# SOME FUNCTIONAL PROPERTIES FOR THE NORMALIZED DETERMINANT OF SEQUENCES OF POSITIVE OPERATORS IN HILBERT SPACES

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Abstract. For positive invertible operators $A$ on a Hilbert space $H$ and a fixed unit vector $x \in H$, define the normalized determinant by $\Delta_x(A) := \exp \langle \ln Ax, x\rangle$. We consider the functional

$$D_n(\mathbf{p};\mathbf{A},x) := \frac{\left[\Delta_x\left(\frac{1}{P_n}\sum_{j=1}^n p_j A_j\right)\right]^{P_n}}{\prod_{i=1}^n [\Delta_x(A_j)]^{p_i}},$$

where $\mathbf{A} = (A_1, ..., A_n)$ is an $n$-tuple of selfadjoint positive operators, $\mathbf{p} \in \mathcal{P}_n^+$ the set of nonnegative $n$-tuples and $x \in H$, $\|x\| = 1$.

In this paper we show among other that, for any $\mathbf{p}, \mathbf{q} \in \mathcal{P}_n^+$ and $x \in H$, $\|x\| = 1$ we have

$$D_n(\mathbf{p}+\mathbf{q};\mathbf{A},x) \geq D_n(\mathbf{p};\mathbf{A},x)\, D_n(\mathbf{q};\mathbf{A},x) \geq 1.$$

Moreover, if $\mathbf{p}, \mathbf{q} \in \mathcal{P}_n^+$ with $\mathbf{p} \geq \mathbf{q}$, then also

$$D_n(\mathbf{p};\mathbf{A},x) \geq D_n(\mathbf{q};\mathbf{A},x) \geq 1$$

for all $x \in H$, $\|x\| = 1$. Some upper bounds for $D_n(\mathbf{p};\mathbf{A},x)$ under boundedness assumptions for $\mathbf{A}$ are also provided.

## 1. Introduction

Let $B(H)$ be the space of all bounded linear operators on a Hilbert space $H$, and $I$ stands for the identity operator on $H$. An operator $A$ in $B(H)$ is said to be positive (in symbol: $A \geq 0$) if $\langle Ax, x\rangle \geq 0$ for all $x \in H$. In particular, $A > 0$ means that $A$ is positive and invertible. For a pair $A$, $B$ of selfadjoint operators the order relation $A \geq B$ means as usual that $A - B$ is positive.

In 1998, Fujii et al. [7], [8], introduced the *normalized determinant* $\Delta_x(A)$ for positive invertible operators $A$ on a Hilbert space $H$ and a fixed unit vector $x \in H$, namely $\|x\| = 1$, defined by $\Delta_x(A) := \exp \langle \ln Ax, x\rangle$ and discussed it as a continuous geometric mean and observed some inequalities around the determinant from this point of view.

Some of the fundamental properties of normalized determinant are as follows, [7].

For each unit vector $x \in H$, see also [9], we have:

(i) *continuity*: the map $A \to \Delta_x(A)$ is norm continuous;
(ii) *bounds*: $\left\langle A^{-1}x, x\right\rangle^{-1} \leq \Delta_x(A) \leq \langle Ax, x\rangle$;

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(iii) *continuous mean*: $\langle A^p x, x\rangle^{1/p} \downarrow \Delta_x(A)$ for $p \downarrow 0$ and $\langle A^p x, x\rangle^{1/p} \uparrow \Delta_x(A)$ for $p \uparrow 0$;
(iv) *power equality:* $\Delta_x(A^t) = \Delta_x(A)^t$ for all $t > 0$;
(v) *homogeneity*: $\Delta_x(tA) = t\Delta_x(A)$ and $\Delta_x(tI) = t$ for all $t > 0$;
(vi) *monotonicity*: $0 < A \le B$ implies $\Delta_x(A) \le \Delta_x(B)$;
(vii) *multiplicativity*: $\Delta_x(AB) = \Delta_x(A)\Delta_x(B)$ for commuting $A$ and $B$;
(viii) *Ky Fan type inequality*: $\Delta_x((1-\alpha)A + \alpha B) \ge \Delta_x(A)^{1-\alpha}\Delta_x(B)^{\alpha}$ for $0 < \alpha < 1$.

We define the logarithmic mean of two positive numbers $a$, $b$ by

$$L(a,b) := \begin{cases} \frac{b-a}{\ln b - \ln a} \text{ if } b \neq a, \\ \\ a \text{ if } b = a. \end{cases} \tag{1.1}$$

In [7] the authors obtained the following additive reverse inequality for the operator $A$ which satisfy the condition $0 < mI \le A \le MI$, where $m, M$ are positive numbers,

$$0 \le \langle Ax, x\rangle - \Delta_x(A) \le L(m,M)\left[\ln L(m,M) + \frac{M\ln m - m\ln M}{M-m} - 1\right] \tag{1.2}$$

for all $x \in H$, $\|x\| = 1$.

We recall that *Specht's ratio* is defined by [13]

$$S(h) := \begin{cases} \frac{h^{\frac{1}{h-1}}}{e\ln\left(h^{\frac{1}{h-1}}\right)} \text{ if } h \in (0,1)\cup(1,\infty), \\ \\ 1 \text{ if } h = 1. \end{cases} \tag{1.3}$$

It is well known that $\lim_{h\to 1} S(h) = 1$, $S(h) = S\left(\frac{1}{h}\right) > 1$ for $h > 0$, $h \neq 1$. The function is decreasing on $(0,1)$ and increasing on $(1,\infty)$.

In [8], the authors obtained the following multiplicative reverse inequality as well

$$1 \le \frac{\langle Ax, x\rangle}{\Delta_x(A)} \le S\left(\frac{M}{m}\right) \tag{1.4}$$

for $0 < mI \le A \le MI$ and $x \in H$, $\|x\| = 1$.

In this paper we obtain several refinements and reverses for the normalized determinant of a sequence of operators that have the spectra in a positive interval $[m, M]$. For this purpose we used some Jensen's type inequalities for twice differentiable functions obtained by the author in [3].

We consider the functional

$$D_n(\mathbf{p};\mathbf{A},x) := \frac{\left[\Delta_x\left(\frac{1}{P_n}\sum_{j=1}^n p_j A_j\right)\right]^{P_n}}{\prod_{i=1}^n \left[\Delta_x(A_j)\right]^{p_i}},$$

where $\mathbf{A} = (A_1, ..., A_n)$ is an $n$-tuple of selfadjoint positive operators, $\mathbf{p} \in \mathcal{P}_n^+$ the set of nonnegative $n$-tuples and $x \in H$, $\|x\| = 1$.

In this paper we show among other that, for any $\mathbf{p}, \mathbf{q} \in \mathcal{P}_n^+$ and $x \in H$, $\|x\| = 1$ we have

$$D_n(\mathbf{p}+\mathbf{q};\mathbf{A},x) \ge D_n(\mathbf{p};\mathbf{A},x)\, D_n(\mathbf{q};\mathbf{A},x) \ge 1.$$

Moreover, if $\mathbf{p},\mathbf{q}\in\mathcal{P}_n^+$ with $\mathbf{p}\geq\mathbf{q}$, then also

$$D_n(\mathbf{p};\mathbf{A},x)\geq D_n(\mathbf{q};\mathbf{A},x)\geq 1$$

for all $x\in H$, $\|x\|=1$.

Some upper bounds for $D_n(\mathbf{p};\mathbf{A},x)$ under boundedness assumptions for $\mathbf{A}$ are also provided.

## 2. Main Results

We consider the functional

$$J_n(\mathbf{p};\mathbf{A},f,I):=\sum_{j=1}^n p_j f(A_j)-P_n f\left(\frac{1}{P_n}\sum_{j=1}^n p_j A_j\right) \tag{2.1}$$

where $\mathbf{p}=(p_1,...,p_n)$, $p_j\geq 0$ with $j\in\{1,...,n\}$ and $P_n>0$, $\mathbf{A}=(A_1,...,A_n)$ is an $n$-tuple of selfadjoint operators with $Sp(A_j)\subseteq I$ for $j\in\{1,...,n\}$ and $f:I\to\mathbb{R}$ is a operator convex function defined on the interval $I$.

We denote by $\mathcal{P}_n^+$ the set of all $n$-tuples $\mathbf{p}=(p_1,...,p_n)$, $p_j\geq 0$ with $j\in\{1,...,n\}$ and $P_n>0$. For $\mathbf{p},\mathbf{q}\in\mathcal{P}_n^+$ we denote $\mathbf{p}\geq\mathbf{q}$ if $p_j\geq q_j$ for any $j\in\{1,...,n\}$.

In [4] we obtained the following result:

**Lemma 1.** *Assume that $f:I\to\mathbb{R}$ is an operator convex function and $\mathbf{A}=(A_1,...,A_n)$ an $n$-tuple of selfadjoint operators with $Sp(A_j)\subseteq I$, then for any $\mathbf{p},\mathbf{q}\in\mathcal{P}_n^+$ we have*

$$J_n(\mathbf{p}+\mathbf{q};\mathbf{A},f,I)\geq J_n(\mathbf{p};\mathbf{A},f,I)+J_n(\mathbf{q};\mathbf{A},f,I)\geq 0, \tag{2.2}$$

*i.e., $J_n(\cdot;\mathbf{A},f,I)$ is a super-additive functional in the operator order.*

**Theorem 1.** *Moreover, if $\mathbf{p},\mathbf{q}\in\mathcal{P}_n^+$ with $\mathbf{p}\geq\mathbf{q}$, then also*

$$J_n(\mathbf{p};\mathbf{A},f,I)\geq J_n(\mathbf{q};\mathbf{A},f,I)\geq 0, \tag{2.3}$$

*i.e., $J_n(\cdot;\mathbf{A},f,I)$ is a monotonic functional in the operator order.*

**Corollary 1.** *Assume that the function $f:I\to\mathbb{R}$ is operator convex and the $n$-tuple of selfadjoint operators $(A_1,...,A_n)$ satisfies the condition $Sp(A_j)\subseteq I$ for any $j\in\{1,...,n\}$. If $\mathbf{p},\mathbf{q}\in\mathcal{P}_n^+$ and there exists the positive constants $m,M$ such that*

$$m\mathbf{q}\leq\mathbf{p}\leq M\mathbf{q} \tag{2.4}$$

*then*

$$mJ_n(\mathbf{q};\mathbf{A},f,I)\leq J_n(\mathbf{p};\mathbf{A},f,I)\leq MJ_n(\mathbf{q};\mathbf{A},f,I) \tag{2.5}$$

*in the operator order.*

**Remark 1.** *We observe that if all $q_j>0$ then we have the inequality*

$$\begin{aligned}\min_{j\in\{1,\dots,n\}}\left\{\frac{p_j}{q_j}\right\}J_n(\mathbf{q};\mathbf{A},f,I)&\leq J_n(\mathbf{p};\mathbf{A},f,I)\\ &\leq\max_{j\in\{1,\dots,n\}}\left\{\frac{p_j}{q_j}\right\}J_n(\mathbf{q};\mathbf{A},f,I)\end{aligned} \tag{2.6}$$

*in the operator order.*

*In particular, if $\mathbf{q}$ is the uniform distribution, i.e., $q_j=\frac{1}{n}, j\in\{1,...,n\}$, then we have the inequalities*

$$n\min_{j\in\{1,\dots,n\}}\{p_j\}J_n(\mathbf{A},f,I)\leq J_n(\mathbf{p};\mathbf{A},f,I)\leq n\max_{j\in\{1,\dots,n\}}\{p_j\}J_n(\mathbf{A},f,I), \tag{2.7}$$

*where*

$$J_n(\mathbf{A}, f, I) := \frac{1}{n}\sum_{j=1}^{n} f(A_j) - f\left(\frac{1}{n}\sum_{j=1}^{n} A_j\right). \tag{2.8}$$

*For $n = 2$ and by choosing $p_1 = \alpha, p_2 = 1-\alpha$ with $\alpha \in [0,1]$, we get from (2.7) the inequality*

$$\begin{aligned} &2\min\{\alpha, 1-\alpha\}\left[\frac{f(A)+f(B)}{2} - f\left(\frac{A+B}{2}\right)\right] \\ &\leq (1-\alpha) f(A) + \alpha f(B) - f((1-\alpha)A + \alpha B) \\ &\leq 2\max\{\alpha, 1-\alpha\}\left[\frac{f(A)+f(B)}{2} - f\left(\frac{A+B}{2}\right)\right], \end{aligned} \tag{2.9}$$

*in the operator order, where $f : I \to \mathbb{R}$ is an operator convex function and $A$ and $B$ are two bounded selfadjoint operators on the complex Hilbert space $H$ with $Sp(A), Sp(B) \subseteq I$.*

We define the functional

$$D_n(\mathbf{p}; \mathbf{A}, x) := \frac{\left[\Delta_x(\frac{1}{P_n}\sum_{j=1}^{n} p_j A_j)\right]^{P_n}}{\prod_{i=1}^{n}[\Delta_x(A_j)]^{p_i}}, \tag{2.10}$$

where $\mathbf{A} = (A_1, ..., A_n)$ is an $n$-tuple of selfadjoint positive operators $\mathbf{p} \in \mathcal{P}_n^+$ and $x \in H$, $\|x\| = 1$.

Our first main results is as follows:

**Theorem 2.** *Assume that $\mathbf{A} = (A_1, ..., A_n)$ an $n$-tuple of selfadjoint positive operators, then for any $\mathbf{p}, \mathbf{q} \in \mathcal{P}_n^+$ and $x \in H$, $\|x\| = 1$ we have*

$$D_n(\mathbf{p}+\mathbf{q}; \mathbf{A}, x) \geq D_n(\mathbf{p}; \mathbf{A}, x) D_n(\mathbf{q}; \mathbf{A}, x) \geq 1, \tag{2.11}$$

*i.e., $D_n(\cdot; \mathbf{A}, x)$ is a super-multiplicative functional.*

*Moreover, if $\mathbf{p}, \mathbf{q} \in \mathcal{P}_n^+$ with $\mathbf{p} \geq \mathbf{q}$, then also*

$$D_n(\mathbf{p}; \mathbf{A}, x) \geq D_n(\mathbf{q}; \mathbf{A}, x) \geq 1 \tag{2.12}$$

*for all $x \in H$, $\|x\| = 1$, i.e., $D_n(\cdot; \mathbf{A}, x)$ is a monotonic non-decreasing functional.*

*Proof.* For the operator convex function $f(t) = -\ln t$, $t > 0$, we have have

$$\begin{aligned} J_n(\mathbf{p}; \mathbf{A}, -\ln) &:= J_n(\mathbf{p}; \mathbf{A}, -\ln, (0,\infty)) \\ &= P_n \ln\left(\frac{1}{P_n}\sum_{j=1}^{n} p_j A_j\right) - \sum_{j=1}^{n} p_j \ln(A_j). \end{aligned}$$

For $x \in H$, $\|x\| = 1$ we have

$$\langle J_n(\mathbf{p}; \mathbf{A}, -\ln)x, x\rangle = P_n\left\langle \ln\left(\frac{1}{P_n}\sum_{j=1}^{n} p_j A_j\right)x, x\right\rangle - \sum_{j=1}^{n} p_j \langle \ln(A_j)x, x\rangle.$$

If we take the exponential, then we get

$$\begin{aligned}
(2.13)\qquad & \exp\left[\left\langle J_n\left(\mathbf{p};\mathbf{A},-\ln\right)x,x\right\rangle\right] \\
&= \exp\left[P_n\left\langle \ln\left(\frac{1}{P_n}\sum_{j=1}^{n}p_jA_j\right)x,x\right\rangle-\sum_{j=1}^{n}p_j\left\langle \ln\left(A_j\right)x,x\right\rangle\right] \\
&= \frac{\exp\left[P_n\left\langle \ln\left(\frac{1}{P_n}\sum_{j=1}^{n}p_jA_j\right)x,x\right\rangle\right]}{\exp\left[\sum_{j=1}^{n}p_j\left\langle \ln\left(A_j\right)x,x\right\rangle\right]} \\
&= \frac{\left(\exp\left\langle \ln\left(\frac{1}{P_n}\sum_{j=1}^{n}p_jA_j\right)x,x\right\rangle\right)^{P_n}}{\prod_{i=1}^{n}\left[\exp\left\langle \ln\left(A_j\right)x,x\right\rangle\right]^{p_i}} = \frac{\left[\Delta_x\left(\frac{1}{P_n}\sum_{j=1}^{n}p_jA_j\right)\right]^{P_n}}{\prod_{i=1}^{n}\left[\Delta_x\left(A_j\right)\right]^{p_i}} \\
&= D_n\left(\mathbf{p};\mathbf{A},x\right).
\end{aligned}$$

For $\mathbf{p},\mathbf{q}\in\mathcal{P}_n^+$ we have by (2.13) and Lemma 1

$$\begin{aligned}
D_n\left(\mathbf{p}+\mathbf{q};\mathbf{A},x\right) &= \exp\left[\left\langle J_n\left(\mathbf{p}+\mathbf{q};\mathbf{A},-\ln\right)x,x\right\rangle\right] \\
&\geq \exp\left[\left\langle J_n\left(\mathbf{p};\mathbf{A},-\ln\right)x,x\right\rangle+\left\langle J_n\left(\mathbf{q};\mathbf{A},-\ln\right)x,x\right\rangle\right] \\
&= \exp\left\langle J_n\left(\mathbf{p};\mathbf{A},-\ln\right)x,x\right\rangle\exp\left\langle J_n\left(\mathbf{q};\mathbf{A},-\ln\right)x,x\right\rangle \\
&= D_n\left(\mathbf{p};\mathbf{A},x\right)D_n\left(\mathbf{q};\mathbf{A},x\right),
\end{aligned}$$

for all $x\in H$, $\|x\|=1$, which proves (2.11).

The property (2.12) follows in a similar way by (2.3). □

**Corollary 2.** *With the assumptions of Theorem 2, if $\mathbf{p},\mathbf{q}\in\mathcal{P}_n^+$ and there exists the positive constants $m,M$ such that $m\mathbf{q}\leq\mathbf{p}\leq M\mathbf{q}$, then*

$$(2.14)\qquad 1\leq\left[D_n\left(\mathbf{q};\mathbf{A},x\right)\right]^m\leq D_n\left(\mathbf{p};\mathbf{A},x\right)\leq\left[D_n\left(\mathbf{q};\mathbf{A},x\right)\right]^M$$

*for all $x\in H$, $\|x\|=1$.*

**Remark 2.** *We observe that if all $q_j>0$ then we have the inequality*

$$\begin{aligned}
(2.15)\qquad 1 &\leq\left[D_n\left(\mathbf{q};\mathbf{A},x\right)\right]^{\min_{j\in\{1,\dots,n\}}\left\{\frac{p_j}{q_j}\right\}} \\
&\leq D_n\left(\mathbf{p};\mathbf{A},x\right)\leq\left[D_n\left(\mathbf{q};\mathbf{A},x\right)\right]^{\max_{j\in\{1,\dots,n\}}\left\{\frac{p_j}{q_j}\right\}}
\end{aligned}$$

*for all $x\in H$, $\|x\|=1$*

*In particular, if $\mathbf{q}$ is the uniform distribution, i.e., $q_j=\frac{1}{n}, j\in\{1,...,n\}$, then we have the inequalities*

$$\begin{aligned}
(2.16)\qquad 1 &\leq\left[D_n\left(\mathbf{A},x\right)\right]^{n\min_{j\in\{1,\dots,n\}}\{p_j\}} \\
&\leq D_n\left(\mathbf{p};\mathbf{A},x\right)\leq\left[D_n\left(\mathbf{A},x\right)\right]^{n\max_{j\in\{1,\dots,n\}}\{p_j\}}
\end{aligned}$$

*for all $x\in H$, $\|x\|=1$, where*

$$D_n\left(\mathbf{A},x\right):=\frac{\Delta_x\left(\frac{1}{n}\sum_{j=1}^{n}A_j\right)}{\prod_{i=1}^{n}\left[\Delta_x\left(A_j\right)\right]^{1/n}}.$$

*For $n=2$ and by choosing $p_1=\alpha, p_2=1-\alpha$ with $\alpha\in[0,1]$, we get from (2.16) the inequality for two positive operators $A$, $B$*

$$1\leq \left(\frac{\Delta_x\left(\frac{A+B}{2}\right)}{\left[\Delta_x(A)\right]^{1/2}\left[\Delta_x(B)\right]^{1/2}}\right)^{2\min\{\alpha,1-\alpha\}} \tag{2.17}$$
$$\leq \frac{\Delta_x((1-\alpha)A+\alpha B)}{\left[\Delta_x(A)\right]^{1-\alpha}\left[\Delta_x(B)\right]^{\alpha}}\leq \left(\frac{\Delta_x\left(\frac{A+B}{2}\right)}{\left[\Delta_x(A)\right]^{1/2}\left[\Delta_x(B)\right]^{1/2}}\right)^{2\max\{\alpha,1-\alpha\}}.$$

*This provides a refinement and a reverse of Ky Fan's inequality (viii) from Introduction.*

Let $\mathcal{P}_f(\mathbb{N})$ be the family of finite parts of the set of natural numbers $\mathbb{N}$, $\mathcal{A}(H)$ the linear space of all sequences of selfadjoint operators defined on the complex Hilbert space, i.e.,

$$\mathcal{A}(H)=\left\{\mathbf{A}=(A_k)_{k\in\mathbb{N}}\mid A_k \text{ are selfadjoint operators on } H \text{ for all } k\in\mathbb{N}\right\}$$

and $\mathcal{S}_+(\mathbb{R})$ the family of nonnegative real sequences.

We consider the functional

$$J(K,\mathbf{p};\mathbf{A},f,I):=\sum_{k\in K}p_kf(A_k)-P_Kf\left(\frac{1}{P_K}\sum_{k\in K}p_kA_k\right) \tag{2.18}$$

where $K\in\mathcal{P}_f(\mathbb{N})$, $\mathbf{p}\in\mathcal{S}_+(\mathbb{R})$, $\mathbf{A}\in\mathcal{A}(H)$ with $P_K:=\sum_{k\in K}p_k>0$ and $f:I\to\mathbb{R}$ is a operator convex function on the interval $I$.

**Lemma 2.** *Let $f:I\to\mathbb{R}$ be an operator convex function on the interval $I$ and $\mathbf{p}\in\mathcal{S}_+(\mathbb{R})$, $\mathbf{A}\in\mathcal{A}(H)$. Assume that $Sp(A_k)\subseteq I$ for any $k\in\mathbb{N}$.*

*If $K,L\in\mathcal{P}_f(\mathbb{N})\setminus\{\emptyset\}$ with $K\cap L=\emptyset$ and $P_K,P_L>0$, then we have the inequality*

$$J(K\cup L,\mathbf{p};\mathbf{A},f,I)\geq J(K,\mathbf{p};\mathbf{A},f,I)+J(L,\mathbf{p};\mathbf{A},f,I)\geq 0, \tag{2.19}$$

*i.e., $J(\cdot,\mathbf{p};\mathbf{A},f,I)$ is super-additive as an index set functional in the operator order.*

*If $\emptyset\neq K\subset L$ then we have*

$$J(L,\mathbf{p};\mathbf{A},f,I)\geq J(K,\mathbf{p};\mathbf{A},f,I)\geq 0, \tag{2.20}$$

*i.e., $J(\cdot,\mathbf{p};\mathbf{A},f,I)$ is monotonic as an index set functional in the operator order.*

In particular, we have:

**Corollary 3.** *Let $f:I\to\mathbb{R}$ be an operator convex function on the interval $I$ and $\mathbf{p}=(p_1,...,p_n)$, $\mathbf{A}=(A_1,...,A_n)$ with $p_k>0$, $A_k$ selfadjoint operators and such that $Sp(A_k)\subseteq I$ for any $k\in\{1,...,n\}$, $n\geq 2$. Then we have the inequality*

$$J_k(\mathbf{p};\mathbf{A},f,I)\geq J_{k-1}(\mathbf{p};\mathbf{A},f,I)\geq 0 \tag{2.21}$$

*for any $k\in\{1,...,n\}$ with $n\geq k\geq 2$.*

*We also have that*

$$J_n(\mathbf{p};\mathbf{A},f,I)\geq p_jf(A_j)+p_kf(A_k)-(p_j+p_k)f\left(\frac{p_jA_j+p_kA_k}{p_j+p_k}\right)\geq 0 \tag{2.22}$$

*for any $k,j\in\{1,...,n\}$ in the operator order.*

We define the functional

$$D_K(\mathbf{p};\mathbf{A},x) := \frac{\left[\Delta_x\left(\frac{1}{P_K}\sum_{k\in K} p_k A_k\right)\right]^{P_K}}{\prod_{k\in K}\left[\Delta_x(A_k)\right]^{p_k}}, \tag{2.23}$$

where $\mathbf{A} \in \mathcal{A}(H)$ is a sequence of selfadjoint positive operators, $K \in \mathcal{P}_f(\mathbb{N})$, $\mathbf{p} \in \mathcal{S}_+(\mathbb{R})$ and $x \in H$, $\|x\| = 1$.

**Theorem 3.** *Assume that $\mathbf{A} \in \mathcal{A}(H)$ is a sequence of selfadjoint positive operators, $\mathbf{p} \in \mathcal{S}_+(\mathbb{R})$ and $x \in H$ with $\|x\| = 1$. If $K, L \in \mathcal{P}_f(\mathbb{N}) \setminus \{\emptyset\}$ with $K \cap L = \emptyset$ and $P_K, P_L > 0$, then we have the inequality*

$$D_{K\cup L}(\mathbf{p};\mathbf{A},x) \geq D_K(\mathbf{p};\mathbf{A},x)\, D_L(\mathbf{p};\mathbf{A},x) \geq 1, \tag{2.24}$$

*i.e., $D_\cdot(\mathbf{p};\mathbf{A},x)$ is super-multiplicative as an index set functional.*

*If $\emptyset \neq K \subset L$, then we have*

$$D_L(\mathbf{p};\mathbf{A},x) \geq D_K(\mathbf{p};\mathbf{A},x) \geq 1 \tag{2.25}$$

*i.e., $D_\cdot(\mathbf{p};\mathbf{A},x)$ is monotonic as an index set functional.*

The proof is similar to the one in Theorem 2 by employing the inequalities in Lemma 2.

**Corollary 4.** *Assume that $\mathbf{A} = (A_1, ..., A_n)$ is an $n$-tuple of selfadjoint positive operators, $\mathbf{p} \in \mathcal{P}_n^+$ and $x \in H$ with $\|x\| = 1$. Then we have the inequality*

$$D_k(\mathbf{p};\mathbf{A},x) \geq D_{k-1}(\mathbf{p};\mathbf{A},x) \geq 1 \tag{2.26}$$

*for any $k \in \{1, ..., n\}$ with $n \geq k \geq 2$.*

*Also, we have*

$$D_n(\mathbf{p};\mathbf{A},x) \geq \max_{k,j\in\{1,\ldots,n\}} \frac{\left[\Delta_x\left(\frac{p_k A_k + p_j A_j}{p_j + p_k}\right)\right]^{p_j+p_k}}{\left[\Delta_x(A_k)\right]^{p_k}\left[\Delta_x(A_j)\right]^{p_j}} \geq 1. \tag{2.27}$$

## 3. Related Results

In [4] we also obtained the following result:

**Lemma 3.** *If the function $f : [m, M] \to \mathbb{R}$ is operator convex and if the $n$-tuple of selfadjoint operators $(A_1, ..., A_n)$ has the property that $\operatorname{Sp}(A_j) \subseteq [m, M]$ for any $j \in \{1, ..., n\}$, then for any $p_j \geq 0$ with $j \in \{1, ..., n\}$ and $P_n := \sum_{j=1}^n p_j > 0$ we have*

$$\begin{aligned} 0 &\leq \frac{1}{P_n}\sum_{j=1}^n p_j f(A_j) - f\left(\frac{1}{P_n}\sum_{j=1}^n p_j A_j\right) \\ &\leq \frac{2}{M-m}\left[\frac{f(m)+f(M)}{2} - f\left(\frac{m+M}{2}\right)\right] \\ &\quad \times \left(\frac{1}{2}(M-m)1_H + \left|\frac{1}{P_n}\sum_{j=1}^n p_j A_j - \frac{m+M}{2}1_H\right|\right) \\ &\leq \frac{2}{M-m}\left[\frac{f(m)+f(M)}{2} - f\left(\frac{m+M}{2}\right)\right]1_H \end{aligned} \tag{3.1}$$

*in the operator order.*

We also have:

**Theorem 4.** *Assume that* $\mathbf{A}=(A_1,...,A_n)$ *an* $n$*-tuple of selfadjoint operators with spectra in* $[m,M]\subset(0,\infty)$, *then for any* $\mathbf{p}\in\mathcal{P}_n^+$ *and* $x\in H$, $\|x\|=1$ *we have*

$$
\begin{aligned}
1 &\le D_n(\mathbf{p};\mathbf{A},x) \\
&\le \left(\frac{m+M}{2\sqrt{mM}}\right)^{\left(1+\frac{2}{M-m}\left\langle\left|\frac{1}{P_n}\sum_{j=1}^n p_jA_j-\frac{m+M}{2}1_H\right|x,x\right\rangle\right)} \\
&\le \left(\frac{m+M}{2\sqrt{mM}}\right)^{\frac{2}{M-m}}.
\end{aligned}
\tag{3.2}
$$

*Proof.* We write the inequality (3.1) for the operator convex function $f(t)=-\ln t$, $t>0$ to get

$$
\begin{aligned}
0 &\le \ln\left(\frac{1}{P_n}\sum_{j=1}^n p_jA_j\right)-\frac{1}{P_n}\sum_{j=1}^n p_j\ln A_j \\
&\le \ln\left(\frac{m+M}{2\sqrt{mM}}\right)^{\frac{2}{M-m}} \\
&\times\left(\frac{1}{2}(M-m)1_H+\left|\frac{1}{P_n}\sum_{j=1}^n p_jA_j-\frac{m+M}{2}1_H\right|\right) \\
&\le \ln\left(\frac{m+M}{2\sqrt{mM}}\right)^{\frac{2}{M-m}}1_H.
\end{aligned}
\tag{3.3}
$$

If we take the inner product for $x\in H$, $\|x\|=1$, then we get

$$
\begin{aligned}
0 &\le \left\langle\ln\left(\frac{1}{P_n}\sum_{j=1}^n p_jA_j\right)x,x\right\rangle-\frac{1}{P_n}\sum_{j=1}^n p_j\langle\ln A_jx,x\rangle \\
&\le \ln\left(\frac{m+M}{2\sqrt{mM}}\right)^{\frac{2}{M-m}} \\
&\times\left(\frac{1}{2}(M-m)1_H+\left\langle\left|\frac{1}{P_n}\sum_{j=1}^n p_jA_j-\frac{m+M}{2}1_H\right|x,x\right\rangle\right) \\
&\le \ln\left(\frac{m+M}{2\sqrt{mM}}\right)^{\frac{2}{M-m}}.
\end{aligned}
\tag{3.4}
$$

If we take the exponential in (3.4), then we get

$$
1\le\frac{\exp\left\langle\ln\left(\frac{1}{P_n}\sum_{j=1}^n p_jA_j\right)x,x\right\rangle}{\exp\left(\frac{1}{P_n}\sum_{j=1}^n p_j\langle\ln A_jx,x\rangle\right)}
$$

$$\leq \exp\left[\ln\left(\frac{m+M}{2\sqrt{mM}}\right)^{\frac{2}{M-m}} \times\left(\frac{1}{2}(M-m)1_H+\left\langle\left|\frac{1}{P_n}\sum_{j=1}^{n}p_jA_j-\frac{m+M}{2}1_H\right|x,x\right\rangle\right)\right] \leq\left(\frac{m+M}{2\sqrt{mM}}\right)^{\frac{2}{M-m}},$$

namely

$$\begin{aligned} 1 &\leq \frac{\left[\Delta_x(\frac{1}{P_n}\sum_{j=1}^{n}p_jA_j)\right]^{P_n}}{\prod_{i=1}^{n}\left[\Delta_x(A_j)\right]^{p_i}} \\ &\leq \left(\frac{m+M}{2\sqrt{mM}}\right)^{\left(1+\frac{2}{M-m}\left\langle\left|\frac{1}{P_n}\sum_{j=1}^{n}p_jA_j-\frac{m+M}{2}1_H\right|x,x\right\rangle\right)} \\ &\leq \left(\frac{m+M}{2\sqrt{mM}}\right)^{\frac{2}{M-m}} \end{aligned}$$

and the inequality (3.2) is proved. □

**Remark 3.** *The case of two operators is as follows: if* $0<m\leq A,\ B\leq M$ *and* $\alpha\in[0,1]$*, then*

$$\begin{aligned} 1 &\leq \frac{\Delta_x((1-\alpha)A+\alpha B)}{\left[\Delta_x(A)\right]^{1-\alpha}\left[\Delta_x(B)\right]^{\alpha}} \\ &\leq \left(\frac{m+M}{2\sqrt{mM}}\right)^{\left(1+\frac{2}{M-m}\left\langle\left|(1-\alpha)A+\alpha B-\frac{m+M}{2}1_H\right|x,x\right\rangle\right)} \\ &\leq \left(\frac{m+M}{2\sqrt{mM}}\right)^{\frac{2}{M-m}} \end{aligned} \tag{3.5}$$

*for all* $x\in H$, $\|x\|=1$.

**Corollary 5.** *If* $0<m\leq A,\ B\leq M$ *and* $x\in H$, $\|x\|=1$*, then*

$$\begin{aligned} L\left(\Delta_x(A),\Delta_x(B)\right) &\leq \int_0^1\Delta_x((1-t)A+tB)dt \\ &\leq \left(\frac{m+M}{2\sqrt{mM}}\right)^{\frac{2}{M-m}} L\left(\Delta_x(A),\Delta_x(B)\right), \end{aligned} \tag{3.6}$$

*where* $L(\cdot,\cdot)$ *is the logarithmic mean (1.1).*

*Proof.* From (3.5) we have

$$\begin{aligned} \left[\Delta_x(A)\right]^{1-t}\left[\Delta_x(B)\right]^{t} &\leq \Delta_x((1-t)A+tB) \\ &\leq \left(\frac{m+M}{2\sqrt{mM}}\right)^{\frac{2}{M-m}}\left[\Delta_x(A)\right]^{1-t}\left[\Delta_x(B)\right]^{t}, \end{aligned}$$

for all $t\in[0,1]$.

If we take the integral over $t$, then we get

$$\int_0^1 [\Delta_x(A)]^{1-t}[\Delta_x(B)]^t dt \le \int_0^1 \Delta_x((1-t)A+tB)dt \le \left(\frac{m+M}{2\sqrt{mM}}\right)^{\frac{2}{M-m}} \int_0^1 [\Delta_x(A)]^{1-t}[\Delta_x(B)]^t dt. \tag{3.7}$$

Since

$$\int_0^1 [\Delta_x(A)]^{1-t}[\Delta_x(B)]^t dt = L(\Delta_x(A),\Delta_x(B)),$$

hence by (3.7) we derive (3.6). □

In we obtained among other, the following reverse of Jensen's inequality:

**Lemma 4.** *Let $f:[m,M]\to\mathbb{R}$ be an operator convex function on $[m,M]$ and $A_j$ selfadjoint operators with the spectrum $\operatorname{Sp}(A_j)\subset[m,M]$ for $j=1,...,k$. If $C_j\in\mathcal{B}(H)$ for $j=1,...,n$ satisfying the condition $\sum_{j=1}^n C_j^*C_j=1_H$, then*

$$\begin{aligned} 0 &\le \sum_{j=1}^n C_j^* f(A_j)C_j - f\left(\sum_{j=1}^n C_j^*A_jC_j\right) \\ &\le \frac{f'_-(M)-f'_+(m)}{M-m}\left(M1_H-\sum_{j=1}^n C_j^*A_jC_j\right)\left(\sum_{j=1}^n C_j^*A_jC_j - m1_H\right) \\ &\le \frac{1}{4}(M-m)\left[f'_-(M)-f'_+(m)\right]1_H. \end{aligned} \tag{3.8}$$

By the use of this lemma we can state the following result as well:

**Theorem 5.** *Assume that $\mathbf{A}=(A_1,...,A_n)$ an $n$-tuple of selfadjoint operators with spectra in $[m,M]\subset(0,\infty)$, then for any $\mathbf{p}\in\mathcal{P}_n^+$ and $x\in H$, $\|x\|=1$ we have*

$$\begin{aligned} 1 &\le D_n(\mathbf{p};\mathbf{A},x) \\ &\le \exp\left[\frac{1}{mM}\left(M-\frac{1}{P_n}\sum_{j=1}^n p_j\langle A_jx,x\rangle\right)\left(\frac{1}{P_n}\sum_{j=1}^n p_j\langle A_jx,x\rangle - m\right)\right] \\ &\le \exp\left[\frac{1}{4mM}(M-m)^2\right]. \end{aligned} \tag{3.9}$$

*Proof.* If we take in (3.8) $C_j=\sqrt{\frac{p_j}{P_n}}I$, $j=1,...,n$, then we get

$$\begin{aligned} 0 &\le \frac{1}{P_n}\sum_{j=1}^n p_j f(A_j) - f\left(\sum_{j=1}^n p_jA_j\right) \\ &\le \frac{f'_-(M)-f'_+(m)}{M-m} \\ &\times \left(M1_H-\frac{1}{P_n}\sum_{j=1}^n p_jA_j\right)\left(\frac{1}{P_n}\sum_{j=1}^n p_jA_j - m1_H\right) \\ &\le \frac{1}{4}(M-m)\left[f'_-(M)-f'_+(m)\right]1_H. \end{aligned}$$

If we take the inner product for $x \in H$, $\|x\| = 1$, then we get

$$\begin{aligned}
0 &\le \frac{1}{P_n}\sum_{j=1}^{n} p_j \langle f(A_j)x,x\rangle - \left\langle f\left(\sum_{j=1}^{n} p_j A_j\right)x,x\right\rangle \\
&\le \frac{f'_-(M)-f'_+(m)}{M-m} \\
&\times \left(M - \frac{1}{P_n}\sum_{j=1}^{n} p_j \langle A_j x,x\rangle\right)\left(\frac{1}{P_n}\sum_{j=1}^{n} p_j \langle A_j x,x\rangle - m1_H\right) \\
&\le \frac{1}{4}(M-m)\left[f'_-(M)-f'_+(m)\right].
\end{aligned}$$

If we write this inequality for the operator convex function $f(t) = -\ln t$, $t > 0$, then we get

$$\begin{aligned}
0 &\le \left\langle \ln\left(\sum_{j=1}^{n} p_j A_j\right)x,x\right\rangle - \frac{1}{P_n}\sum_{j=1}^{n} p_j \langle f(A_j)x,x\rangle \\
&\le \frac{1}{mM}\left(M - \frac{1}{P_n}\sum_{j=1}^{n} p_j \langle A_j x,x\rangle\right)\left(\frac{1}{P_n}\sum_{j=1}^{n} p_j \langle A_j x,x\rangle - m\right) \\
&\le \frac{1}{4mM}(M-m)^2 .
\end{aligned} \tag{3.10}$$

Now, if we take the exponential in (3.10), then we get

$$\begin{aligned}
1 &\le \frac{\exp\left\langle \ln\left(\sum_{j=1}^{n} p_j A_j\right)x,x\right\rangle}{\exp\left(\frac{1}{P_n}\sum_{j=1}^{n} p_j \langle f(A_j)x,x\rangle\right)} \\
&\le \exp\left[\frac{1}{mM}\left(M - \frac{1}{P_n}\sum_{j=1}^{n} p_j \langle A_j x,x\rangle\right)\left(\frac{1}{P_n}\sum_{j=1}^{n} p_j \langle A_j x,x\rangle - m\right)\right] \\
&\le \exp\left[\frac{1}{4mM}(M-m)^2\right],
\end{aligned}$$

which proves the desired result (3.9). □

Finally, by the use of [5]

**Lemma 5.** *Let $f : [m,M] \to \mathbb{R}$ be an operator convex function on $[m,M]$ and $A_j$ selfadjoint operators with the spectrum $\operatorname{Sp}(A_j) \subset [m,M]$ for $j = 1,...,k$. If*

*$C_j \in \mathcal{B}(H)$ for $j = 1, ..., k$ satisfying the condition $\sum_{j=1}^{k} C_j^* C_j = 1_H$, then*

$$0 \leq \sum_{j=1}^{k} C_j^* f(A_j) C_j - f\left(\sum_{j=1}^{k} C_j^* A_j C_j\right) \tag{3.11}$$
$$\leq \frac{1}{2} \|f''\|_{[m,M],\infty} \left(M 1_H - \sum_{j=1}^{k} C_j^* A_j C_j\right) \left(\sum_{j=1}^{k} C_j^* A_j C_j - m 1_H\right)$$
$$\leq \frac{1}{8} \|f''\|_{[m,M],\infty} (M - m)^2 1_H$$

we can also state:

**Theorem 6.** *With the assumptions of Theorem 5, we have*

$$1 \leq D_n(\mathbf{p}; \mathbf{A}, x) \tag{3.12}$$
$$\leq \exp\left[\frac{1}{2m^2}\left(M - \frac{1}{P_n}\sum_{j=1}^{n} p_j \langle A_j x, x\rangle\right)\left(\frac{1}{P_n}\sum_{j=1}^{n} p_j \langle A_j x, x\rangle - m\right)\right]$$
$$\leq \exp\left[\frac{1}{8}\left(\frac{M}{m} - 1\right)^2\right].$$

The case of two operators is as follows: if $0 < m \leq A$, $B \leq M$ and $\alpha \in [0, 1]$, then from (3.9) we obtain

$$1 \leq \frac{\Delta_x((1-\alpha)A + \alpha B)}{[\Delta_x(A)]^{1-\alpha}[\Delta_x(B)]^{\alpha}} \tag{3.13}$$
$$\leq \exp\left[\frac{1}{mM}(M - (1-\alpha)\langle Ax, x\rangle - \alpha\langle Bx, x\rangle)\right.$$
$$\left.\times((1-\alpha)\langle Ax, x\rangle + \alpha\langle Bx, x\rangle - m)\right]$$
$$\leq \exp\left[\frac{1}{4mM}(M - m)^2\right]$$

for all $x \in H$, $\|x\| = 1$, while from (3.12) we derive

$$1 \leq \frac{\Delta_x((1-\alpha)A + \alpha B)}{[\Delta_x(A)]^{1-\alpha}[\Delta_x(B)]^{\alpha}} \tag{3.14}$$
$$\leq \exp\left[\frac{1}{2m^2}(M - (M - (1-\alpha)\langle Ax, x\rangle - \alpha\langle Bx, x\rangle))\right.$$
$$\left.\times((1-\alpha)\langle Ax, x\rangle + \alpha\langle Bx, x\rangle - m)\right]$$
$$\leq \exp\left[\frac{1}{8}\left(\frac{M}{m} - 1\right)^2\right]$$

for all $x \in H$, $\|x\| = 1$.

We observe that if $M > 2m$ then the bound in (3.13) is better than the one from (3.14). If $M < 2m$, then the conclusion is the other way around.

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