# SOME NEW INEQUALITIES FOR THE TRACE CLASS $P$-DETERMINANT OF POSITIVE OPERATORS IN HILBERT SPACES

SILVESTRU SEVER DRAGOMIR[1,2]

Abstract. Let $H$ be a complex Hilbert space. For a given operator $P \geq 0$ with $P \in \mathcal{B}_1(H)$, the trace class associated to $\mathcal{B}(H)$ and $\operatorname{tr}(P) = 1$, we define the $P$-*determinant* of the positive invertible operator $A$ by

$$\Delta_P(A) := \exp \operatorname{tr}(P \ln A).$$

In this paper we show among others that, if $P_i \geq 0$ with $P_i \in \mathcal{B}_1(H)$ and $\operatorname{tr}(P_i) = 1$ for $i \in \{1, ..., n\}$, $p_i \geq 0$ with $\sum_{i=1}^n p_i = 1$ and $0 < mI \leq A_i \leq MI$ for $i \in \{1, ..., n\}$, then

$$\begin{aligned} 1 &\leq \frac{\sum_{j=1}^n p_j \operatorname{tr}(P_j A_j)}{\prod_{j=1}^n [\Delta_{Pj}(A_j)]^{p_j}} \\ &\leq \exp\left[\frac{1}{2m^2}\left(M - \sum_{j=1}^n p_j \operatorname{tr}(P_j A_j)\right)\left(\sum_{j=1}^n p_j \operatorname{tr}(P_j A_j) - m\right)\right] \\ &\leq \exp\left[\frac{1}{8}\left(\frac{M}{m} - 1\right)^2\right]. \end{aligned}$$

## 1. Introduction

In 1952, in the paper [5], B. Fuglede and R. V. Kadison introduced the determinant of a (invertible) operator and established its fundamental properties. The notion generalizes the usual determinant and can be considered for any operator in a finite von Neumann algebra $(M, \tau)$ with a faithful normal trace.

Let $T \in M$ be normal and $|T| := (T^*T)^{1/2}$ its modulus. By the spectral theorem one can represent $T$ as an integral

$$T = \int_{\operatorname{Sp}(T)} \lambda dE(\lambda),$$

where $E(\lambda)$ is a projection valued measure and $\operatorname{Sp}(T)$ is the spectrum of $T$. The measure $\mu_T := \tau \circ E$ becomes a probability measure on the complex plane and has the support in the spectrum $\operatorname{Sp}(T)$.

For any $T \in M$ the Fuglede-Kadison determinant (*FK-determinant*) is defined by

$$\Delta_{FK}(T) := \exp\left(\int_0^\infty \ln t d\mu_{|T|}\right).$$

1991 *Mathematics Subject Classification.* 47A63, 26D15, 46C05.

*Key words and phrases.* Positive operators, Trace class operators, Determinants, Inequalities.



*RGMIA Res. Rep. Coll.* **25** (2022), Art. 15, 16 pp. Received 05/03/22

If $T$ is invertible, then

$$\Delta_{FK}(T) := \exp\left(\tau\left(\ln\left(|T|\right)\right)\right),$$

where $\ln(|T|)$ is defined by the use of functional calculus.

Let $B(H)$ be the space of all bounded linear operators on a Hilbert space $H$, and $I$ stands for the identity operator on $H$. An operator $A$ in $B(H)$ is said to be positive (in symbol: $A \geq 0$) if $\langle Ax, x\rangle \geq 0$ for all $x \in H$. In particular, $A > 0$ means that $A$ is positive and invertible. For a pair $A$, $B$ of selfadjoint operators the order relation $A \geq B$ means as usual that $A - B$ is positive.

In 1998, Fujii et al. [6], [7], introduced the *normalized determinant* $\Delta_x(A)$ for positive invertible operators $A$ on a Hilbert space $H$ and a fixed unit vector $x \in H$, namely $\|x\| = 1$, defined by

$$\Delta_x(A) := \exp\langle \ln Ax, x\rangle$$

and discussed it as a continuous geometric mean and observed some inequalities around the determinant from this point of view. For some recent results, see [10].

We need now some preparations for trace of operators in Hilbert spaces.

Let $(H, \langle\cdot,\cdot\rangle)$ be a complex Hilbert space and $\{e_i\}_{i\in I}$ an *orthonormal basis* of $H$. We say that $A \in \mathcal{B}(H)$ is a *Hilbert-Schmidt operator* if

$$\sum_{i\in I}\|Ae_i\|^2 < \infty. \tag{1.1}$$

It is well know that, if $\{e_i\}_{i\in I}$ and $\{f_j\}_{j\in J}$ are orthonormal bases for $H$ and $A \in \mathcal{B}(H)$ then

$$\sum_{i\in I}\|Ae_i\|^2 = \sum_{j\in I}\|Af_j\|^2 = \sum_{j\in I}\|A^*f_j\|^2 \tag{1.2}$$

showing that the definition (1.1) is independent of the orthonormal basis and $A$ is a Hilbert-Schmidt operator iff $A^*$ is a Hilbert-Schmidt operator.

Let $\mathcal{B}_2(H)$ the set of *Hilbert-Schmidt operators* in $\mathcal{B}(H)$. For $A \in \mathcal{B}_2(H)$ we define

$$\|A\|_2 := \left(\sum_{i\in I}\|Ae_i\|^2\right)^{1/2} \tag{1.3}$$

for $\{e_i\}_{i\in I}$ an orthonormal basis of $H$.

Using the triangle inequality in $l^2(I)$, one checks that $\mathcal{B}_2(H)$ is a *vector space* and that $\|\cdot\|_2$ is a norm on $\mathcal{B}_2(H)$, which is usually called in the literature as the *Hilbert-Schmidt norm.*

Denote *the modulus* of an operator $A \in \mathcal{B}(H)$ by $|A| := (A^*A)^{1/2}$.

Because $\||A|x\| = \|Ax\|$ for all $x \in H$, $A$ is Hilbert-Schmidt iff $|A|$ is Hilbert-Schmidt and $\|A\|_2 = \||A|\|_2$. From (1.2) we have that if $A \in \mathcal{B}_2(H)$, then $A^* \in \mathcal{B}_2(H)$ and $\|A\|_2 = \|A^*\|_2$.

The following theorem collects some of the most important properties of Hilbert-Schmidt operators:

**Theorem 1.** *We have:*

*(i)* $(\mathcal{B}_2(H), \|\cdot\|_2)$ *is a Hilbert space with inner product*

$$\langle A, B\rangle_2 := \sum_{i\in I}\langle Ae_i, Be_i\rangle = \sum_{i\in I}\langle B^*Ae_i, e_i\rangle \tag{1.4}$$

*and the definition does not depend on the choice of the orthonormal basis* $\{e_i\}_{i\in I}$;

*(ii) We have the inequalities*

$$\|A\| \le \|A\|_2 \tag{1.5}$$

*for any* $A \in \mathcal{B}_2(H)$ *and, if* $A \in \mathcal{B}_2(H)$ *and* $T \in \mathcal{B}(H)$, *then* $AT, TA \in \mathcal{B}_2(H)$ *with*

$$\|AT\|_2, \ \ \|TA\|_2 \le \|T\| \|A\|_2 \tag{1.6}$$

*(iii)* $\mathcal{B}_2(H)$ *is an operator ideal in* $\mathcal{B}(H)$, *i.e.*

$$\mathcal{B}(H)\,\mathcal{B}_2(H)\,\mathcal{B}(H) \subseteq \mathcal{B}_2(H).$$

If $\{e_i\}_{i\in I}$ an orthonormal basis of $H$, we say that $A \in \mathcal{B}(H)$ is *trace class* if

$$\|A\|_1 := \sum_{i\in I} \langle |A| e_i, e_i\rangle < \infty. \tag{1.7}$$

The definition of $\|A\|_1$ does not depend on the choice of the orthonormal basis $\{e_i\}_{i\in I}$. We denote by $\mathcal{B}_1(H)$ the set of trace class operators in $\mathcal{B}(H)$.

The following proposition holds:

**Proposition 1.** *If* $A \in \mathcal{B}(H)$, *then the following are equivalent:*

*(i)* $A \in \mathcal{B}_1(H)$;

*(ii)* $|A|^{1/2} \in \mathcal{B}_2(H)$.

The following properties are also well known:

**Theorem 2.** *With the above notations:*

*(i) We have*

$$\|A\|_1 = \|A^*\|_1 \ \text{ and } \ \|A\|_2 \le \|A\|_1 \tag{1.8}$$

*for any* $A \in \mathcal{B}_1(H)$;

*(ii)* $\mathcal{B}_1(H)$ *is an operator ideal in* $\mathcal{B}(H)$, *i.e.*

$$\mathcal{B}(H)\,\mathcal{B}_1(H)\,\mathcal{B}(H) \subseteq \mathcal{B}_1(H);$$

*(iii) We have*

$$\mathcal{B}_2(H)\,\mathcal{B}_2(H) = \mathcal{B}_1(H);$$

*(iv) We have*

$$\|A\|_1 = \sup\left\{\langle A, B\rangle_2 \ \mid B \in \mathcal{B}_2(H), \ \ \|B\|_2 \le 1\right\};$$

*(v)* $(\mathcal{B}_1(H), \|\cdot\|_1)$ *is a Banach space.*

We define the *trace* of a trace class operator $A \in \mathcal{B}_1(H)$ to be

$$\operatorname{tr}(A) := \sum_{i\in I} \langle Ae_i, e_i\rangle, \tag{1.9}$$

where $\{e_i\}_{i\in I}$ an orthonormal basis of $H$. Note that this coincides with the usual definition of the trace if $H$ is finite-dimensional. We observe that the series (1.9) converges absolutely and it is independent from the choice of basis.

The following result collects some properties of the trace:

**Theorem 3.** *We have:*

*(i) If* $A \in \mathcal{B}_1(H)$ *then* $A^* \in \mathcal{B}_1(H)$ *and*

$$\operatorname{tr}(A^*) = \overline{\operatorname{tr}(A)}; \tag{1.10}$$

*(ii) If* $A \in \mathcal{B}_1(H)$ *and* $T \in \mathcal{B}(H)$, *then* $AT, TA \in \mathcal{B}_1(H)$,

$$\operatorname{tr}(AT) = \operatorname{tr}(TA) \text{ and } |\operatorname{tr}(AT)| \le \|A\|_1 \|T\|; \tag{1.11}$$

*(iii)* $\operatorname{tr}(\cdot)$ *is a bounded linear functional on* $\mathcal{B}_1(H)$ *with* $\|\operatorname{tr}\| = 1$;

*(iv) If* $A, B \in \mathcal{B}_2(H)$ *then* $AB, BA \in \mathcal{B}_1(H)$ *and* $\operatorname{tr}(AB) = \operatorname{tr}(BA)$.

Now, if we assume that $P \ge 0$ and $P \in \mathcal{B}_1(H)$, then for all $T \in \mathcal{B}(H)$, $PT$, $TP \in \mathcal{B}_1(H)$ and $\operatorname{tr}(PT) = \operatorname{tr}(TP)$. Also, since $P^{1/2} \in \mathcal{B}_2(H)$, $TP^{1/2} \in \mathcal{B}_2(H)$, hence $P^{1/2}TP^{1/2}$ and $TP^{1/2}P^{1/2} = TP \in \mathcal{B}_1(H)$ with $\operatorname{tr}\left(P^{1/2}TP^{1/2}\right) = \operatorname{tr}(TP)$. Therefore, if $P \ge 0$ and $P \in \mathcal{B}_1(H)$,

$$\operatorname{tr}(PT) = \operatorname{tr}(TP) = \operatorname{tr}\left(P^{1/2}TP^{1/2}\right)$$

for all $T \in \mathcal{B}(H)$.

If $T \ge 0$, then $P^{1/2}TP^{1/2} \ge 0$, which implies that $\operatorname{tr}(PT) \ge 0$ that shows that the functional $\mathcal{B}(H) \ni T \longmapsto \operatorname{tr}(PT)$ is linear and isotonic functional. Also, by (1.11), if $T_n \to T$ for $n \to \infty$ in $\mathcal{B}(H)$ then $\lim_{n\to\infty} \operatorname{tr}(PT_n) = \operatorname{tr}(PT)$, namely $\mathcal{B}(H) \ni T \longmapsto \operatorname{tr}(PT)$ is also continuous in the norm topology.

For a survey on recent trace inequalities see [3] and the references therein.

Now, for a given $P \ge 0$ with $P \in \mathcal{B}_1(H)$ and $\operatorname{tr}(P) = 1$, we define the $P$-*determinant* of the positive invertible operator $A$ by

$$\Delta_P(A) := \exp\operatorname{tr}(P\ln A) = \exp\operatorname{tr}((\ln A)P) = \exp\operatorname{tr}\left(P^{1/2}(\ln A)P^{1/2}\right). \tag{1.12}$$

Assume that $P \ge 0$ with $P \in \mathcal{B}_1(H)$ and $\operatorname{tr}(P) = 1$. We observe that we have the following elementary properties:

(i) *continuity*: the map $A \to \Delta_P(A)$ is norm continuous;
(ii) *power equality:* $\Delta_P(A^t) = \Delta_P(A)^t$ for all $t > 0$;
(iii) *homogeneity*: $\Delta_P(tA) = t\Delta_x(A)$ and $\Delta_P(tI) = t$ for all $t > 0$;
(iv) *monotonicity*: $0 < A \le B$ implies $\Delta_P(A) \le \Delta_P(B)$.

In the recent paper [4] we obtained the following results:

**Theorem 4.** *Let* $P \ge 0$ *with* $P \in \mathcal{B}_1(H)$ *and* $\operatorname{tr}(P) = 1$, *then for all* $A, B > 0$ *and* $t \in [0,1]$,

$$\Delta_P((1-t)A + tB) \ge [\Delta_P(A)]^{1-t}[\Delta_P(B)]^t.$$

and

**Theorem 5.** *Let* $P \ge 0$ *with* $P \in \mathcal{B}_1(H)$ *and* $\operatorname{tr}(P) = 1$, *then for all* $A > 0$ *and* $a > 0$ *we have the double inequality*

$$a\exp\left[1 - a\operatorname{tr}\left(PA^{-1}\right)\right] \le \Delta_P(A) \le a\exp\left[a^{-1}\operatorname{tr}(PA) - 1\right]. \tag{1.13}$$

*In particular*

$$1 \le \frac{\operatorname{tr}(PA)}{\Delta_P(A)} \le \exp\left[\operatorname{tr}(PA)\operatorname{tr}\left(PA^{-1}\right) - 1\right] \tag{1.14}$$

*and*

$$1 \le \frac{\Delta_P(A)}{\left[\operatorname{tr}\left(PA^{-1}\right)\right]^{-1}} \le \exp\left[\operatorname{tr}\left(PA^{-1}\right)\operatorname{tr}(PA) - 1\right]. \tag{1.15}$$

*The first inequalities in (1.14) and 1.15) are best possible from (1.13).*

Motivated by the above results, in this paper we show among others that, if $P_i \geq 0$ with $P_i \in \mathcal{B}_1(H)$ and $\operatorname{tr}(P_i) = 1$ for $i \in \{1, ..., n\}$, $p_i \geq 0$ with $\sum_{i=1}^n p_i = 1$ and $0 < mI \leq A_i \leq MI$ for $i \in \{1, ..., n\}$, then

$$\begin{aligned} 1 &\leq \frac{\sum_{j=1}^n p_j \operatorname{tr}(P_j A_j)}{\prod_{j=1}^n [\Delta_{Pj}(A_j)]^{p_j}} \\ &\leq \exp\left[\frac{1}{2m^2}\left(M - \sum_{j=1}^n p_j \operatorname{tr}(P_j A_j)\right)\left(\sum_{j=1}^n p_j \operatorname{tr}(P_j A_j) - m\right)\right] \\ &\leq \exp\left[\frac{1}{8}\left(\frac{M}{m} - 1\right)^2\right]. \end{aligned}$$

## 2. Some Trace Inequalities

We use the following result that was obtained in [1]:

**Lemma 1.** *If $f : [a, b] \to \mathbb{R}$ is a convex function on $[a, b]$, then*

$$\begin{aligned} 0 &\leq \frac{(b-t) f(a) + (t-a) f(b)}{b-a} - f(t) \\ &\leq (b-t)(t-a) \frac{f'_-(b) - f'_+(a)}{b-a} \leq \frac{1}{4}(b-a)\left[f'_-(b) - f'_+(a)\right] \end{aligned} \tag{2.1}$$

*for any $t \in [a, b]$.*

*If the lateral derivatives $f'_-(b)$ and $f'_+(a)$ are finite, then the second inequality and the constant $1/4$ are sharp.*

We have the following reverse for the Jensen's trace inequality:

**Theorem 6.** *Assume that $f$ is differentiable convex on the interior $\mathring{I}$ of an interval. Let $Q_j \geq 0$ with $Q_j \in \mathcal{B}_1(H)$ for $j \in \{1, ..., n\}$ and $\sum_{j=1}^n \operatorname{tr}(Q_j) > 0$, then for all $B_j$ with the spectra $\operatorname{Sp}(B_j) \subseteq [m, M] \subset \mathring{I}$ for $j \in \{1, ..., n\}$, we have*

$$\begin{aligned} 0 &\leq \frac{\sum_{j=1}^n \operatorname{tr}[Q_j f(B_j)]}{\sum_{j=1}^n \operatorname{tr}(Q_j)} - f\left(\frac{\sum_{j=1}^n \operatorname{tr}(Q_j B_j)}{\sum_{j=1}^n \operatorname{tr}(Q_j)}\right) \\ &\leq \frac{f'_-(M) - f'_+(m)}{M-m} \\ &\times \left(M - \frac{\sum_{j=1}^n \operatorname{tr}(Q_j B_j)}{\sum_{j=1}^n \operatorname{tr}(Q_j)}\right)\left(\frac{\sum_{j=1}^n \operatorname{tr}(Q_j B_j)}{\sum_{j=1}^n \operatorname{tr}(Q_j)} - m\right) \\ &\leq \frac{1}{4}(M-m)\left[f'_-(M) - f'_+(m)\right]. \end{aligned} \tag{2.2}$$

*Proof.* Utilizing the continuous functional calculus for a selfadjoint operator $T$ with $0 \leq T \leq 1_H$ and the convexity of $f$ on $[m, M]$, we have

$$f(m(1_H - T) + MT) \leq f(m)(1_H - T) + f(M)T \tag{2.3}$$

in the operator order.

If we take in (2.3)

$$0 \leq T = \frac{B_j - m1_H}{M - m} \leq 1_H,$$

then we get

$$f\left(m\left(1_H - \frac{B_j - m1_H}{M - m}\right) + M\frac{B_j - m1_H}{M - m}\right) \leq f(m)\left(1_H - \frac{B_j - m1_H}{M - m}\right) + f(M)\frac{B_j - m1_H}{M - m}. \tag{2.4}$$

Observe that

$$m\left(1_H - \frac{B_j - m1_H}{M - m}\right) + M\frac{B_j - m1_H}{M - m} = \frac{m(M1_H - B_j) + M(B_j - m1_H)}{M - m} = B_j$$

and

$$f(m)\left(1_H - \frac{B_j - m1_H}{M - m}\right) + f(M)\frac{B_j - m1_H}{M - m} = \frac{f(m)(M1_H - B_j) + f(M)(B_j - m1_H)}{M - m}$$

and by (2.4) we get the following inequality of interest

$$f(B_j) \leq \frac{f(m)(M1_H - B_j) + f(M)(B_j - m1_H)}{M - m} \tag{2.5}$$

for all $j \in \{1, ..., n\}$.

If we multiply (2.5) both sides with $Q_j^{1/2}$ we get

$$\begin{aligned}
&\sum_{j=1}^{n} Q_j^{1/2} f(B_j) Q_j^{1/2} \\
&\leq \sum_{j=1}^{n} Q_j^{1/2}\left[\frac{f(m)(M1_H - B_j) + f(M)(B_j - m1_H)}{M - m}\right] Q_j^{1/2} \\
&= \frac{f(m)\sum_{j=1}^{n} Q_j^{1/2}(M1_H - B_j) Q_j^{1/2} + f(M)\sum_{j=1}^{n} Q_j^{1/2}(B_j - m1_H) Q_j^{1/2}}{M - m} \\
&= \frac{1}{M - m}\left[f(m)\left(M\sum_{j=1}^{n} Q_j - \sum_{j=1}^{n} Q_j^{1/2} B_j Q_j^{1/2}\right)\right. \\
&\left. + f(M)\left(\sum_{j=1}^{n} Q_j^{1/2} B_j Q_j^{1/2} - m\sum_{j=1}^{n} Q_j\right)\right]
\end{aligned}$$

which implies, by taking the trace and using its properties, that

$$
\begin{aligned}
&\sum_{j=1}^{n} \operatorname{tr}\left[Q_j f\left(B_j\right)\right] \\
&\leq \frac{1}{M-m}\left[f(m)\left(M \sum_{j=1}^{n} \operatorname{tr}\left(Q_j\right)-\sum_{j=1}^{n} \operatorname{tr}\left(Q_j B_j\right)\right)\right. \\
&\left.+f(M)\left(\sum_{j=1}^{n} \operatorname{tr}\left(Q_j B_j\right)-m \sum_{j=1}^{n} \operatorname{tr}\left(Q_j\right)\right)\right],
\end{aligned}
$$

which gives that

$$
\begin{aligned}
&\frac{\sum_{j=1}^{n} \operatorname{tr}\left[Q_j f\left(B_j\right)\right]}{\sum_{j=1}^{n} \operatorname{tr}\left(Q_j\right)} \\
&\leq \frac{f(m)\left(M-\frac{\sum_{j=1}^{n} \operatorname{tr}\left(Q_j B_j\right)}{\sum_{j=1}^{n} \operatorname{tr}\left(Q_j\right)}\right)+f(M)\left(\frac{\sum_{j=1}^{n} \operatorname{tr}\left(Q_j B_j\right)}{\sum_{j=1}^{n} \operatorname{tr}\left(Q_j\right)}-m\right)}{M-m},
\end{aligned}
$$

namely

$$
\begin{aligned}
0 &\leq \frac{\sum_{j=1}^{n} \operatorname{tr}\left[Q_j f\left(B_j\right)\right]}{\sum_{j=1}^{n} \operatorname{tr}\left(Q_j\right)}-f\left(\frac{\sum_{j=1}^{n} \operatorname{tr}\left(Q_j B_j\right)}{\sum_{j=1}^{n} \operatorname{tr}\left(Q_j\right)}\right) \\
&\leq \frac{f(m)\left(M-\frac{\sum_{j=1}^{n} \operatorname{tr}\left(Q_j B_j\right)}{\sum_{j=1}^{n} \operatorname{tr}\left(Q_j\right)}\right)+f(M)\left(\frac{\sum_{j=1}^{n} \operatorname{tr}\left(Q_j B_j\right)}{\sum_{j=1}^{n} \operatorname{tr}\left(Q_j\right)}-m\right)}{M-m} \\
&-f\left(\frac{\sum_{j=1}^{n} \operatorname{tr}\left(Q_j B_j\right)}{\sum_{j=1}^{n} \operatorname{tr}\left(Q_j\right)}\right).
\end{aligned}
\tag{2.6}
$$

Here the first inequality is Jensen's inequality.

Using the inequality (2.1) for

$$
t=\frac{\sum_{j=1}^{n} \operatorname{tr}\left(Q_j B_j\right)}{\sum_{j=1}^{n} \operatorname{tr}\left(Q_j\right)} \in[m, M],
$$

$a=m$ and $b=M$ we have

$$
\begin{aligned}
&\frac{f(m)\left(M-\frac{\sum_{j=1}^{n} \operatorname{tr}\left(Q_j B_j\right)}{\sum_{j=1}^{n} \operatorname{tr}\left(Q_j\right)}\right)+f(M)\left(\frac{\sum_{j=1}^{n} \operatorname{tr}\left(Q_j B_j\right)}{\sum_{j=1}^{n} \operatorname{tr}\left(Q_j\right)}-m\right)}{M-m} \\
&-f\left(\frac{\sum_{j=1}^{n} \operatorname{tr}\left(Q_j B_j\right)}{\sum_{j=1}^{n} \operatorname{tr}\left(Q_j\right)}\right) \\
&\leq \frac{f_{-}'(M)-f_{+}'(m)}{M-m} \quad M-\frac{\sum_{j=1}^{n} \operatorname{tr}\left(Q_j B_j\right)}{\sum_{j=1}^{n} \operatorname{tr}\left(Q_j\right)}\Bigg)\left(\frac{\sum_{j=1}^{n} \operatorname{tr}\left(Q_j B_j\right)}{\sum_{j=1}^{n} \operatorname{tr}\left(Q_j\right)}-m\right) \\
&\leq \frac{1}{4}(M-m)\left[f_{-}'(M)-f_{+}'(m)\right].
\end{aligned}
\tag{2.7}
$$

By making use of (2.6) and (2.7) we derive (2.2). □

We also have [1]:

**Lemma 2.** *Assume that $f:[a,b]\to\mathbb{R}$ is absolutely continuous on $[a,b]$. If $f'$ is $K$-Lipschitzian on $[a,b]$, then*

$$\begin{aligned}(2.8)\qquad &\left|(1-t)f(a)+tf(b)-f((1-t)a+tb)\right|\\ &\leq\frac{1}{2}K(b-t)(t-a)\leq\frac{1}{8}K(b-a)^2\end{aligned}$$

*for all $t\in[0,1]$.*

*The constants $1/2$ and $1/8$ are the best possible in (2.8).*

**Remark 1.** *If $f:[a,b]\to\mathbb{R}$ is twice differentiable and $f''\in L_\infty[a,b]$, then*

$$\begin{aligned}(2.9)\qquad &\left|(1-t)f(a)+tf(b)-f((1-t)a+tb)\right|\\ &\leq\frac{1}{2}\|f''\|_{[a,b],\infty}(b-t)(t-a)\leq\frac{1}{8}\|f''\|_{[a,b],\infty}(b-a)^2,\end{aligned}$$

*where $\|f''\|_{[a,b],\infty}:=\operatorname{essup}_{t\in[a,b]}|f''(t)|<\infty$. The constants $1/2$ and $1/8$ are the best possible in (2.9).*

**Theorem 7.** *Assume that $f$ is twice differentiable convex on the interior $\mathring{I}$ of the interval $I$ and the derivative $f''$ is bounded on $\mathring{I}$. Let $Q_j\geq0$ with $Q_j\in\mathcal{B}_1(H)$ for $j\in\{1,...,n\}$ and $\sum_{j=1}^n\operatorname{tr}(Q_j)>0$, then for all $B_j$ with the spectra $\operatorname{Sp}(B_j)\subseteq[m,M]\subset\mathring{I}$ for $j\in\{1,...,n\}$, we have*

$$\begin{aligned}(2.10)\qquad 0&\leq\frac{\sum_{j=1}^n\operatorname{tr}[Q_jf(B_j)]}{\sum_{j=1}^n\operatorname{tr}(Q_j)}-f\left(\frac{\sum_{j=1}^n\operatorname{tr}(Q_jB_j)}{\sum_{j=1}^n\operatorname{tr}(Q_j)}\right)\\ &\leq\frac{1}{2}\|f''\|_{[m,M],\infty}\left.M-\frac{\sum_{j=1}^n\operatorname{tr}(Q_jB_j)}{\sum_{j=1}^n\operatorname{tr}(Q_j)}\right)\left(\frac{\sum_{j=1}^n\operatorname{tr}(Q_jB_j)}{\sum_{j=1}^n\operatorname{tr}(Q_j)}-m\right)\\ &\leq\frac{1}{8}\|f''\|_{[m,M],\infty}(M-m)^2.\end{aligned}$$

*Proof.* From (2.9) and the continuous functional calculus, we get

$$\begin{aligned}(2.11)\qquad 0&\leq\frac{f(m)(M1_H-B_j)+f(M)(B_j-m1_H)}{M-m}-f(B_j)\\ &\leq\frac{1}{2}\|f''\|_{[m,M],\infty}(M1_H-B_j)(B_j-m1_H)\\ &\leq\frac{1}{8}\|f''\|_{[m,M],\infty}(M-m)^2 1_H\end{aligned}$$

where $B_j$ are selfadjoint operators with the spectra $\operatorname{Sp}(B_j)\subset[m,M]$, $j\in\{1,...,n\}$.

Now, by employing a similar argument to the one in the proof of Theorem 6 we derive the desired result (2.10). □

We also have the following scalar inequality of interest:

**Lemma 3.** *Let $f:[a,b]\to\mathbb{R}$ be a convex function on $[a,b]$ and $t\in[0,1]$, then*

$$\begin{aligned}(2.12)\qquad &2\min\{t,1-t\}\left[\frac{f(a)+f(b)}{2}-f\left(\frac{a+b}{2}\right)\right]\\ &\leq(1-t)f(a)+tf(b)-f((1-t)a+tb)\\ &\leq2\max\{t,1-t\}\left[\frac{f(a)+f(b)}{2}-f\left(\frac{a+b}{2}\right)\right].\end{aligned}$$

The proof follows, for instance, by Corollary 1 from [2] for $n=2$, $p_1=1-t$, $p_2=t$, $t\in[0,1]$ and $x_1=a$, $x_2=b$.

**Theorem 8.** *Assume that $f$ is convex on the interior $\mathring{I}$ of an interval $I$. Let $Q_j\geq 0$ with $Q_j\in\mathcal{B}_1(H)$ for $j\in\{1,...,n\}$ and $\sum_{j=1}^n \operatorname{tr}(Q_j)>0$, then for all $B_j$ with the spectra $\operatorname{Sp}(B_j)\subseteq[m,M]\subset\mathring{I}$ for $j\in\{1,...,n\}$, we have*

$$
\begin{aligned}
0 &\leq \frac{2}{M-m}\left[\frac{f(m)+f(M)}{2}-f\left(\frac{m+M}{2}\right)\right] \\
&\times\left(\frac{1}{2}(M-m)-\frac{\sum_{j=1}^k \operatorname{tr}\left(Q_j\left|B_j-\frac{1}{2}(m+M)1_H\right|\right)}{\sum_{j=1}^k \operatorname{tr}(Q_j)}\right) \\
&\leq \frac{f(m)\left(M-\frac{\sum_{j=1}^k \operatorname{tr}(Q_jB_j)}{\sum_{j=1}^k \operatorname{tr}(Q_j)}\right)+f(M)\left(\frac{\sum_{j=1}^k \operatorname{tr}(Q_jB_j)}{\sum_{j=1}^k \operatorname{tr}(Q_j)}-m\right)}{M-m} \\
&-\frac{\sum_{j=1}^k \operatorname{tr}(Q_jf(B_j))}{\sum_{j=1}^k \operatorname{tr}(Q_j)} \\
&\leq \frac{2}{M-m}\left[\frac{f(m)+f(M)}{2}-f\left(\frac{m+M}{2}\right)\right] \\
&\times\left(\frac{1}{2}(M-m)+\frac{1}{\sum_{j=1}^k \operatorname{tr}(Q_j)}\sum_{j=1}^k \operatorname{tr}\left(Q_j\left|B_j-\frac{1}{2}(m+M)1_H\right|\right)\right) \\
&\leq 2\left[\frac{f(m)+f(M)}{2}-f\left(\frac{m+M}{2}\right)\right].
\end{aligned}
\tag{2.13}
$$

*Proof.* We have from (2.12) that

$$
\begin{aligned}
0 &\leq 2\left(\frac{1}{2}-\left|t-\frac{1}{2}\right|\right)\left[\frac{f(m)+f(M)}{2}-f\left(\frac{m+M}{2}\right)\right] \\
&\leq (1-t)f(m)+tf(M)-f((1-t)m+tM) \\
&\leq 2\left(\frac{1}{2}+\left|t-\frac{1}{2}\right|\right)\left[\frac{f(m)+f(M)}{2}-f\left(\frac{m+M}{2}\right)\right],
\end{aligned}
\tag{2.14}
$$

for all $t\in[0,1]$.

Utilizing the continuous functional calculus for a selfadjoint operator $T$ with $0\leq T\leq 1_H$ we get from (2.14) that

$$
\begin{aligned}
0 &\leq 2\left[\frac{f(m)+f(M)}{2}-f\left(\frac{m+M}{2}\right)\right]\left(\frac{1}{2}1_H-\left|T-\frac{1}{2}1_H\right|\right) \\
&\leq (1-T)f(m)+Tf(M)-f((1-T)m+TM) \\
&\leq 2\left[\frac{f(m)+f(M)}{2}-f\left(\frac{m+M}{2}\right)\right]\left(\frac{1}{2}1_H+\left|T-\frac{1}{2}1_H\right|\right),
\end{aligned}
\tag{2.15}
$$

in the operator order.

If we take in (2.15)

$$0\leq T=\frac{B_j-m1_H}{M-m}\leq 1_H,$$

then we get

$$(2.16)\quad 0\leq \frac{2}{M-m}\left[\frac{f(m)+f(M)}{2}-f\left(\frac{m+M}{2}\right)\right]$$
$$\times\left(\frac{1}{2}(M-m)1_H-\left|B_j-\frac{1}{2}(m+M)1_H\right|\right)$$
$$\leq \frac{f(m)(M1_H-B_j)+f(M)(B_j-m1_H)}{M-m}-f(B_j)$$
$$\leq \frac{2}{M-m}\left[\frac{f(m)+f(M)}{2}-f\left(\frac{m+M}{2}\right)\right]$$
$$\times\left(\frac{1}{2}(M-m)1_H+\left|B_j-\frac{1}{2}(m+M)1_H\right|\right).$$

If we multiply both sides by $Q_j^{1/2}$ we derive

$$0\leq \frac{2}{M-m}\left[\frac{f(m)+f(M)}{2}-f\left(\frac{m+M}{2}\right)\right]$$
$$\times\left(\frac{1}{2}(M-m)Q_j-Q_j^{1/2}\left|B_j-\frac{1}{2}(m+M)1_H\right|Q_j^{1/2}\right)$$
$$\leq \frac{f(m)\left(M1_H-Q_j^{1/2}B_jQ_j^{1/2}\right)+f(M)\left(Q_j^{1/2}B_jQ_j^{1/2}-m1_H\right)}{M-m}$$
$$-Q_j^{1/2}f(B_j)Q_j^{1/2}$$
$$\leq \frac{2}{M-m}\left[\frac{f(m)+f(M)}{2}-f\left(\frac{m+M}{2}\right)\right]$$
$$\times\left(\frac{1}{2}(M-m)Q_j+Q_j^{1/2}\left|B_j-\frac{1}{2}(m+M)1_H\right|Q_j^{1/2}\right).$$

Now, by taking the trace and summing over $j$ from 1 to $n$, we derive

$$0\leq \frac{2}{M-m}\left[\frac{f(m)+f(M)}{2}-f\left(\frac{m+M}{2}\right)\right]$$
$$\times\left(\frac{1}{2}(M-m)\sum_{j=1}^{k}\operatorname{tr}(Q_j)-\sum_{j=1}^{k}\operatorname{tr}\left(Q_j\left|B_j-\frac{1}{2}(m+M)1_H\right|\right)\right)$$
$$\leq \frac{1}{M-m}\left[f(m)\left(M\sum_{j=1}^{k}\operatorname{tr}(Q_j)-\sum_{j=1}^{k}\operatorname{tr}(Q_jB_j)\right)\right.$$
$$\left.+f(M)\left(\sum_{j=1}^{k}\operatorname{tr}(Q_jB_j)-m\sum_{j=1}^{k}\operatorname{tr}(Q_j)\right)\right]-\sum_{j=1}^{k}\operatorname{tr}(Q_jf(B_j))$$
$$\leq \frac{2}{M-m}\left[\frac{f(m)+f(M)}{2}-f\left(\frac{m+M}{2}\right)\right]$$
$$\times\left(\frac{1}{2}(M-m)\sum_{j=1}^{k}\operatorname{tr}(Q_j)+\sum_{j=1}^{k}\operatorname{tr}\left(Q_j\left|B_j-\frac{1}{2}(m+M)1_H\right|\right)\right).$$

This proves (2.13). □

We also have:

**Proposition 2.** *With the assumptions of Theorem 8 we have*

$$0\le\frac{\sum_{j=1}^{n}\operatorname{tr}\left[Q_jf\left(B_j\right)\right]}{\sum_{j=1}^{n}\operatorname{tr}\left(Q_j\right)}-f\left(\frac{\sum_{j=1}^{n}\operatorname{tr}\left(Q_jB_j\right)}{\sum_{j=1}^{n}\operatorname{tr}\left(Q_j\right)}\right)\tag{2.17}$$
$$\le\frac{2}{M-m}\left[\frac{f\left(m\right)+f\left(M\right)}{2}-f\left(\frac{m+M}{2}\right)\right]$$
$$\times\left(\frac{1}{2}\left(M-m\right)+\left|\frac{\sum_{j=1}^{n}\operatorname{tr}\left(Q_jB_j\right)}{\sum_{j=1}^{n}\operatorname{tr}\left(Q_j\right)}-\frac{1}{2}\left(m+M\right)\right|\right)$$
$$\le 2\left[\frac{f\left(m\right)+f\left(M\right)}{2}-f\left(\frac{m+M}{2}\right)\right].$$

*Proof.* From (2.6) we have

$$0\le\frac{\sum_{j=1}^{n}\operatorname{tr}\left[Q_jf\left(B_j\right)\right]}{\sum_{j=1}^{n}\operatorname{tr}\left(Q_j\right)}-f\left(\frac{\sum_{j=1}^{n}\operatorname{tr}\left(Q_jB_j\right)}{\sum_{j=1}^{n}\operatorname{tr}\left(Q_j\right)}\right)\tag{2.18}$$
$$\le\frac{f\left(m\right)\left(M-\frac{\sum_{j=1}^{n}\operatorname{tr}(Q_jB_j)}{\sum_{j=1}^{n}\operatorname{tr}(Q_j)}\right)+f\left(M\right)\left(\frac{\sum_{j=1}^{n}\operatorname{tr}(Q_jB_j)}{\sum_{j=1}^{n}\operatorname{tr}(Q_j)}-m\right)}{M-m}$$
$$-f\left(\frac{\sum_{j=1}^{n}\operatorname{tr}\left(Q_jB_j\right)}{\sum_{j=1}^{n}\operatorname{tr}\left(Q_j\right)}\right).$$

From the second part of the scalar version of (2.16) we also have the scalar inequality

$$\frac{f\left(m\right)\left(M-\frac{\sum_{j=1}^{k}\operatorname{tr}(Q_jB_j)}{\sum_{j=1}^{k}\operatorname{tr}(Q_j)}\right)+f\left(M\right)\left(\frac{\sum_{j=1}^{k}\operatorname{tr}(Q_jB_j)}{\sum_{j=1}^{k}\operatorname{tr}(Q_j)}-m\right)}{M-m}\tag{2.19}$$
$$-f\left(\frac{\sum_{j=1}^{n}\operatorname{tr}\left(Q_jB_j\right)}{\sum_{j=1}^{n}\operatorname{tr}\left(Q_j\right)}\right)$$
$$\le\frac{2}{M-m}\left[\frac{f\left(m\right)+f\left(M\right)}{2}-f\left(\frac{m+M}{2}\right)\right]$$
$$\times\left(\frac{1}{2}\left(M-m\right)1_H+\left|\frac{\sum_{j=1}^{n}\operatorname{tr}\left(Q_jB_j\right)}{\sum_{j=1}^{n}\operatorname{tr}\left(Q_j\right)}-\frac{1}{2}\left(m+M\right)\right|\right)$$
$$\le 2\left[\frac{f\left(m\right)+f\left(M\right)}{2}-f\left(\frac{m+M}{2}\right)\right].$$

By utilizing (2.18) and (2.19) we obtain the desired result (2.17). □

## 3. Determinant Inequalities

Our first main result is as follows:

**Theorem 9.** *Assume that* $P_j\ge 0$ *with* $P_j\in\mathcal{B}_1\left(H\right)$ *and* $\operatorname{tr}\left(P_j\right)=1$ *for* $j\in\{1,...,n\}$. *If* $p_j\ge 0$ *with* $\sum_{j=1}^{n}p_j=1$ *and* $A_j$ *with the property that* $0<mI\le$

$A_j \leq MI,\ j \in \{1, ..., n\}$ *for* $j \in \{1, ..., n\}$, *then*

$$1 \leq \frac{\sum_{j=1}^{n} p_j \operatorname{tr}(P_j A_j)}{\prod_{j=1}^{n} [\Delta_{P_j}(A_j)]^{p_j}} \tag{3.1}$$
$$\leq \exp\left[\frac{1}{Mm}\left(M - \sum_{j=1}^{n} p_j \operatorname{tr}(P_j A_j)\right)\left(\sum_{j=1}^{n} p_j \operatorname{tr}(P_j A_j) - m\right)\right]$$
$$\leq \exp\left[\frac{1}{4Mm}(M-m)^2\right].$$

*Also,*

$$1 \leq \frac{\sum_{j=1}^{n} p_j \operatorname{tr}(P_j A_j)}{\prod_{j=1}^{n} [\Delta_{P_j}(A_j)]^{p_j}} \tag{3.2}$$
$$\leq \exp\left[\frac{1}{2m^2}\left(M - \sum_{j=1}^{n} p_j \operatorname{tr}(P_j A_j)\right)\left(\sum_{j=1}^{n} p_j \operatorname{tr}(P_j A_j) - m\right)\right]$$
$$\leq \exp\left[\frac{1}{8}\left(\frac{M}{m} - 1\right)^2\right].$$

*Proof.* If we take $f(t) = -\ln t$, $t > 0$, $A_j = A_j$, $Q_j = p_j P_j$, $j \in \{1, ..., n\}$, in (2.2), then we get

$$0 \leq \ln\left(\sum_{j=1}^{n} p_j \operatorname{tr}(P_j A_j)\right) - \sum_{j=1}^{n} p_j \operatorname{tr}[P_j f(A_j)] \tag{3.3}$$
$$\leq \frac{1}{Mm}\left(M - \sum_{j=1}^{n} p_j \operatorname{tr}(P_j A_j)\right)\left(\sum_{j=1}^{n} p_j \operatorname{tr}(P_j A_j) - m\right)$$
$$\leq \frac{1}{4Mm}(M-m)^2.$$

If we take the exponential in (3.3), then we get

$$1 \leq \frac{\exp\ln\left(\sum_{j=1}^{n} p_j \operatorname{tr}(P_j A_j)\right)}{\exp\left(\sum_{j=1}^{n} p_j \operatorname{tr}[P_j \ln(A_j)]\right)}$$
$$\leq \exp\left[\frac{1}{Mm}\left(M - \sum_{j=1}^{n} p_j \operatorname{tr}(P_j A_j)\right)\left(\sum_{j=1}^{n} p_j \operatorname{tr}(P_j A_j) - m\right)\right]$$
$$\leq \exp\left[\frac{1}{4Mm}(M-m)^2\right],$$

which is equivalent to

$$\begin{aligned}
1 &\leq \frac{\sum_{j=1}^{n} p_j \operatorname{tr}(P_j A_j)}{\prod_{j=1}^{n} (\exp \operatorname{tr}(P_j \ln A_j))^{p_j}} \\
&\leq \exp\left[\frac{1}{Mm}\left(M - \sum_{j=1}^{n} p_j \operatorname{tr}(P_j A_j)\right)\left(\sum_{j=1}^{n} p_j \operatorname{tr}(P_j A_j) - m\right)\right] \\
&\leq \exp\left[\frac{1}{4Mm}(M-m)^2\right],
\end{aligned}$$

and the inequality (3.1) is proved.

The proof of (3.2) follows by (2.10) in a similar way and the details are omitted. □

**Theorem 10.** *With the assumptions of Theorem 9, we have*

$$\begin{aligned}
(3.4) \qquad 1 &\leq \left.\frac{\frac{m+M}{2}}{\sqrt{mM}}\right)^{1-\frac{2}{M-m}\sum_{j=1}^{k} p_j \operatorname{tr}\left(P_j\left|A_j - \frac{1}{2}(m+M)1_H\right|\right)} \\
&\leq \frac{\prod_{j=1}^{n} [\Delta_{Pj}(A_j)]^{p_j}}{m^{\frac{M-\sum_{j=1}^{n} p_j \operatorname{tr}(P_j A_j)}{M-m}} M^{\frac{\sum_{j=1}^{n} p_j \operatorname{tr}(P_j A_j) - m}{M-m}}} \\
&\leq \left.\frac{\frac{m+M}{2}}{\sqrt{mM}}\right)^{1+\frac{2}{M-m}\sum_{j=1}^{k} p_j \operatorname{tr}\left(P_j\left|A_j - \frac{1}{2}(m+M)1_H\right|\right)} \leq \left.\frac{\frac{m+M}{2}}{\sqrt{mM}}\right)^{2}.
\end{aligned}$$

*Also,*

$$\begin{aligned}
(3.5) \qquad 1 &\leq \frac{\sum_{j=1}^{n} p_j \operatorname{tr}(P_j A_j)}{\prod_{j=1}^{n} [\Delta_{Pj}(A_j)]^{p_j}} \\
&\leq \left.\frac{\frac{m+M}{2}}{\sqrt{mM}}\right)^{1+\frac{2}{M-m}\left|\sum_{j=1}^{n} p_j \operatorname{tr}(P_j A_j) - \frac{1}{2}(m+M)\right|} \leq \left.\frac{\frac{m+M}{2}}{\sqrt{mM}}\right)^{2}.
\end{aligned}$$

*Proof.* If we take $f(t) = -\ln t$, $t > 0$, $A_j = A_j$, $Q_j = p_j P_j$, $j \in \{1, ..., n\}$, in (2.13), then we get

$$\begin{aligned}
0 &\leq \frac{2}{M-m} \ln \left.\frac{\frac{m+M}{2}}{\sqrt{mM}}\right) \\
&\times \left(\frac{1}{2}(M-m) - \sum_{j=1}^{k} p_j \operatorname{tr}\left(P_j\left|A_j - \frac{1}{2}(m+M)1_H\right|\right)\right)
\end{aligned}$$

$$
\begin{aligned}
&\le \sum_{j=1}^{k} \operatorname{tr}\left(Q_j \ln A_j\right) \\
&- \frac{\left(M-\sum_{j=1}^{n} p_j \operatorname{tr}\left(P_j A_j\right)\right) \ln m+\left(\sum_{j=1}^{n} p_j \operatorname{tr}\left(P_j A_j\right)-m\right) \ln M}{M-m} \\
&\le \frac{2}{M-m} \ln \left.\quad \frac{\frac{m+M}{2}}{\sqrt{mM}}\right) \\
&\times\left(\frac{1}{2}(M-m)+\sum_{j=1}^{k} p_j \operatorname{tr}\left(P_j\left|A_j-\frac{1}{2}(m+M) 1_H\right|\right)\right) \\
&\le \ln \left.\quad \frac{\frac{m+M}{2}}{\sqrt{mM}}\right)^2 .
\end{aligned}
$$

This is equivalent to

$$
\begin{aligned}
0 &\le \ln \left.\quad \frac{\frac{m+M}{2}}{\sqrt{mM}}\right)^{\left(1-\frac{2}{M-m} \sum_{j=1}^{k} p_j \operatorname{tr}\left(P_j\left|A_j-\frac{1}{2}(m+M) 1_H\right|\right)\right)} \\
&\le \sum_{j=1}^{k} p_j \operatorname{tr}\left(P_j \ln A_j\right)-\ln m^{\frac{M-\sum_{j=1}^{n} p_j \operatorname{tr}\left(P_j A_j\right)}{M-m}} M^{\frac{\sum_{j=1}^{n} p_j \operatorname{tr}\left(P_j A_j\right)-m}{M-m}} \\
&\le \ln \left.\quad \frac{\frac{m+M}{2}}{\sqrt{mM}}\right)^{\left(1+\frac{2}{M-m} \sum_{j=1}^{k} p_j \operatorname{tr}\left(P_j\left|A_j-\frac{1}{2}(m+M) 1_H\right|\right)\right)} \le \ln \left.\quad \frac{\frac{m+M}{2}}{\sqrt{mM}}\right)^2 .
\end{aligned}
$$

By taking the exponential, we derive

$$
\begin{aligned}
1 &\le \left.\quad \frac{\frac{m+M}{2}}{\sqrt{mM}}\right)^{1-\frac{2}{M-m} \sum_{j=1}^{k} p_j \operatorname{tr}\left(P_j\left|A_j-\frac{1}{2}(m+M) 1_H\right|\right)} \\
&\le \frac{\prod_{j=1}^{n}\left(\exp \operatorname{tr}\left(P_j \ln A_j\right)\right)^{p_j}}{m^{\frac{M-\sum_{j=1}^{n} p_j \operatorname{tr}\left(P_j A_j\right)}{M-m}} M^{\frac{\sum_{j=1}^{n} p_j \operatorname{tr}\left(P_j A_j\right)-m}{M-m}}} \\
&\le \left.\quad \frac{\frac{m+M}{2}}{\sqrt{mM}}\right)^{1+\frac{2}{M-m} \sum_{j=1}^{k} p_j \operatorname{tr}\left(P_j\left|A_j-\frac{1}{2}(m+M) 1_H\right|\right)} \le \left.\quad \frac{\frac{m+M}{2}}{\sqrt{mM}}\right)^2 ,
\end{aligned}
$$

which proves (3.4).

Inequality (3.5) follows in a similar way from (2.17). $\square$

**Remark 2.** *The case of one operator is as follows: Let $P \ge 0$ with $P \in \mathcal{B}_1(H)$ and $\operatorname{tr}(P)=1$, then for all $A$ satisfying the condition $0<mI \le A \le MI$ we have the inequalities*

$$
\begin{aligned}
1 \le \frac{\operatorname{tr}(PA)}{\Delta_P(A)} &\le \exp \left[\frac{1}{Mm}(M-\operatorname{tr}(PA))(\operatorname{tr}(PA)-m)\right] \qquad (3.6)\\
&\le \exp \left[\frac{1}{4Mm}(M-m)^2\right],
\end{aligned}
$$

$$
\begin{aligned}
(3.7) \qquad 1 &\leq \frac{\operatorname{tr}(PA)}{\Delta_P(A)} \leq \exp\left[\frac{1}{2m^2}\left(M-\operatorname{tr}(PA)\right)\left(\operatorname{tr}(PA)-m\right)\right] \\
&\leq \exp\left[\frac{1}{8}\left(\frac{M}{m}-1\right)^2\right],
\end{aligned}
$$

$$
\begin{aligned}
(3.8) \qquad 1 &\leq \left.\frac{\frac{m+M}{2}}{\sqrt{mM}}\right)^{1-\frac{2}{M-m}\operatorname{tr}\left(P\left|A-\frac{1}{2}(m+M)1_H\right|\right)} \\
&\leq \frac{\Delta_P(A)}{m^{\frac{M-\operatorname{tr}(PA)}{M-m}}M^{\frac{\operatorname{tr}(PA)-m}{M-m}}} \\
&\leq \left.\frac{\frac{m+M}{2}}{\sqrt{mM}}\right)^{1+\frac{2}{M-m}\operatorname{tr}\left(P\left|A-\frac{1}{2}(m+M)1_H\right|\right)} \leq \left.\frac{\frac{m+M}{2}}{\sqrt{mM}}\right)^{2},
\end{aligned}
$$

*and*

$$
\begin{aligned}
(3.9) \qquad 1 &\leq \frac{\operatorname{tr}(PA)}{\Delta_P(A)} \\
&\leq \left.\frac{\frac{m+M}{2}}{\sqrt{mM}}\right)^{1+\frac{2}{M-m}\left|\operatorname{tr}(PA)-\frac{1}{2}(m+M)\right|} \leq \left.\frac{\frac{m+M}{2}}{\sqrt{mM}}\right)^{2}.
\end{aligned}
$$

**Remark 3.** *The case of two operators is as follows: Let $P, Q \geq 0$ with $P, Q \in \mathcal{B}_1(H)$ and $\operatorname{tr}(P) = \operatorname{tr}(Q) = 1$, then for all $A$, $B$ satisfying the condition $0 < mI \leq A, B \leq MI$ we have the inequalities*

$$
\begin{aligned}
(3.10) \qquad 1 &\leq \frac{(1-t)\operatorname{tr}(PA)+t\operatorname{tr}(QB)}{[\Delta_P(A)]^{(1-t)}[\Delta_Q(B)]^t} \\
&\leq \exp\left[\frac{1}{Mm}\left(M-(1-t)\operatorname{tr}(PA)-t\operatorname{tr}(QB)\right)\right. \\
&\quad \left.\times\left((1-t)\operatorname{tr}(PA)+t\operatorname{tr}(QB)-m\right)\right] \\
&\leq \exp\left[\frac{1}{4Mm}(M-m)^2\right],
\end{aligned}
$$

$$
\begin{aligned}
(3.11) \qquad 1 &\leq \frac{(1-t)\operatorname{tr}(PA)+t\operatorname{tr}(QB)}{[\Delta_P(A)]^{(1-t)}[\Delta_Q(B)]^t} \\
&\leq \exp\frac{1}{2m^2}\left(M-(1-t)\operatorname{tr}(PA)-t\operatorname{tr}(QB)\right) \\
&\quad \left.\times\left((1-t)\operatorname{tr}(PA)+t\operatorname{tr}(QB)-m\right)\right] \\
&\leq \exp\left[\frac{1}{8}\left(\frac{M}{m}-1\right)^2\right],
\end{aligned}
$$

$$
\begin{aligned}
(3.12)\qquad 1 &\leq \left(\frac{\frac{m+M}{2}}{\sqrt{mM}}\right)^{1-\frac{2}{M-m}\left[(1-t)\operatorname{tr}\left(P\left|A-\frac{1}{2}(m+M)1_H\right|\right)+\operatorname{tr}\left(Q\left|B-\frac{1}{2}(m+M)1_H\right|\right)\right]} \\
&\leq \frac{\left[\Delta_P(A)\right]^{(1-t)}\left[\Delta_Q(B)\right]^t}{m^{\frac{M-(1-t)\operatorname{tr}(PA)-t\operatorname{tr}(QB)}{M-m}}M^{\frac{(1-t)\operatorname{tr}(PA)+t\operatorname{tr}(QB)-m}{M-m}}} \\
&\leq \left(\frac{\frac{m+M}{2}}{\sqrt{mM}}\right)^{1+\frac{2}{M-m}\left[(1-t)\operatorname{tr}\left(P\left|A-\frac{1}{2}(m+M)1_H\right|\right)+\operatorname{tr}\left(Q\left|B-\frac{1}{2}(m+M)1_H\right|\right)\right]} \\
&\leq \left(\frac{\frac{m+M}{2}}{\sqrt{mM}}\right)^2,
\end{aligned}
$$

*and*

$$
\begin{aligned}
(3.13)\qquad 1 &\leq \frac{(1-t)\operatorname{tr}(PA)+t\operatorname{tr}(QB)}{\left[\Delta_P(A)\right]^{(1-t)}\left[\Delta_Q(B)\right]^t} \\
&\leq \left(\frac{\frac{m+M}{2}}{\sqrt{mM}}\right)^{1+\frac{2}{M-m}\left|(1-t)\operatorname{tr}(PA)+t\operatorname{tr}(QB)-\frac{1}{2}(m+M)\right|} \leq \left(\frac{\frac{m+M}{2}}{\sqrt{mM}}\right)^2
\end{aligned}
$$

*for all* $t \in [0,1]$.

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[1]Mathematics, College of Engineering & Science, Victoria University, PO Box 14428, Melbourne City, MC 8001, Australia.

*E-mail address*: sever.dragomir@vu.edu.au

*URL*: http://rgmia.org/dragomir

[2]DST-NRF Centre of Excellence in the Mathematical, and Statistical Sciences, School of Computer Science, & Applied Mathematics, University of the Witwatersrand,, Private Bag 3, Johannesburg 2050, South Africa