sqrt{M}}\right)^2 \\
&- \left(\frac{1}{\sqrt{mM}} \left(\sum_{i=1}^{n} p_i \operatorname{tr}\left(P_i A_i\right)\right)^{1/2} - \left(\sum_{i=1}^{n} p_i \operatorname{tr}\left(P_i A_i\right)\right)^{-1/2}\right)^2 \\
&\leq \left(\frac{1}{\sqrt{m}} - \frac{1}{\sqrt{M}}\right)^2 .
\end{aligned}$$

If we multiply by $\sum_{i=1}^{n} p_i \operatorname{tr}\left(P_i A_i\right) > 0$ we get

$$\begin{aligned}
&\sum_{i=1}^{n} p_i \operatorname{tr}\left(P_i A_i^{-1}\right) \sum_{i=1}^{n} p_i \operatorname{tr}\left(P_i A_i\right) - 1 \\
&\leq \left(\frac{1}{\sqrt{m}} - \frac{1}{\sqrt{M}}\right)^2 \sum_{i=1}^{n} p_i \operatorname{tr}\left(P_i A_i\right) \leq M \left(\frac{1}{\sqrt{m}} - \frac{1}{\sqrt{M}}\right)^2 \sum_{i=1}^{n} p_i \operatorname{tr}\left(P_i\right) \\
&= M \left(\frac{1}{\sqrt{m}} - \frac{1}{\sqrt{M}}\right)^2 = \frac{\left(\sqrt{M} - \sqrt{m}\right)^2}{m},
\end{aligned}$$

which proves the desired results. □

Remark 4. *If* $0 < mI \leq A \leq MI$ *and* $P \geq 0$ *with* $P \in \mathcal{B}_1(H)$ *and* $\operatorname{tr}(P) = 1$, *then* $i \in \{1, ..., n\}$, *then, see also* [4],

$$1 \leq \frac{\operatorname{tr}(PA)}{\Delta_P(A)} \leq \exp\left[\operatorname{tr}(PA) \operatorname{tr}\left(PA^{-1}\right) - 1\right] \leq \exp\left[\frac{(M-m)^2}{4mM}\right] \tag{3.7}$$

and

$$1 \leq \frac{\Delta_P(A)}{\left[\operatorname{tr}\left(PA^{-1}\right)\right]^{-1}} \leq \exp\left[\operatorname{tr}\left(PA^{-1}\right) \operatorname{tr}(PA) - 1\right] \leq \exp\left[\frac{(M-m)^2}{4mM}\right]. \tag{3.8}$$

Also

$$\begin{aligned}
1 &\leq \frac{\operatorname{tr}(PA)}{\Delta_P(A)} \leq \exp\left[\operatorname{tr}(PA) \operatorname{tr}\left(PA^{-1}\right) - 1\right] \\
&\leq \exp\left[\frac{\left(\sqrt{M} - \sqrt{m}\right)^2}{mM} \operatorname{tr}(PA)\right] \leq \exp\left[\frac{\left(\sqrt{M} - \sqrt{m}\right)^2}{m}\right]
\end{aligned} \tag{3.9}$$

and

$$1 \leq \frac{\Delta_P(A)}{\left[\operatorname{tr}\left(PA^{-1}\right)\right]^{-1}} \leq \exp\left[\operatorname{tr}\left(PA^{-1}\right)\operatorname{tr}(PA) - 1\right] \tag{3.10}$$
$$\leq \exp\left[\frac{\left(\sqrt{M}-\sqrt{m}\right)^2}{mM}\operatorname{tr}(PA)\right] \leq \exp\left[\frac{\left(\sqrt{M}-\sqrt{m}\right)^2}{m}\right].$$

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