BOUNDS FOR THE TRACE CLASS P-DETERMINANT OF POSITIVE OPERATORS IN HILBERT SPACES VIA JENSEN'S TYPE INEQUALITIES FOR TWICE DIFFERENTIABLE FUNCTIONS

SILVESTRU SEVER DRAGOMIR[1,2]

Abstract. Let H be a complex Hilbert space. For a given operator $P \geq 0$ with $P \in \mathcal{B}_1(H)$, the trace class associated to $\mathcal{B}(H)$ and $\operatorname{tr}(P) = 1$, we define the P-*determinant* of the positive invertible operator A by

$$\Delta_P(A) := \exp \operatorname{tr}(P \ln A).$$

In this paper we show among others that, if $P_i \geq 0$ with $P_i \in \mathcal{B}_1(H)$ and $\operatorname{tr}(P_i) = 1$ for $i \in \{1, ..., n\}$, $p_i \geq 0$ with $\sum_{i=1}^n p_i = 1$, $p \in (-\infty, 0) \cup (1, \infty)$ and that A_j are operators such that $0 < m \leq A_j \leq M$, for $i \in \{1, ..., n\}$, then

$$\begin{aligned} &\exp\left\{\gamma_p\left[\sum_{i=1}^n p_i \operatorname{tr}\left(P_i A_i^p\right) - \left(\sum_{i=1}^n p_i \operatorname{tr}(P_i A_i)\right)^p\right]\right\} \\ &\leq \frac{\sum_{i=1}^n p_i \operatorname{tr}(P_i A_i)}{\prod_{i=1}^n \left[\Delta_{P_i}(A_i)\right]^{p_i}} \\ &\leq \exp\left\{\Gamma_p\left[\sum_{i=1}^n p_i \operatorname{tr}\left(P_i A_i^p\right) - \sum_{i=1}^n p_i \operatorname{tr}(P_i A_i)\Big)^p\right]\right\}, \end{aligned}$$

where

$$\gamma_p := \begin{cases} \frac{M^{-p}}{p(p-1)}, \ p \in (1, \infty), \\ \\ \frac{m^{-p}}{p(p-1)}, \ p \in (-\infty, 0) \end{cases}, \quad \Gamma_p := \begin{cases} \frac{m^{-p}}{p(p-1)}, \ p \in (1, \infty), \\ \\ \frac{M^{-p}}{p(p-1)}, \ p \in (-\infty, 0). \end{cases}$$

1. Introduction

In 1952, in the paper [5], B. Fuglede and R. V. Kadison introduced the determinant of a (invertible) operator and established its fundamental properties. The notion generalizes the usual determinant and can be considered for any operator in a finite von Neumann algebra (M, τ) with a faithful normal trace.

Let $T \in M$ be normal and $|T| := (T^*T)^{1/2}$ its modulus. By the spectral theorem one can represent T as an integral

$$T = \int_{\operatorname{Sp}(T)} \lambda dE(\lambda),$$

where $E(\lambda)$ is a projection valued measure and $\operatorname{Sp}(T)$ is the spectrum of T. The measure $\mu_T := \tau \circ E$ becomes a probability measure on the complex plane and has the support in the spectrum $\operatorname{Sp}(T)$.

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For any $T \in M$ the Fuglede-Kadison determinant (*FK-determinant*) is defined by

$$\Delta_{FK}(T) := \exp\left(\int_0^\infty \ln t d\mu_{|T|}\right).$$

If T is invertible, then

$$\Delta_{FK}(T) := \exp(\tau(\ln(|T|))),$$

where $\ln(|T|)$ is defined by the use of functional calculus.

Let $B(H)$ be the space of all bounded linear operators on a Hilbert space H, and I stands for the identity operator on H. An operator A in $B(H)$ is said to be positive (in symbol: $A \geq 0$) if $\langle Ax, x\rangle \geq 0$ for all $x \in H$. In particular, $A > 0$ means that A is positive and invertible. For a pair A, B of selfadjoint operators the order relation $A \geq B$ means as usual that $A - B$ is positive.

In 1998, Fujii et al. [6], [7], introduced the *normalized determinant* $\Delta_x(A)$ for positive invertible operators A on a Hilbert space H and a fixed unit vector $x \in H$, namely $\|x\| = 1$, defined by

$$\Delta_x(A) := \exp\langle \ln Ax, x\rangle$$

and discussed it as a continuous geometric mean and observed some inequalities around the determinant from this point of view. For some recent results, see [10].

We need now some preparations for trace of operators in Hilbert spaces.

Let $(H, \langle\cdot,\cdot\rangle)$ be a complex Hilbert space and $\{e_i\}_{i\in I}$ an *orthonormal basis* of H. We say that $A \in \mathcal{B}(H)$ is a *Hilbert-Schmidt operator* if

$$\sum_{i\in I} \|Ae_i\|^2 < \infty. \tag{1.1}$$

It is well know that, if $\{e_i\}_{i\in I}$ and $\{f_j\}_{j\in J}$ are orthonormal bases for H and $A \in \mathcal{B}(H)$ then

$$\sum_{i\in I} \|Ae_i\|^2 = \sum_{j\in I} \|Af_j\|^2 = \sum_{j\in I} \|A^*f_j\|^2 \tag{1.2}$$

showing that the definition (1.1) is independent of the orthonormal basis and A is a Hilbert-Schmidt operator iff A^* is a Hilbert-Schmidt operator.

Let $\mathcal{B}_2(H)$ the set of *Hilbert-Schmidt operators* in $\mathcal{B}(H)$. For $A \in \mathcal{B}_2(H)$ we define

$$\|A\|_2 := \left(\sum_{i\in I} \|Ae_i\|^2\right)^{1/2} \tag{1.3}$$

for $\{e_i\}_{i\in I}$ an orthonormal basis of H.

Using the triangle inequality in $l^2(I)$, one checks that $\mathcal{B}_2(H)$ is a *vector space* and that $\|\cdot\|_2$ is a norm on $\mathcal{B}_2(H)$, which is usually called in the literature as the *Hilbert-Schmidt norm.*

Denote *the modulus* of an operator $A \in \mathcal{B}(H)$ by $|A| := (A^*A)^{1/2}$.

Because $\||A|x\| = \|Ax\|$ for all $x \in H$, A is Hilbert-Schmidt iff $|A|$ is Hilbert-Schmidt and $\|A\|_2 = \||A|\|_2$. From (1.2) we have that if $A \in \mathcal{B}_2(H)$, then $A^* \in \mathcal{B}_2(H)$ and $\|A\|_2 = \|A^*\|_2$.

The following theorem collects some of the most important properties of Hilbert-Schmidt operators:

Theorem 1. *We have:*

(i) $(\mathcal{B}_2(H), \|\cdot\|_2)$ *is a Hilbert space with inner product*

$$\langle A, B\rangle_2 := \sum_{i\in I} \langle Ae_i, Be_i\rangle = \sum_{i\in I} \langle B^* Ae_i, e_i\rangle \tag{1.4}$$

and the definition does not depend on the choice of the orthonormal basis $\{e_i\}_{i\in I}$*;*

(ii) We have the inequalities

$$\|A\| \le \|A\|_2 \tag{1.5}$$

for any $A \in \mathcal{B}_2(H)$ *and, if* $A \in \mathcal{B}_2(H)$ *and* $T \in \mathcal{B}(H)$, *then* $AT, TA \in \mathcal{B}_2(H)$ *with*

$$\|AT\|_2, \ \|TA\|_2 \le \|T\| \|A\|_2 \tag{1.6}$$

(iii) $\mathcal{B}_2(H)$ *is an operator ideal in* $\mathcal{B}(H)$, *i.e.*

$$\mathcal{B}(H)\mathcal{B}_2(H)\mathcal{B}(H) \subseteq \mathcal{B}_2(H).$$

If $\{e_i\}_{i\in I}$ an orthonormal basis of H, we say that $A \in \mathcal{B}(H)$ is *trace class* if

$$\|A\|_1 := \sum_{i\in I} \langle |A| e_i, e_i\rangle < \infty. \tag{1.7}$$

The definition of $\|A\|_1$ does not depend on the choice of the orthonormal basis $\{e_i\}_{i\in I}$. We denote by $\mathcal{B}_1(H)$ the set of trace class operators in $\mathcal{B}(H)$.

The following proposition holds:

Proposition 1. *If* $A \in \mathcal{B}(H)$, *then the following are equivalent:*

(i) $A \in \mathcal{B}_1(H)$;

(ii) $|A|^{1/2} \in \mathcal{B}_2(H)$.

The following properties are also well known:

Theorem 2. *With the above notations:*

(i) We have

$$\|A\|_1 = \|A^*\|_1 \ \text{ and } \ \|A\|_2 \le \|A\|_1 \tag{1.8}$$

for any $A \in \mathcal{B}_1(H)$;

(ii) $\mathcal{B}_1(H)$ *is an operator ideal in* $\mathcal{B}(H)$, *i.e.*

$$\mathcal{B}(H)\mathcal{B}_1(H)\mathcal{B}(H) \subseteq \mathcal{B}_1(H);$$

(iii) We have

$$\mathcal{B}_2(H)\mathcal{B}_2(H) = \mathcal{B}_1(H);$$

(iv) We have

$$\|A\|_1 = \sup\{\langle A, B\rangle_2 \mid B \in \mathcal{B}_2(H), \ \|B\|_2 \le 1\};$$

(v) $(\mathcal{B}_1(H), \|\cdot\|_1)$ *is a Banach space.*

We define the *trace* of a trace class operator $A \in \mathcal{B}_1(H)$ to be

$$\operatorname{tr}(A) := \sum_{i\in I} \langle Ae_i, e_i\rangle, \tag{1.9}$$

where $\{e_i\}_{i\in I}$ an orthonormal basis of H. Note that this coincides with the usual definition of the trace if H is finite-dimensional. We observe that the series (1.9) converges absolutely and it is independent from the choice of basis.

The following result collects some properties of the trace:

Theorem 3. *We have:*

(i) If $A \in \mathcal{B}_1(H)$ *then* $A^* \in \mathcal{B}_1(H)$ *and*

$$\operatorname{tr}(A^*) = \overline{\operatorname{tr}(A)}; \tag{1.10}$$

(ii) If $A \in \mathcal{B}_1(H)$ *and* $T \in \mathcal{B}(H)$, *then* $AT,\ TA \in \mathcal{B}_1(H)$,

$$\operatorname{tr}(AT) = \operatorname{tr}(TA) \text{ and } |\operatorname{tr}(AT)| \le \|A\|_1 \|T\|; \tag{1.11}$$

(iii) $\operatorname{tr}(\cdot)$ *is a bounded linear functional on* $\mathcal{B}_1(H)$ *with* $\|\operatorname{tr}\| = 1$;

(iv) If $A,\ B \in \mathcal{B}_2(H)$ *then* $AB,\ BA \in \mathcal{B}_1(H)$ *and* $\operatorname{tr}(AB) = \operatorname{tr}(BA)$.

Now, if we assume that $P \ge 0$ and $P \in \mathcal{B}_1(H)$, then for all $T \in \mathcal{B}(H)$, PT, $TP \in \mathcal{B}_1(H)$ and $\operatorname{tr}(PT) = \operatorname{tr}(TP)$. Also, since $P^{1/2} \in \mathcal{B}_2(H)$, $TP^{1/2} \in \mathcal{B}_2(H)$, hence $P^{1/2}TP^{1/2}$ and $TP^{1/2}P^{1/2} = TP \in \mathcal{B}_1(H)$ with $\operatorname{tr}\left(P^{1/2}TP^{1/2}\right) = \operatorname{tr}(TP)$. Therefore, if $P \ge 0$ and $P \in \mathcal{B}_1(H)$,

$$\operatorname{tr}(PT) = \operatorname{tr}(TP) = \operatorname{tr}\left(P^{1/2}TP^{1/2}\right)$$

for all $T \in \mathcal{B}(H)$.

If $T \ge 0$, then $P^{1/2}TP^{1/2} \ge 0$, which implies that $\operatorname{tr}(PT) \ge 0$ that shows that the functional $\mathcal{B}(H) \ni T \longmapsto \operatorname{tr}(PT)$ is linear and isotonic functional. Also, by (1.11), if $T_n \to T$ for $n \to \infty$ in $\mathcal{B}(H)$ then $\lim_{n\to\infty} \operatorname{tr}(PT_n) = \operatorname{tr}(PT)$, namely $\mathcal{B}(H) \ni T \longmapsto \operatorname{tr}(PT)$ is also continuous in the norm topology.

For a survey on recent trace inequalities see [3] and the references therein.

Now, for a given $P \ge 0$ with $P \in \mathcal{B}_1(H)$ and $\operatorname{tr}(P) = 1$, we define the P-*determinant* of the positive invertible operator A by

$$\Delta_P(A) := \exp \operatorname{tr}(P \ln A) = \exp \operatorname{tr}((\ln A) P) = \exp \operatorname{tr}\left(P^{1/2}(\ln A) P^{1/2}\right). \tag{1.12}$$

Assume that $P \ge 0$ with $P \in \mathcal{B}_1(H)$ and $\operatorname{tr}(P) = 1$. We observe that we have the following elementary properties:

(i) *continuity*: the map $A \to \Delta_P(A)$ is norm continuous;
(ii) *power equality:* $\Delta_P(A^t) = \Delta_P(A)^t$ for all $t > 0$;
(iii) *homogeneity*: $\Delta_P(tA) = t\Delta_x(A)$ and $\Delta_P(tI) = t$ for all $t > 0$;
(iv) *monotonicity*: $0 < A \le B$ implies $\Delta_P(A) \le \Delta_P(B)$.

In the recent paper [4] we obtained the following results:

Theorem 4. *Let* $P \ge 0$ *with* $P \in \mathcal{B}_1(H)$ *and* $\operatorname{tr}(P) = 1$, *then for all* $A,\ B > 0$ *and* $t \in [0,1]$,

$$\Delta_P((1-t)A + tB) \ge [\Delta_P(A)]^{1-t}[\Delta_P(B)]^t. \tag{1.13}$$

and

Theorem 5. *Let* $P \ge 0$ *with* $P \in \mathcal{B}_1(H)$ *and* $\operatorname{tr}(P) = 1$, *then for all* $A > 0$ *and* $a > 0$ *we have the double inequality*

$$a \exp\left[1 - a \operatorname{tr}\left(PA^{-1}\right)\right] \le \Delta_P(A) \le a \exp\left[a^{-1}\operatorname{tr}(PA) - 1\right]. \tag{1.14}$$

In particular

$$1 \le \frac{\operatorname{tr}(PA)}{\Delta_P(A)} \le \exp\left[\operatorname{tr}(PA)\operatorname{tr}\left(PA^{-1}\right) - 1\right] \tag{1.15}$$

and

$$1 \le \frac{\Delta_P(A)}{\left[\operatorname{tr}\left(PA^{-1}\right)\right]^{-1}} \le \exp\left[\operatorname{tr}\left(PA^{-1}\right)\operatorname{tr}(PA) - 1\right]. \tag{1.16}$$

The first inequalities in (1.15) and 1.16) are best possible from (1.14).

Motivated by the above results, in this paper we show among others that, if $P_i \geq 0$ with $P_i \in \mathcal{B}_1(H)$ and $\operatorname{tr}(P_i) = 1$ for $i \in \{1, ..., n\}$, $p_i \geq 0$ with $\sum_{i=1}^n p_i = 1$, $p \in (-\infty, 0) \cup (1, \infty)$ and that A_j are operators such that $0 < m \leq A_j \leq M$, for $i \in \{1, ..., n\}$, then

$$\begin{aligned}
&\exp\left\{\gamma_p\left[\sum_{i=1}^n p_i \operatorname{tr}(P_i A_i^p) - \left(\sum_{i=1}^n p_i \operatorname{tr}(P_i A_i)\right)^p\right]\right\} \\
&\leq \frac{\sum_{i=1}^n p_i \operatorname{tr}(P_i A_i)}{\prod_{i=1}^n [\Delta_{P_i}(A_i)]^{p_i}} \\
&\leq \exp\left\{\Gamma_p\left[\sum_{i=1}^n p_i \operatorname{tr}(P_i A_i^p) - \left(\sum_{i=1}^n p_i \operatorname{tr}(P_i A_i)\right)^p\right]\right\},
\end{aligned}$$

where

$$\gamma_p := \begin{cases} \frac{M^{-p}}{p(p-1)}, & p \in (1, \infty), \\ \\ \frac{m^{-p}}{p(p-1)}, & p \in (-\infty, 0) \end{cases}, \quad \Gamma_p := \begin{cases} \frac{m^{-p}}{p(p-1)}, & p \in (1, \infty), \\ \\ \frac{M^{-p}}{p(p-1)}, & p \in (-\infty, 0). \end{cases}$$

2. Main Results

The following result is of interest in itself as well:

Lemma 1. *Assume that f is twice differentiable on the interior $\mathring{I}$ of the interval $I \subset (0, \infty)$ and the second derivative f'' is continuous on $\mathring{I}$ and for $p \in (-\infty, 0) \cup (1, \infty)$ satisfies the condition*

$$\gamma \leq \frac{t^{2-p}}{p(p-1)} f''(t) \leq \Gamma \text{ for any } t \in \mathring{I}, \tag{2.1}$$

where $\gamma < \Gamma$ are constants. If $Q_i \geq 0$ with $Q_i \in \mathcal{B}_1(H)$ for $i \in \{1, ..., n\}$ and $\sum_{i=1}^n \operatorname{tr}(Q_i) > 0$, then for all B_i with the spectra $\operatorname{Sp}(B_i) \subset \mathring{I}$ for $i \in \{1, ..., n\}$ and $a \in \mathring{I}$,

$$\begin{aligned}
&\gamma\left[\frac{\sum_{i=1}^n \operatorname{tr}(Q_i B_i^p)}{\sum_{i=1}^n \operatorname{tr}(Q_i)} - a^p - pa^{p-1}\left(\frac{\sum_{i=1}^n \operatorname{tr}(Q_i B_i)}{\sum_{i=1}^n \operatorname{tr}(Q_i)} - a\right)\right] \\
&\leq \frac{\sum_{i=1}^n \operatorname{tr}[Q_i f(B_i)]}{\sum_{i=1}^n \operatorname{tr}(Q_i)} - f(a) - f'(a)\left(\frac{\sum_{i=1}^n \operatorname{tr}(Q_i B_i)}{\sum_{i=1}^n \operatorname{tr}(Q_i)} - a\right) \\
&\leq \Gamma\left[\frac{\sum_{i=1}^n \operatorname{tr}(Q_i B_i^p)}{\sum_{i=1}^n \operatorname{tr}(Q_i)} - a^p - pa^{p-1}\left(\frac{\sum_{i=1}^n \operatorname{tr}(Q_i B_i)}{\sum_{i=1}^n \operatorname{tr}(Q_i)} - a\right)\right].
\end{aligned} \tag{2.2}$$

Proof. We use the Taylor's expansion for twice differentiable functions

$$f(x) = f(a) + (x-a) f'(a) + (x-a)^2 \int_0^1 f''(sa + (1-s)x) s ds \tag{2.3}$$

that holds for all x, $a \in \mathring{I}$.

Since

$$\gamma p(p-1) t^{p-2} \leq f''(t) \leq p(p-1) \Gamma t^{p-2} \text{ for any } t \in \mathring{I},$$

hence

$$p(p-1)\gamma\int_0^1 (sa+(1-s)x)^{p-2}\,sds \le \int_0^1 f''(sa+(1-s)x)\,sds$$
$$\le p(p-1)\Gamma\int_0^1 (sa+(1-s)x)^{p-2}\,sds,$$

which, by (2.3) gives that

$$\begin{aligned} (2.4)\qquad & p(p-1)\gamma(x-a)^2\int_0^1 (sa+(1-s)x)^{p-2}\,sds \\ & \le f(x)-f(a)-(x-a)f'(a) \\ & \le p(p-1)\Gamma(x-a)^2\int_0^1 (sa+(1-s)x)^{p-2}\,sds \end{aligned}$$

for all $x,\ a\in \mathring{I}$.

Using integration by parts, we get

$$\begin{aligned} & \int_0^1 (sa+(1-s)x)^{p-2}\,sds \\ & = \frac{1}{(p-1)(a-x)}\int_0^1 sd\left[(sa+(1-s)x)^{p-1}\right] \\ & = \frac{1}{(p-1)(a-x)}\left[a^{p-1}-\int_0^1 (sa+(1-s)x)^{p-1}\,ds\right] \end{aligned}$$

$$\begin{aligned} & = \frac{1}{(p-1)(a-x)}\left[a^{p-1}-\frac{1}{p(a-x)}\int_0^1 d(sa+(1-s)x)^{p}\right] \\ & = \frac{1}{(p-1)(a-x)}\left[a^{p-1}-\frac{1}{p(a-x)}(a^p-x^p)\right] \\ & = \frac{1}{p(p-1)(a-x)^2}\left[x^p-a^p-p(x-a)a^{p-1}\right] \end{aligned}$$

and by (2.4) we get

$$\begin{aligned} (2.5)\qquad \gamma\left[x^p-a^p-p(x-a)a^{p-1}\right] & \le f(x)-f(a)-(x-a)f'(a) \\ & \le \Gamma\left[x^p-a^p-p(x-a)a^{p-1}\right] \end{aligned}$$

for all $x,\ a\in \mathring{I}$.

Now, by using the continuous functional calculus for the selfadjoint operators , we get from (2.5) that

$$\begin{aligned} (2.6)\qquad \gamma\left[B_i^p-a^pI-pa^{p-1}(B_i-aI)\right] & \le f(B_i)-f(a)I-f'(a)(B_i-aI) \\ & \le \Gamma\left[B_i^p-a^pI-pa^{p-1}(B_i-aI)\right] \end{aligned}$$

for B_i with the spectra $\operatorname{Sp}(B_i)\subset \mathring{I}$ for $i\in\{1,...,n\}$ and $a\in \mathring{I}$.

If we multiply both sides by $Q_i^{1/2}$ we get

$$\begin{aligned}
&\gamma\left[Q_i^{1/2}B_i^pQ_i^{1/2}-a^pQ_i-pa^{p-1}\left(Q_i^{1/2}B_iQ_i^{1/2}-aQ_i\right)\right]\\
&\le Q_i^{1/2}f\left(B_i\right)Q_i^{1/2}-f\left(a\right)Q_i-f'\left(a\right)\left(Q_i^{1/2}B_iQ_i^{1/2}-aQ_i\right)\\
&\le \Gamma\left[Q_i^{1/2}B_i^pQ_i^{1/2}-a^pQ_i-pa^{p-1}\left(Q_i^{1/2}B_iQ_i^{1/2}-aQ_i\right)\right]
\end{aligned}$$

for $i\in\{1,...,n\}$ and $a\in\mathring{I}$.

Now, if we take the trace and use its properties, we derive

$$\begin{aligned}
&\gamma\left[\operatorname{tr}\left(Q_iB_i^p\right)-a^p\operatorname{tr}\left(Q_i\right)-pa^{p-1}\left(\operatorname{tr}\left(Q_iB_i\right)-a\operatorname{tr}\left(Q_i\right)\right)\right]\\
&\le \operatorname{tr}\left[Q_if\left(B_i\right)\right]-f\left(a\right)\operatorname{tr}\left(Q_i\right)-f'\left(a\right)\left(\operatorname{tr}\left(Q_iB_i\right)-a\operatorname{tr}\left(Q_i\right)\right)\\
&\le \Gamma\left[\operatorname{tr}\left(Q_iB_i^p\right)-a^p\operatorname{tr}\left(Q_i\right)-pa^{p-1}\left(\operatorname{tr}\left(Q_iB_i\right)-a\operatorname{tr}\left(Q_i\right)\right)\right]
\end{aligned}$$

for $i\in\{1,...,n\}$ and $a\in\mathring{I}$.

If we sum over $i\in\{1,...,n\}$ and divide by $\sum_{i=1}^n\operatorname{tr}\left(Q_i\right)>0$, we get (2.2). □

Remark 1. *Assume that f is twice differentiable on the interior $\mathring{I}$ of the interval $I\subset(0,\infty)$ and the second derivative f'' is continuous on $\mathring{I}$ and satisfies the condition*

$$\varphi\le f''\left(t\right)\le\Phi \text{ for any } t\in\mathring{I}, \tag{2.7}$$

where $\varphi<\Phi$ are constants. If $Q_i\ge0$ with $Q_i\in\mathcal{B}_1\left(H\right)$ for $i\in\{1,...,n\}$ and $\sum_{i=1}^n\operatorname{tr}\left(Q_i\right)>0$, then for all B_i with the spectra $\operatorname{Sp}\left(B_i\right)\subset\mathring{I}$ for $i\in\{1,...,n\}$ and $a\in\mathring{I}$,

$$\begin{aligned}
&\frac{1}{2}\varphi\left[\frac{\sum_{i=1}^n\operatorname{tr}\left(Q_iB_i^2\right)}{\sum_{i=1}^n\operatorname{tr}\left(Q_i\right)}-\left(\frac{\sum_{i=1}^n\operatorname{tr}\left(Q_iB_i\right)}{\sum_{i=1}^n\operatorname{tr}\left(Q_i\right)}\right)^2\right.\\
&\left.+\left(a-\frac{\sum_{i=1}^n\operatorname{tr}\left(Q_iB_i\right)}{\sum_{i=1}^n\operatorname{tr}\left(Q_i\right)}\right)^2\right]\\
&\le\frac{\sum_{i=1}^n\operatorname{tr}\left[Q_if\left(B_i\right)\right]}{\sum_{i=1}^n\operatorname{tr}\left(Q_i\right)}-f\left(a\right)-f'\left(a\right)\left(\frac{\sum_{i=1}^n\operatorname{tr}\left(Q_iB_i\right)}{\sum_{i=1}^n\operatorname{tr}\left(Q_i\right)}-a\right)\\
&\le\frac{1}{2}\Phi\left[\frac{\sum_{i=1}^n\operatorname{tr}\left(Q_iB_i^2\right)}{\sum_{i=1}^n\operatorname{tr}\left(Q_i\right)}-\left(\frac{\sum_{i=1}^n\operatorname{tr}\left(Q_iB_i\right)}{\sum_{i=1}^n\operatorname{tr}\left(Q_i\right)}\right)^2\right.\\
&\left.+\left(a-\frac{\sum_{i=1}^n\operatorname{tr}\left(Q_iB_i\right)}{\sum_{i=1}^n\operatorname{tr}\left(Q_i\right)}\right)^2\right].
\end{aligned}\tag{2.8}$$

The proof follows by Lemma 1 for $p=2$ and $\gamma=\frac{1}{2}\varphi$, $\Gamma=\frac{1}{2}\Phi$.

If

$$\frac{\psi}{t^3}\le f''\left(t\right)\le\frac{\Psi}{t^3} \text{ for any } t\in\mathring{I}, \tag{2.9}$$

then

$$\frac{1}{2}\psi\left[\frac{\sum_{i=1}^{n}\operatorname{tr}\left(Q_iB_i^{-1}\right)}{\sum_{i=1}^{n}\operatorname{tr}(Q_i)}+a^{-2}\frac{\sum_{i=1}^{n}\operatorname{tr}(Q_iB_i)}{\sum_{i=1}^{n}\operatorname{tr}(Q_i)}-2a^{-1}\right] \tag{2.10}$$
$$\leq\frac{\sum_{i=1}^{n}\operatorname{tr}[Q_if(B_i)]}{\sum_{i=1}^{n}\operatorname{tr}(Q_i)}-f(a)-f'(a)\left(\frac{\sum_{i=1}^{n}\operatorname{tr}(Q_iB_i)}{\sum_{i=1}^{n}\operatorname{tr}(Q_i)}-a\right)$$
$$\leq\frac{1}{2}\Psi\left[\frac{\sum_{i=1}^{n}\operatorname{tr}\left(Q_iB_i^{-1}\right)}{\sum_{i=1}^{n}\operatorname{tr}(Q_i)}+a^{-2}\frac{\sum_{i=1}^{n}\operatorname{tr}(Q_iB_i)}{\sum_{i=1}^{n}\operatorname{tr}(Q_i)}-2a^{-1}\right].$$

Corollary 1. *With the assumptions of Lemma 1 we have*

$$\gamma\left[\frac{\sum_{i=1}^{n}\operatorname{tr}(Q_iB_i^{p})}{\sum_{i=1}^{n}\operatorname{tr}(Q_i)}-\left(\frac{\sum_{i=1}^{n}\operatorname{tr}(Q_iB_i)}{\sum_{i=1}^{n}\operatorname{tr}(Q_i)}\right)^{p}\right] \tag{2.11}$$
$$\leq\frac{\sum_{i=1}^{n}\operatorname{tr}[Q_if(B_i)]}{\sum_{i=1}^{n}\operatorname{tr}(Q_i)}-f\left(\frac{\sum_{i=1}^{n}\operatorname{tr}(Q_iB_i)}{\sum_{i=1}^{n}\operatorname{tr}(Q_i)}\right)$$
$$\leq\Gamma\left[\frac{\sum_{i=1}^{n}\operatorname{tr}(Q_iB_i^{p})}{\sum_{i=1}^{n}\operatorname{tr}(Q_i)}-\left(\frac{\sum_{i=1}^{n}\operatorname{tr}(Q_iB_i)}{\sum_{i=1}^{n}\operatorname{tr}(Q_i)}\right)^{p}\right].$$

In particular, if f satisfies the condition (2.7), then

$$\frac{1}{2}\varphi\left[\frac{\sum_{i=1}^{n}\operatorname{tr}\left(Q_iB_i^{2}\right)}{\sum_{i=1}^{n}\operatorname{tr}(Q_i)}-\left(\frac{\sum_{i=1}^{n}\operatorname{tr}(Q_iB_i)}{\sum_{i=1}^{n}\operatorname{tr}(Q_i)}\right)^{2}\right] \tag{2.12}$$
$$\leq\frac{\sum_{i=1}^{n}\operatorname{tr}[Q_if(B_i)]}{\sum_{i=1}^{n}\operatorname{tr}(Q_i)}-f\left(\frac{\sum_{i=1}^{n}\operatorname{tr}(Q_iB_i)}{\sum_{i=1}^{n}\operatorname{tr}(Q_i)}\right)$$
$$\leq\frac{1}{2}\Phi\left[\frac{\sum_{i=1}^{n}\operatorname{tr}\left(Q_iB_i^{2}\right)}{\sum_{i=1}^{n}\operatorname{tr}(Q_i)}-\left(\frac{\sum_{i=1}^{n}\operatorname{tr}(Q_iB_i)}{\sum_{i=1}^{n}\operatorname{tr}(Q_i)}\right)^{2}\right].$$

If f satisfies the condition (2.9), then

$$\frac{1}{2}\psi\left[\frac{\sum_{i=1}^{n}\operatorname{tr}\left(Q_iB_i^{-1}\right)}{\sum_{i=1}^{n}\operatorname{tr}(Q_i)}-\left(\frac{\sum_{i=1}^{n}\operatorname{tr}(Q_iB_i)}{\sum_{i=1}^{n}\operatorname{tr}(Q_i)}\right)^{-1}\right] \tag{2.13}$$
$$\leq\frac{\sum_{i=1}^{n}\operatorname{tr}[Q_if(B_i)]}{\sum_{i=1}^{n}\operatorname{tr}(Q_i)}-f\left(\frac{\sum_{i=1}^{n}\operatorname{tr}(Q_iB_i)}{\sum_{i=1}^{n}\operatorname{tr}(Q_i)}\right)$$
$$\leq\frac{1}{2}\Psi\left[\frac{\sum_{i=1}^{n}\operatorname{tr}\left(Q_iB_i^{-1}\right)}{\sum_{i=1}^{n}\operatorname{tr}(Q_i)}-\left(\frac{\sum_{i=1}^{n}\operatorname{tr}(Q_iB_i)}{\sum_{i=1}^{n}\operatorname{tr}(Q_i)}\right)^{-1}\right].$$

We also have:

Lemma 2. *Assume that f is twice differentiable on the interior $\mathring{I}$ of the interval $I\subset(0,\infty)$ with the second derivative f'' is continuous on $\mathring{I}$ and for $p\in(0,1)$ satisfies the condition*

$$\delta\leq\frac{t^{2-p}}{p(1-p)}f''(t)\leq\Delta \text{ for any } t\in\mathring{I} \tag{2.14}$$

for some $\delta < \Delta$. *If* $Q_i \geq 0$ *with* $Q_i \in \mathcal{B}_1(H)$ *for* $i \in \{1, ..., n\}$ *and* $\sum_{i=1}^n \operatorname{tr}(Q_i) > 0$, *then for all* B_i *with the spectra* $\operatorname{Sp}(B_i) \subset \mathring{I}$ *for* $i \in \{1, ..., n\}$ *and* $a \in \mathring{I}$,

$$\begin{aligned}
(2.15)\quad & \delta\left[pa^{p-1}\left(\frac{\sum_{i=1}^n \operatorname{tr}(Q_i B_i)}{\sum_{i=1}^n \operatorname{tr}(Q_i)} - a\right) + a^p - \frac{\sum_{i=1}^n \operatorname{tr}(Q_i B_i^p)}{\sum_{i=1}^n \operatorname{tr}(Q_i)}\right] \\
& \leq \frac{\sum_{i=1}^n \operatorname{tr}[Q_i f(B_i)]}{\sum_{i=1}^n \operatorname{tr}(Q_i)} - f(a) - f'(a)\left(\frac{\sum_{i=1}^n \operatorname{tr}(Q_i B_i)}{\sum_{i=1}^n \operatorname{tr}(Q_i)} - a\right) \\
& \leq \Delta\left[\frac{\sum_{i=1}^n \operatorname{tr}(Q_i B_i^p)}{\sum_{i=1}^n \operatorname{tr}(Q_i)} - a^p - pa^{p-1}\left(\frac{\sum_{i=1}^n \operatorname{tr}(Q_i B_i)}{\sum_{i=1}^n \operatorname{tr}(Q_i)} - a\right)\right].
\end{aligned}$$

In particular,

$$\begin{aligned}
(2.16)\quad & \delta\left[\left(\frac{\sum_{i=1}^n \operatorname{tr}(Q_i B_i)}{\sum_{i=1}^n \operatorname{tr}(Q_i)}\right)^p - \frac{\sum_{i=1}^n \operatorname{tr}(Q_i B_i^p)}{\sum_{i=1}^n \operatorname{tr}(Q_i)}\right] \\
& \leq \frac{\sum_{i=1}^n \operatorname{tr}[Q_i f(B_i)]}{\sum_{i=1}^n \operatorname{tr}(Q_i)} - f\left(\frac{\sum_{i=1}^n \operatorname{tr}(Q_i B_i)}{\sum_{i=1}^n \operatorname{tr}(Q_i)}\right) \\
& \leq \Delta\left[\left(\frac{\sum_{i=1}^n \operatorname{tr}(Q_i B_i)}{\sum_{i=1}^n \operatorname{tr}(Q_i)}\right)^p - \frac{\sum_{i=1}^n \operatorname{tr}(Q_i B_i^p)}{\sum_{i=1}^n \operatorname{tr}(Q_i)}\right].
\end{aligned}$$

Proof. As above, from (2.14) we derive

$$\begin{aligned}
\gamma\left[p(x-a)a^{p-1} + a^p - x^p\right] &\leq f(x) - f(a) - (x-a) f'(a) \\
&\leq \Delta\left[p(x-a)a^{p-1} + a^p - x^p\right]
\end{aligned}$$

for all x, $a \in \mathring{I}$.

By making use of a similar argument as in the proof of Lemma 1 we derive the desired result (2.15). □

Remark 2. *If*

$$(2.17)\quad \frac{\varphi}{t^{3/2}} \leq f''(t) \leq \frac{\digamma}{t^{3/2}} \text{ for any } t \in \mathring{I},$$

then

$$\begin{aligned}
(2.18)\quad & 4\varphi\left[\left(\frac{\sum_{i=1}^n \operatorname{tr}(Q_i B_i)}{\sum_{i=1}^n \operatorname{tr}(Q_i)}\right)^{1/2} - \frac{\sum_{i=1}^n \operatorname{tr}\left(Q_i B_i^{1/2}\right)}{\sum_{i=1}^n \operatorname{tr}(Q_i)}\right] \\
& \leq \frac{\sum_{i=1}^n \operatorname{tr}[Q_i f(B_i)]}{\sum_{i=1}^n \operatorname{tr}(Q_i)} - f\left(\frac{\sum_{i=1}^n \operatorname{tr}(Q_i B_i)}{\sum_{i=1}^n \operatorname{tr}(Q_i)}\right) \\
& \leq 4\digamma\left[\left(\frac{\sum_{i=1}^n \operatorname{tr}(Q_i B_i)}{\sum_{i=1}^n \operatorname{tr}(Q_i)}\right)^{1/2} - \frac{\sum_{i=1}^n \operatorname{tr}\left(Q_i B_i^{1/2}\right)}{\sum_{i=1}^n \operatorname{tr}(Q_i)}\right].
\end{aligned}$$

We have the following main result:

Theorem 6. *If* $P_i \geq 0$ *with* $P_i \in \mathcal{B}_1(H)$ *and* $\operatorname{tr}(P_i) = 1$ *for* $i \in \{1, ..., n\}$, $p_i \geq 0$ *with* $\sum_{i=1}^n p_i = 1$, $p \in (-\infty, 0) \cup (1, \infty)$ *and that* A_j *are operators such that* $0 < m \leq A_j \leq M$, *for* $i \in \{1, ..., n\}$, *then for all* $a > 0$ *we have the lower and upper* *bounds*

$$\exp\left\{\gamma_p\left[\sum_{i=1}^n p_i \operatorname{tr}\left(P_i A_i^p\right) - a^p - pa^{p-1}\left(\sum_{i=1}^n p_i \operatorname{tr}\left(P_i A_i\right) - a\right)\right]\right\} \tag{2.19}$$
$$\leq \frac{a\exp\left(a^{-1}\sum_{i=1}^n p_i \operatorname{tr}\left(P_i A_i\right) - 1\right)}{\prod_{i=1}^n \left[\Delta_{P_i}\left(A_i\right)\right]^{p_i}}$$
$$\leq \exp\left\{\Gamma_p\left[\sum_{i=1}^n p_i \operatorname{tr}\left(P_i A_i^p\right) - a^p - pa^{p-1}\left(\sum_{i=1}^n p_i \operatorname{tr}\left(P_i A_i\right) - a\right)\right]\right\},$$

where

$$\gamma_p := \begin{cases} \frac{M^{-p}}{p(p-1)} \text{ for } p \in (1, \infty), \\ \\ \frac{m^{-p}}{p(p-1)} \text{ for } p \in (-\infty, 0) \end{cases}$$

and

$$\Gamma_p := \begin{cases} \frac{m^{-p}}{p(p-1)} \text{ for } p \in (1, \infty), \\ \\ \frac{M^{-p}}{p(p-1)} \text{ for } p \in (-\infty, 0). \end{cases}$$

Proof. We consider the convex function $f(t) = -\ln t$, $t \in [m, M] \subset (0, \infty)$. Then

$$g(t) = \frac{t^{2-p}}{p(p-1)} f''(t) = \frac{t^{2-p}}{p(p-1)} \frac{1}{t^2} = \frac{1}{p(p-1)t^p}.$$

For $p \in (1, \infty)$, we have

$$\sup_{t\in[m,M]} g(t) = \frac{m^{-p}}{p(p-1)} \text{ and } \inf_{t\in[m,M]} g(t) = \frac{M^{-p}}{p(p-1)}$$

and for $p \in (-\infty, 0)$

$$\sup_{t\in[m,M]} g(t) = \sup_{t\in[m,M]} \frac{t^{-p}}{p(p-1)} = \frac{M^{-p}}{p(p-1)}$$

and

$$\inf_{t\in[m,M]} g(t) = \inf_{t\in[m,M]} \frac{t^{-p}}{p(p-1)} = \frac{m^{-p}}{p(p-1)}.$$

From (2.2) applied for $f(t) = -\ln t$, $t \in [m, M] \subset (0, \infty)$, we get for $Q_i = p_i P_i$ and $B_i = A_i$ that

$$\gamma_p\left[\sum_{i=1}^n p_i \operatorname{tr}\left(P_i A_i^p\right) - a^p - pa^{p-1}\left(\sum_{i=1}^n p_i \operatorname{tr}\left(P_i A_i\right) - a\right)\right] \tag{2.20}$$
$$\leq -\sum_{i=1}^n p_i \operatorname{tr}\left[P_i \ln\left(A_i\right)\right] + \ln a + a^{-1}\left(\sum_{i=1}^n p_i \operatorname{tr}\left(P_i A_i\right) - a\right)$$
$$\leq \Gamma_p\left[\sum_{i=1}^n p_i \operatorname{tr}\left(P_i A_i^p\right) - a^p - pa^{p-1}\left(\sum_{i=1}^n p_i \operatorname{tr}\left(P_i A_i\right) - a\right)\right]$$

for all $a > 0$.

If we take the exponential in (2.20), then we get

$$\begin{aligned}&\exp\left\{\gamma_p\left[\sum_{i=1}^n p_i\operatorname{tr}(P_iA_i^p)-a^p-pa^{p-1}\left(\sum_{i=1}^n p_i\operatorname{tr}(P_iA_i)-a\right)\right]\right\}\\&\le\exp\left[-\sum_{i=1}^n p_i\operatorname{tr}[P_i\ln(A_i)]+\ln a+a^{-1}\left(\sum_{i=1}^n p_i\operatorname{tr}(P_iA_i)-a\right)\right]\\&\le\exp\left\{\Gamma_p\left[\sum_{i=1}^n p_i\operatorname{tr}(P_iA_i^p)-a^p-pa^{p-1}\left(\sum_{i=1}^n p_i\operatorname{tr}(P_iA_i)-a\right)\right]\right\},\end{aligned}$$

namely

$$\begin{aligned}&\exp\left\{\gamma_p\left[\sum_{i=1}^n p_i\operatorname{tr}(P_iA_i^p)-a^p-pa^{p-1}\left(\sum_{i=1}^n p_i\operatorname{tr}(P_iA_i)-a\right)\right]\right\}\\&\le\frac{\exp\left[\ln a+a^{-1}\left(\sum_{i=1}^n p_i\operatorname{tr}(P_iA_i)-a\right)\right]}{\exp\left(\sum_{i=1}^n p_i\operatorname{tr}[P_i\ln(A_i)]\right)}\\&\le\exp\left\{\Gamma_p\left[\sum_{i=1}^n p_i\operatorname{tr}(P_iA_i^p)-a^p-pa^{p-1}\left(\sum_{i=1}^n p_i\operatorname{tr}(P_iA_i)-a\right)\right]\right\}.\end{aligned}\tag{2.21}$$

Since

$$\begin{aligned}&\exp\left[\ln a+a^{-1}\left(\sum_{i=1}^n p_i\operatorname{tr}(P_iA_i)-a\right)\right]\\&=\exp(\ln a)\exp\left[a^{-1}\left(\sum_{i=1}^n p_i\operatorname{tr}(P_iA_i)-a\right)\right]\\&=a\exp\left(a^{-1}\sum_{i=1}^n p_i\operatorname{tr}(P_iA_i)-1\right)\end{aligned}$$

and

$$\exp\left(\sum_{i=1}^n p_i\operatorname{tr}[P_i\ln(A_i)]\right)=\prod_{i=1}^n[\exp(\operatorname{tr}[P_i\ln(A_i)])]^{p_i}=\prod_{i=1}^n[\Delta_{P_i}(A_i)]^{p_i}$$

then by (2.21) we derive (2.19). □

Corollary 2. *With the assumptions of Theorem 6, we have for $p\in(-\infty,0)\cup(1,\infty)$ that*

$$\begin{aligned}&\exp\left\{\gamma_p\left[\sum_{i=1}^n p_i\operatorname{tr}(P_iA_i^p)-\left(\sum_{i=1}^n p_i\operatorname{tr}(P_iA_i)\right)^p\right]\right\}\\&\le\frac{\sum_{i=1}^n p_i\operatorname{tr}(P_iA_i)}{\prod_{i=1}^n[\Delta_{P_i}(A_i)]^{p_i}}\\&\le\exp\left\{\Gamma_p\left[\sum_{i=1}^n p_i\operatorname{tr}(P_iA_i^p)-\left(\sum_{i=1}^n p_i\operatorname{tr}(P_iA_i)\right)^p\right]\right\}.\end{aligned}\tag{2.22}$$

Remark 3. *For $p=2$ we obtain*

$$\exp\left\{\frac{1}{2M^2}\left[\sum_{i=1}^n p_i \operatorname{tr}\left(P_i A_i^2\right)-\left(\sum_{i=1}^n p_i \operatorname{tr}\left(P_i A_i\right)\right)^2\right]\right\} \tag{2.23}$$
$$\leq \frac{\sum_{i=1}^n p_i \operatorname{tr}\left(P_i A_i\right)}{\prod_{i=1}^n \left[\Delta_{P_i}\left(A_i\right)\right]^{p_i}}$$
$$\leq \exp\left\{\frac{1}{2m^2}\left[\sum_{i=1}^n p_i \operatorname{tr}\left(P_i A_i^2\right)-\left(\sum_{i=1}^n p_i \operatorname{tr}\left(P_i A_i\right)\right)^2\right]\right\},$$

while for $p=-1$,

$$\exp\left\{\frac{m}{2}\left[\sum_{i=1}^n p_i \operatorname{tr}\left(P_i A_i^{-1}\right)-\left(\sum_{i=1}^n p_i \operatorname{tr}\left(P_i A_i\right)\right)^{-1}\right]\right\} \tag{2.24}$$
$$\leq \frac{\sum_{i=1}^n p_i \operatorname{tr}\left(P_i A_i\right)}{\prod_{i=1}^n \left[\Delta_{P_i}\left(A_i\right)\right]^{p_i}}$$
$$\leq \exp\left\{\frac{M}{2}\left[\sum_{i=1}^n p_i \operatorname{tr}\left(P_i A_i^{-1}\right)-\left(\sum_{i=1}^n p_i \operatorname{tr}\left(P_i A_i\right)\right)^{-1}\right]\right\}.$$

We also have:

Theorem 7. *If $P_i \geq 0$ with $P_i \in \mathcal{B}_1(H)$ and $\operatorname{tr}(P_i)=1$ for $i\in\{1,...,n\}$, $p_i \geq 0$ with $\sum_{i=1}^n p_i=1$, $p\in(0,1)$ and that A_j are operators such that $0<m\leq A_j\leq M$, for $i\in\{1,...,n\}$, then for all $a>0$ we have the lower and upper bounds*

$$\exp\left\{\frac{1}{p(1-p)M^p}\right. \tag{2.25}$$
$$\left.\times\left[pa^{p-1}\left(\sum_{i=1}^n p_i \operatorname{tr}\left(P_i A_i\right)-a\right)+a^p-\sum_{i=1}^n p_i \operatorname{tr}\left(P_i A_i^p\right)\right]\right\}$$
$$\leq \frac{a\exp\left(a^{-1}\sum_{i=1}^n p_i \operatorname{tr}\left(P_i A_i\right)-1\right)}{\prod_{i=1}^n \left[\Delta_{P_i}\left(A_i\right)\right]^{p_i}}$$
$$\leq \exp\left\{\frac{1}{p(1-p)m^p}\right.$$
$$\left.\times\left[pa^{p-1}\left(\sum_{i=1}^n p_i \operatorname{tr}\left(P_i A_i\right)-a\right)+a^p-\sum_{i=1}^n p_i \operatorname{tr}\left(P_i A_i^p\right)\right]\right\}.$$

In particular,

$$\exp\left\{\frac{1}{p(1-p)M^p}\left[\left(\sum_{i=1}^n p_i \operatorname{tr}(P_iA_i)\right)^p-\sum_{i=1}^n p_i \operatorname{tr}(P_iA_i^p)\right]\right\} \tag{2.26}$$
$$\leq \frac{\sum_{i=1}^n p_i \operatorname{tr}(P_iA_i)}{\prod_{i=1}^n [\Delta_{P_i}(A_i)]^{p_i}}$$
$$\leq \exp\left\{\frac{1}{p(1-p)m^p}\left[\left(\sum_{i=1}^n p_i \operatorname{tr}(P_iA_i)\right)^p-\sum_{i=1}^n p_i \operatorname{tr}(P_iA_i^p)\right]\right\}.$$

Proof. If we take $f(t)=-\ln t$, then

$$h(t)=\frac{t^{2-p}}{p(1-p)}\frac{1}{t^2}$$
$$=\frac{1}{p(1-p)t^p}\in\left[\frac{1}{p(1-p)M^p},\frac{1}{p(1-p)m^p}\right].$$

From (2.15) applied for $f(t)=-\ln t$, $t\in[m,M]\subset(0,\infty)$, we get for $Q_i=p_iP_i$ and $B_i=A_i$ the desired result (2.25). □

Remark 4. *If we take $p=1/2$ in (2.26) then we get*

$$\exp\left\{\frac{4}{\sqrt{M}}\left[\left(\sum_{i=1}^n p_i \operatorname{tr}(P_iA_i)\right)^{1/2}-\sum_{i=1}^n p_i \operatorname{tr}\left(P_iA_i^{1/2}\right)\right]\right\} \tag{2.27}$$
$$\leq \frac{\sum_{i=1}^n p_i \operatorname{tr}(P_iA_i)}{\prod_{i=1}^n [\Delta_{P_i}(A_i)]^{p_i}}$$
$$\leq \exp\left\{\frac{4}{\sqrt{m}}\left[\left(\sum_{i=1}^n p_i \operatorname{tr}(P_iA_i)\right)^{1/2}-\sum_{i=1}^n p_i \operatorname{tr}\left(P_iA_i^{1/2}\right)\right]\right\}.$$

3. Related Results

We also have some simpler upper bounds as follows:

Proposition 2. *If $P_i\geq 0$ with $P_i\in\mathcal{B}_1(H)$ and $\operatorname{tr}(P_i)=1$ for $i\in\{1,...,n\}$, $p_i\geq 0$ with $\sum_{i=1}^n p_i=1$ and that A_j are operators such that $0<m\leq A_j\leq M$, for*

$i \in \{1, ..., n\}$, *then*

$$\frac{\sum_{i=1}^{n} p_i \operatorname{tr}(P_i A_i)}{\prod_{i=1}^{n} [\Delta_{P_i}(A_i)]^{p_i}} \tag{3.1}$$
$$\leq \exp\left\{\frac{1}{2m^2}\left[\sum_{i=1}^{n} p_i \operatorname{tr}\left(P_i A_i^2\right) - \left(\sum_{i=1}^{n} p_i \operatorname{tr}(P_i A_i)\right)^2\right]\right\}$$
$$\leq \exp\left[\frac{1}{2m^2}\left(M - \sum_{i=1}^{n} p_i \operatorname{tr}(P_i A_i)\right)\left(\sum_{i=1}^{n} p_i \operatorname{tr}(P_i A_i) - m\right)\right]$$
$$\leq \exp\left[\frac{1}{8}\left(\frac{M}{m} - 1\right)^2\right].$$

Proof. We observe that

$$\sum_{i=1}^{n} p_i \operatorname{tr}\left(P_i A_i^2\right) - \left(\sum_{i=1}^{n} p_i \operatorname{tr}(P_i A_i)\right)^2 \tag{3.2}$$
$$= \left(M - \sum_{i=1}^{n} p_i \operatorname{tr}(P_i A_i)\right)\left(\sum_{i=1}^{n} p_i \operatorname{tr}(P_i A_i) - m\right)$$
$$- \sum_{i=1}^{n} p_i \operatorname{tr}[P_i (MI - A_i)(A_i - mI)].$$

Since $(M - t)(m - t) \geq 0$ for all $t \in [m, M]$, then by the continuous functional calculus for selfadjoint operators we get that

$$(MI - A_i)(A_i - mI) \geq 0, \ i \in \{1, ..., n\}.$$

If we multiply this inequality both sides by $P_i^{1/2} \geq 0$ we get

$$P_i^{1/2}(MI - A_i)(A_i - mI) P_i^{1/2} \geq 0, \ i \in \{1, ..., n\},$$

and by taking the trace, we derive

$$\operatorname{tr}[P_i (MI - A_i)(A_i - mI)] \geq 0, \ i \in \{1, ..., n\},$$

which implies that

$$\sum_{i=1}^{n} p_i \operatorname{tr}[P_i (MI - A_i)(A_i - mI)] \geq 0$$

and by (3.2) we obtain

$$\sum_{i=1}^{n} p_i \operatorname{tr}\left(P_i A_i^2\right) - \left(\sum_{i=1}^{n} p_i \operatorname{tr}(P_i A_i)\right)^2$$
$$\leq \left(M - \sum_{i=1}^{n} p_i \operatorname{tr}(P_i A_i)\right)\left(\sum_{i=1}^{n} p_i \operatorname{tr}(P_i A_i) - m\right)$$
$$\leq \frac{1}{4}(M - m)^2.$$

By utilizing (2.23) we derive the desired result (3.1). □

We also have:

Proposition 3. *If $P_i \geq 0$ with $P_i \in \mathcal{B}_1(H)$ and $\operatorname{tr}(P_i) = 1$ for $i \in \{1, ..., n\}$, $p_i \geq 0$ with $\sum_{i=1}^n p_i = 1$ and that A_j are operators such that $0 < m \leq A_j \leq M$, for $i \in \{1, ..., n\}$, then*

$$\frac{\sum_{i=1}^n p_i \operatorname{tr}(P_i A_i)}{\prod_{i=1}^n [\Delta_{P_i}(A_i)]^{p_i}} \leq \exp\left\{\frac{M}{2}\left[\sum_{i=1}^n p_i \operatorname{tr}\left(P_i A_i^{-1}\right) - \left(\sum_{i=1}^n p_i \operatorname{tr}(P_i A_i)\right)^{-1}\right]\right\} \leq \exp\left[\frac{1}{2}\left(\sqrt{\frac{M}{m}} - 1\right)^2\right]. \tag{3.3}$$

Proof. If $t \in [m, M] \subset (0, \infty)$, then $(M - t)\left(m^{-1} - t^{-1}\right) \geq 0$. Since $0 < mI \leq A_i \leq MI$, $i \in \{1, ..., n\}$ hence by using the functional calculus for selfadjoint operators we get

$$(MI - A_i)\left(m^{-1}I - A_i^{-1}\right) \geq 0$$

for all $i \in \{1, ..., n\}$, which is equivalent to

$$(M + m) I \geq MmA_i^{-1} + A_i \tag{3.4}$$

for all $i \in \{1, ..., n\}$.

If we multiply (3.4) both sides by $P_i^{1/2}$ we get

$$(M + m) P_i \geq MmP_i^{1/2} A_i^{-1} P_i^{1/2} + P_i^{1/2} A_i P_i^{1/2}$$

for all $i \in \{1, ..., n\}$.

If we take the trace and use its properties, we get

$$M + m \geq Mm \operatorname{tr}\left(P_i A_i^{-1}\right) + \operatorname{tr}(P_i A_i)$$

for all $i \in \{1, ..., n\}$.

If we multiply by $p_i \geq 0$ and summing over i from 1 to n, we get

$$M + m \geq Mm \sum_{i=1}^n p_i \operatorname{tr}\left(P_i A_i^{-1}\right) + \sum_{i=1}^n p_i \operatorname{tr}(P_i A_i). \tag{3.5}$$

From (3.5) we get

$$\sum_{i=1}^n p_i \operatorname{tr}\left(P_i A_i^{-1}\right) \leq \frac{1}{m} + \frac{1}{M} - \frac{1}{mM} \sum_{i=1}^n p_i \operatorname{tr}(P_i A_i),$$

which implies that

$$\begin{aligned}
&\sum_{i=1}^{n} p_i \operatorname{tr}\left(P_i A_i^{-1}\right) - \left(\sum_{i=1}^{n} p_i \operatorname{tr}\left(P_i A_i\right)\right)^{-1} \\
&\leq \frac{1}{m} + \frac{1}{M} - \frac{1}{mM} \sum_{i=1}^{n} p_i \operatorname{tr}\left(P_i A_i\right) - \left(\sum_{i=1}^{n} p_i \operatorname{tr}\left(P_i A_i\right)\right)^{-1} \\
&= \left(\frac{1}{\sqrt{m}} - \frac{1}{\sqrt{M}}\right)^2 \\
&- \left(\frac{1}{\sqrt{mM}} \left(\sum_{i=1}^{n} p_i \operatorname{tr}\left(P_i A_i\right)\right)^{1/2} - \left(\sum_{i=1}^{n} p_i \operatorname{tr}\left(P_i A_i\right)\right)^{-1/2}\right)^2 \\
&\leq \left(\frac{1}{\sqrt{m}} - \frac{1}{\sqrt{M}}\right)^2 .
\end{aligned}$$

By making use of (2.24) we derive (3.3). □

Remark 5. *If $0 < mI \leq A \leq MI$ and $P \geq 0$ with $P \in \mathcal{B}_1(H)$ and $\operatorname{tr}(P) = 1$, then we have the one operator inequalities*

$$\begin{aligned}
\frac{\operatorname{tr}(PA)}{\Delta_P(A)} &\leq \exp\left\{\frac{1}{2m^2}\left[\operatorname{tr}\left(PA^2\right) - [\operatorname{tr}(PA)]^2\right]\right\} \\
&\leq \exp\left[\frac{1}{2m^2}\left(M - [\operatorname{tr}(PA)]\right)\left([\operatorname{tr}(PA)] - m\right)\right] \\
&\leq \exp\left[\frac{1}{8}\left(\frac{M}{m} - 1\right)^2\right]
\end{aligned} \tag{3.6}$$

and

$$\begin{aligned}
\frac{\operatorname{tr}(PA)}{\Delta_P(A)} &\leq \exp\left\{\frac{M}{2}\left[\operatorname{tr}\left(PA^{-1}\right) - [\operatorname{tr}(PA)]^{-1}\right]\right\} \\
&\leq \exp\left[\frac{1}{2}\left(\sqrt{\frac{M}{m}} - 1\right)^2\right].
\end{aligned} \tag{3.7}$$

References

[1] S. S. Dragomir, Jensen's type trace inequalities for convex functions of selfadjoint operators in Hilbert spaces, *Facta Univ. Ser. Math. Inform.*, 31 (2016), no. 5, 981–998. Preprint *RGMIA Res. Rep. Coll.*, **17** (2014), Art. 116. [https://rgmia.org/papers/v17/v17a116.pdf].

[2] S. S. Dragomir, Some Slater's type trace inequalities for convex functions of selfadjoint operators in Hilbert spaces, *Toyama Math. J.*, **38** (2016), 75–99. Preprint *RGMIA Res. Rep. Coll.*, **17** (2014), Art. 117. [https://rgmia.org/papers/v17/v17a117.pdf]

[3] S. S. Dragomir, Trace inequalities for operators in Hilbert spaces: a survey of recent results, *Aust. J. Math. Anal. Appl.* Vol. **19** (2022), No. 1, Art. 1, 202 pp. [Online https://ajmaa.org/searchroot/files/pdf/v19n1/v19i1p1.pdf].

[4] S. S. Dragomir, Some properties of trace class P-determinant of positive operators in Hilbert spaces, Preprint *RGMIA Res. Rep. Coll.* **25** (2022), Art.

[5] B. Fuglede and R. V. Kadison, Determinant theory in finite factors, *Ann. of Math.* (2) **55** (1952), 520–530.

[6] J. I. Fujii and Y. Seo, Determinant for positive operators, *Sci. Math.,* **1** (1998), 153–156.

[7] J. I. Fujii, S. Izumino and Y. Seo, Determinant for positive operators and Specht's Theorem, *Sci. Math.*, **1** (1998), 307–310.
[8] S. Furuichi, Refined Young inequalities with Specht's ratio, *J. Egyptian Math. Soc.* **20** (2012), 46–49.
[9] T. Furuta, J. Mičić-Hot, J. Pečarić and Y. Seo, *Mond-Pečarić Method in Operator Inequalities*, Element, Croatia.
[10] S. Hiramatsu and Y. Seo, Determinant for positive operators and Oppenheim's inequality, *J. Math. Inequal.*, Volume 15 (2021), Number 4, 1637–1645.

[1]Mathematics, College of Engineering & Science, Victoria University, PO Box 14428, Melbourne City, MC 8001, Australia.
E-mail address: `sever.dragomir@vu.edu.au`
URL: `http://rgmia.org/dragomir`

[2]DST-NRF Centre of Excellence in the Mathematical, and Statistical Sciences, School of Computer Science, & Applied Mathematics, University of the Witwatersrand,, Private Bag 3, Johannesburg 2050, South Africa