SOME FUNCTIONAL PROPERTIES FOR THE TRACE CLASS P-DETERMINANT OF SEQUENCES OF POSITIVE OPERATORS IN HILBERT SPACES

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Abstract. Let H be a complex Hilbert space. For a given operator $P \geq 0$ with $P \in \mathcal{B}_1(H)$, the trace class associated to $\mathcal{B}(H)$ and $\operatorname{tr}(P) = 1$, we define the P-*determinant* of the positive invertible operator A by

$$\Delta_P(A) := \exp \operatorname{tr}(P \ln A).$$

We define the determinant functional

$$D_n(\mathbf{q}; \mathbf{A}, Q) := \frac{\left[\operatorname{tr}\left(\frac{1}{Q_n}\sum_{j=1}^{n} q_j Q A_j\right)\right]^{Q_n}}{\prod_{i=1}^{n}\left[\Delta_Q(A_j)\right]^{q_i}},$$

where $\mathbf{A} = (A_1, ..., A_n)$ is an n-tuple of selfadjoint positive operators, $\mathbf{q} \in \mathcal{P}_n^+$, the set of positive n-tuple and $Q \in \mathcal{B}_1(H)$, $Q > 0$ with $\operatorname{tr} Q = 1$.

In this paper we show among others that, if p, $\mathbf{q} \in \mathcal{P}_n^+$, then we have

$$D_n(\mathbf{p} + \mathbf{q}; \mathbf{A}, Q) \geq D_n(\mathbf{p}; \mathbf{A}, Q)\, D_n(\mathbf{q}; \mathbf{A}, Q) \geq 1$$

i.e., the functional $J_n(\cdot, Q, \mathbf{A})$ is super-multiplicative on $\mathcal{P}_n^+$. For p, $\mathbf{q} \in \mathcal{P}_n^+$ with $\mathbf{p} \geq \mathbf{q}$,

$$D_n(\mathbf{p}; Q, \mathbf{A}) \geq D_n(\mathbf{q}; Q, \mathbf{A}) \geq 1,$$

i.e., the functional $J_n(\cdot, Q, \mathbf{A})$ is monotonic non-decreasing on $\mathcal{P}_n^+$.

1. Introduction

In 1952, in the paper [7], B. Fuglede and R. V. Kadison introduced the determinant of a (invertible) operator and established its fundamental properties. The notion generalizes the usual determinant and can be considered for any operator in a finite von Neumann algebra (M, τ) with a faithful normal trace.

Let $T \in M$ be normal and $|T| := (T^*T)^{1/2}$ its modulus. By the spectral theorem one can represent T as an integral

$$T = \int_{\operatorname{Sp}(T)} \lambda dE(\lambda),$$

where $E(\lambda)$ is a projection valued measure and $\operatorname{Sp}(T)$ is the spectrum of T. The measure $\mu_T := \tau \circ E$ becomes a probability measure on the complex plane and has the support in the spectrum $\operatorname{Sp}(T)$.

For any $T \in M$ the Fuglede-Kadison determinant (*FK-determinant*) is defined by

$$\Delta_{FK}(T) := \exp\left(\int_0^\infty \ln t d\mu_{|T|}\right).$$

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If T is invertible, then

$$\Delta_{FK}(T) := \exp(\tau(\ln(|T|))),$$

where $\ln(|T|)$ is defined by the use of functional calculus.

Let $B(H)$ be the space of all bounded linear operators on a Hilbert space H, and I stands for the identity operator on H. An operator A in $B(H)$ is said to be positive (in symbol: $A \geq 0$) if $\langle Ax, x\rangle \geq 0$ for all $x \in H$. In particular, $A > 0$ means that A is positive and invertible. For a pair A, B of selfadjoint operators the order relation $A \geq B$ means as usual that $A - B$ is positive.

In 1998, Fujii et al. [8], [9], introduced the *normalized determinant* $\Delta_x(A)$ for positive invertible operators A on a Hilbert space H and a fixed unit vector $x \in H$, namely $\|x\| = 1$, defined by

$$\Delta_x(A) := \exp\langle \ln Ax, x\rangle$$

and discussed it as a continuous geometric mean and observed some inequalities around the determinant from this point of view. For some recent results, see [12].

We need now some preparations for trace of operators in Hilbert spaces.

Let $(H, \langle\cdot,\cdot\rangle)$ be a complex Hilbert space and $\{e_i\}_{i\in I}$ an *orthonormal basis* of H. We say that $A \in \mathcal{B}(H)$ is a *Hilbert-Schmidt operator* if

$$\sum_{i\in I}\|Ae_i\|^2 < \infty. \tag{1.1}$$

It is well know that, if $\{e_i\}_{i\in I}$ and $\{f_j\}_{j\in J}$ are orthonormal bases for H and $A \in \mathcal{B}(H)$ then

$$\sum_{i\in I}\|Ae_i\|^2 = \sum_{j\in I}\|Af_j\|^2 = \sum_{j\in I}\|A^*f_j\|^2 \tag{1.2}$$

showing that the definition (1.1) is independent of the orthonormal basis and A is a Hilbert-Schmidt operator iff A^* is a Hilbert-Schmidt operator.

Let $\mathcal{B}_2(H)$ the set of *Hilbert-Schmidt operators* in $\mathcal{B}(H)$. For $A \in \mathcal{B}_2(H)$ we define

$$\|A\|_2 := \left(\sum_{i\in I}\|Ae_i\|^2\right)^{1/2} \tag{1.3}$$

for $\{e_i\}_{i\in I}$ an orthonormal basis of H.

Using the triangle inequality in $l^2(I)$, one checks that $\mathcal{B}_2(H)$ is a *vector space* and that $\|\cdot\|_2$ is a norm on $\mathcal{B}_2(H)$, which is usually called in the literature as the *Hilbert-Schmidt norm.*

Denote *the modulus* of an operator $A \in \mathcal{B}(H)$ by $|A| := (A^*A)^{1/2}$.

Because $\||A|x\| = \|Ax\|$ for all $x \in H$, A is Hilbert-Schmidt iff $|A|$ is Hilbert-Schmidt and $\|A\|_2 = \||A|\|_2$. From (1.2) we have that if $A \in \mathcal{B}_2(H)$, then $A^* \in \mathcal{B}_2(H)$ and $\|A\|_2 = \|A^*\|_2$.

The following theorem collects some of the most important properties of Hilbert-Schmidt operators:

Theorem 1. *We have:*

(i) $(\mathcal{B}_2(H), \|\cdot\|_2)$ *is a Hilbert space with inner product*

$$\langle A, B\rangle_2 := \sum_{i\in I}\langle Ae_i, Be_i\rangle = \sum_{i\in I}\langle B^*Ae_i, e_i\rangle \tag{1.4}$$

and the definition does not depend on the choice of the orthonormal basis $\{e_i\}_{i\in I}$*;*

(ii) We have the inequalities

$$\|A\| \le \|A\|_2 \tag{1.5}$$

for any $A \in \mathcal{B}_2(H)$ *and, if* $A \in \mathcal{B}_2(H)$ *and* $T \in \mathcal{B}(H)$*, then* $AT, TA \in \mathcal{B}_2(H)$ *with*

$$\|AT\|_2\,,\ \ \|TA\|_2 \le \|T\|\,\|A\|_2 \tag{1.6}$$

(iii) $\mathcal{B}_2(H)$ *is an operator ideal in* $\mathcal{B}(H)$*, i.e.*

$$\mathcal{B}(H)\,\mathcal{B}_2(H)\,\mathcal{B}(H) \subseteq \mathcal{B}_2(H)\,.$$

If $\{e_i\}_{i\in I}$ an orthonormal basis of H, we say that $A \in \mathcal{B}(H)$ is *trace class* if

$$\|A\|_1 := \sum_{i\in I} \langle |A|\, e_i, e_i\rangle < \infty. \tag{1.7}$$

The definition of $\|A\|_1$ does not depend on the choice of the orthonormal basis $\{e_i\}_{i\in I}$. We denote by $\mathcal{B}_1(H)$ the set of trace class operators in $\mathcal{B}(H)$.

The following proposition holds:

Proposition 1. *If* $A \in \mathcal{B}(H)$*, then the following are equivalent:*

(i) $A \in \mathcal{B}_1(H)$*;*

(ii) $|A|^{1/2} \in \mathcal{B}_2(H)$*.*

The following properties are also well known:

Theorem 2. *With the above notations:*

(i) We have

$$\|A\|_1 = \|A^*\|_1 \ \ \text{and}\ \ \|A\|_2 \le \|A\|_1 \tag{1.8}$$

for any $A \in \mathcal{B}_1(H)$*;*

(ii) $\mathcal{B}_1(H)$ *is an operator ideal in* $\mathcal{B}(H)$*, i.e.*

$$\mathcal{B}(H)\,\mathcal{B}_1(H)\,\mathcal{B}(H) \subseteq \mathcal{B}_1(H)\,;$$

(iii) We have

$$\mathcal{B}_2(H)\,\mathcal{B}_2(H) = \mathcal{B}_1(H)\,;$$

(iv) We have

$$\|A\|_1 = \sup\left\{\langle A, B\rangle_2 \ \mid\ B \in \mathcal{B}_2(H)\,,\ \ \|B\|_2 \le 1\right\};$$

(v) $(\mathcal{B}_1(H), \|\cdot\|_1)$ *is a Banach space.*

We define the *trace* of a trace class operator $A \in \mathcal{B}_1(H)$ to be

$$\operatorname{tr}(A) := \sum_{i\in I} \langle Ae_i, e_i\rangle\,, \tag{1.9}$$

where $\{e_i\}_{i\in I}$ an orthonormal basis of H. Note that this coincides with the usual definition of the trace if H is finite-dimensional. We observe that the series (1.9) converges absolutely and it is independent from the choice of basis.

The following result collects some properties of the trace:

Theorem 3. *We have:*

(i) If $A \in \mathcal{B}_1(H)$ *then* $A^* \in \mathcal{B}_1(H)$ *and*

$$\operatorname{tr}(A^*) = \overline{\operatorname{tr}(A)}; \tag{1.10}$$

(ii) If $A \in \mathcal{B}_1(H)$ *and* $T \in \mathcal{B}(H)$, *then* $AT,\ TA \in \mathcal{B}_1(H)$,

$$\operatorname{tr}(AT) = \operatorname{tr}(TA) \text{ and } |\operatorname{tr}(AT)| \leq \|A\|_1 \|T\|; \tag{1.11}$$

(iii) $\operatorname{tr}(\cdot)$ *is a bounded linear functional on* $\mathcal{B}_1(H)$ *with* $\|\operatorname{tr}\| = 1$;

(iv) If $A,\ B \in \mathcal{B}_2(H)$ *then* $AB,\ BA \in \mathcal{B}_1(H)$ *and* $\operatorname{tr}(AB) = \operatorname{tr}(BA)$.

Now, if we assume that $P \geq 0$ and $P \in \mathcal{B}_1(H)$, then for all $T \in \mathcal{B}(H)$, PT, $TP \in \mathcal{B}_1(H)$ and $\operatorname{tr}(PT) = \operatorname{tr}(TP)$. Also, since $P^{1/2} \in \mathcal{B}_2(H)$, $TP^{1/2} \in \mathcal{B}_2(H)$, hence $P^{1/2}TP^{1/2}$ and $TP^{1/2}P^{1/2} = TP \in \mathcal{B}_1(H)$ with $\operatorname{tr}\left(P^{1/2}TP^{1/2}\right) = \operatorname{tr}(TP)$. Therefore, if $P \geq 0$ and $P \in \mathcal{B}_1(H)$,

$$\operatorname{tr}(PT) = \operatorname{tr}(TP) = \operatorname{tr}\left(P^{1/2}TP^{1/2}\right)$$

for all $T \in \mathcal{B}(H)$.

If $T \geq 0$, then $P^{1/2}TP^{1/2} \geq 0$, which implies that $\operatorname{tr}(PT) \geq 0$ that shows that the functional $\mathcal{B}(H) \ni T \longmapsto \operatorname{tr}(PT)$ is linear and isotonic functional. Also, by (1.11), if $T_n \to T$ for $n \to \infty$ in $\mathcal{B}(H)$ then $\lim_{n\to\infty} \operatorname{tr}(PT_n) = \operatorname{tr}(PT)$, namely $\mathcal{B}(H) \ni T \longmapsto \operatorname{tr}(PT)$ is also continuous in the norm topology.

For a survey on recent trace inequalities see [5] and the references therein.

Now, for a given $P \geq 0$ with $P \in \mathcal{B}_1(H)$ and $\operatorname{tr}(P) = 1$, we define the P-*determinant* of the positive invertible operator A by

$$\Delta_P(A) := \exp \operatorname{tr}(P \ln A) = \exp \operatorname{tr}((\ln A) P) = \exp \operatorname{tr}\left(P^{1/2}(\ln A) P^{1/2}\right). \tag{1.12}$$

Assume that $P \geq 0$ with $P \in \mathcal{B}_1(H)$ and $\operatorname{tr}(P) = 1$. We observe that we have the following elementary properties:

(i) *continuity*: the map $A \to \Delta_P(A)$ is norm continuous;
(ii) *power equality:* $\Delta_P(A^t) = \Delta_P(A)^t$ for all $t > 0$;
(iii) *homogeneity*: $\Delta_P(tA) = t\Delta_x(A)$ and $\Delta_P(tI) = t$ for all $t > 0$;
(iv) *monotonicity*: $0 < A \leq B$ implies $\Delta_P(A) \leq \Delta_P(B)$.

In the recent paper [6] we obtained the following results:

Theorem 4. *Let* $P \geq 0$ *with* $P \in \mathcal{B}_1(H)$ *and* $\operatorname{tr}(P) = 1$, *then for all* $A,\ B > 0$ *and* $t \in [0,1]$ *we have the Ky Fan's type inequality*

$$\Delta_P((1-t)A + tB) \geq [\Delta_P(A)]^{1-t}[\Delta_P(B)]^t. \tag{1.13}$$

and

Theorem 5. *Let* $P \geq 0$ *with* $P \in \mathcal{B}_1(H)$ *and* $\operatorname{tr}(P) = 1$, *then for all* $A > 0$ *and* $a > 0$ *we have the double inequality*

$$a \exp\left[1 - a \operatorname{tr}\left(PA^{-1}\right)\right] \leq \Delta_P(A) \leq a \exp\left[a^{-1}\operatorname{tr}(PA) - 1\right]. \tag{1.14}$$

In particular

$$1 \leq \frac{\operatorname{tr}(PA)}{\Delta_P(A)} \leq \exp\left[\operatorname{tr}(PA)\operatorname{tr}\left(PA^{-1}\right) - 1\right] \tag{1.15}$$

and

$$1 \leq \frac{\Delta_P(A)}{\left[\operatorname{tr}\left(PA^{-1}\right)\right]^{-1}} \leq \exp\left[\operatorname{tr}\left(PA^{-1}\right)\operatorname{tr}(PA) - 1\right]. \tag{1.16}$$

The first inequalities in (1.15) and 1.16) are best possible from (1.14).

We define the determinant functional

$$D_n(\mathbf{q};\mathbf{A},Q):=\frac{\left[\operatorname{tr}\left(\frac{1}{Q_n}\sum_{j=1}^n q_jQA_j\right)\right]^{Q_n}}{\prod_{i=1}^n\left[\Delta_Q(A_j)\right]^{q_i}},$$

where $\mathbf{A}=(A_1,...,A_n)$ is an n-tuple of selfadjoint positive operators, $\mathbf{q}\in\mathcal{P}_n^+$, the set of positive n-tuple and $Q\in\mathcal{B}_1(H)$, $Q>0$ with $\operatorname{tr}Q=1$.

In this paper we show among others that, if p, $\mathbf{q}\in\mathcal{P}_n^+$, then we have

$$D_n(\mathbf{p}+\mathbf{q};\mathbf{A},Q)\geq D_n(\mathbf{p};\mathbf{A},Q)\,D_n(\mathbf{q};\mathbf{A},Q)\geq 1$$

i.e., the functional $J_n(\cdot,Q,\mathbf{A})$ is super-multiplicative on $\mathcal{P}_n^+$. For p, $\mathbf{q}\in\mathcal{P}_n^+$ with $\mathbf{p}\geq\mathbf{q}$,

$$D_n(\mathbf{p};Q,\mathbf{A})\geq D_n(\mathbf{q};Q,\mathbf{A})\geq 1,$$

i.e., the functional $J_n(\cdot,Q,\mathbf{A})$ is monotonic non-decreasing on $\mathcal{P}_n^+$.

2. Functional Properties for Trace

Consider a convex function f on the interval I. We define

$$\mathcal{B}_1^{++}(H):=\{Q\in\mathcal{B}_1(H)\mid Q>0\}$$

and consider the n-tuples

$$\mathbf{Q}:=(Q_1,...,Q_n)\in\left[\mathcal{B}_1^{++}(H)\right]^n:=\mathcal{B}_1^{++}(H)\times...\times\mathcal{B}_1^{++}(H)$$

and $\mathbf{A}:=(A_1,...,A_n)$ with $Sp(A_j)\subseteq I$, $j\in\{1,...,n\}$. We have the following Jensen type trace inequality for convex function f,

$$f\left(\frac{1}{\sum_{j=1}^n\operatorname{tr}(Q_j)}\sum_{j=1}^n\operatorname{tr}(Q_jA_j)\right)\leq\frac{1}{\sum_{j=1}^n\operatorname{tr}(Q_j)}\sum_{j=1}^n\operatorname{tr}\left[Q_jf(A_j)\right],\tag{2.1}$$

and can introduce the *Jensen's gap functional*

$$J_n(\mathbf{Q},\mathbf{A},f):=\sum_{j=1}^n\operatorname{tr}\left[Q_jf(A_j)\right]-\sum_{j=1}^n\operatorname{tr}(Q_j)\,f\left(\frac{1}{\sum_{j=1}^n\operatorname{tr}(Q_j)}\sum_{j=1}^n\operatorname{tr}(Q_jA_j)\right).$$

We have the following functional properties:

Theorem 6. *Assume that f is convex on the interval I and $\mathbf{A}:=(A_1,...,A_n)$ with $Sp(A_j)\subseteq I$, $j\in\{1,...,n\}$.*

(i) *For all $\mathbf{P}$, $\mathbf{Q}\in\left[\mathcal{B}_1^{+}(H)\right]^n$ we have*

$$J_n(\mathbf{P}+\mathbf{Q},\mathbf{A},f)\geq J_n(\mathbf{P},\mathbf{A},f)+J_n(\mathbf{Q},\mathbf{A},f)\geq 0,\tag{2.2}$$

i.e., the functional $J_n(\cdot,\mathbf{A},f)$ is superadditive on $\left[\mathcal{B}_1^{++}(H)\right]^n$;

(ii) *For all $\mathbf{P}$, $\mathbf{Q}\in\left[\mathcal{B}_1^{+}(H)\right]^n$ with $\mathbf{P}\geq\mathbf{Q}$, namely $P_j\geq Q_j$ for $j\in\{1,...,n\}$,*

$$J_n(\mathbf{P},\mathbf{A},f)\geq J_n(\mathbf{Q},\mathbf{A},f)\geq 0,\tag{2.3}$$

i.e., the functional $J_n(\cdot,\mathbf{A},f)$ is monotonic non-decreasing on $\left[\mathcal{B}_1^{++}(H)\right]^n$.

Proof. (i). If $\mathbf{P}$, $\mathbf{Q}\in\left[\mathcal{B}_1^+(H)\right]^n$, then we have

$$\begin{aligned}
&J_n(\mathbf{P}+\mathbf{Q},\mathbf{A},f) \\
&=\sum_{j=1}^n \operatorname{tr}\left[(P_j+Q_j)f(A_j)\right] \\
&-\sum_{j=1}^n \operatorname{tr}(P_j+Q_j)f\left(\frac{\sum_{j=1}^n \operatorname{tr}((P_j+Q_j)A_j)}{\sum_{j=1}^n \operatorname{tr}(P_j+Q_j)}\right) \\
&=\sum_{j=1}^n \operatorname{tr}\left[P_jf(A_j)\right]+\sum_{j=1}^n \operatorname{tr}\left[Q_jf(A_j)\right] \\
&-\sum_{j=1}^n \left[\operatorname{tr}(P_j)+\operatorname{tr}(Q_j)\right]f\left(\frac{\sum_{j=1}^n \operatorname{tr}(P_jA_j)+\sum_{j=1}^n \operatorname{tr}(Q_jA_j)}{\sum_{j=1}^n\left[\operatorname{tr}(P_j)+\operatorname{tr}(Q_j)\right]}\right).
\end{aligned}\tag{2.4}$$

By the convexity of f we obtain

$$\begin{aligned}
&f\left(\frac{\sum_{j=1}^n \operatorname{tr}(P_jA_j)+\sum_{j=1}^n \operatorname{tr}(Q_jA_j)}{\sum_{j=1}^n\left[\operatorname{tr}(P_j)+\operatorname{tr}(Q_j)\right]}\right) \\
&=f\left(\frac{\sum_{j=1}^n \operatorname{tr}(P_j)\frac{\sum_{j=1}^n \operatorname{tr}(P_jA_j)}{\sum_{j=1}^n \operatorname{tr}(P_j)}+\sum_{j=1}^n \operatorname{tr}(Q_j)\frac{\sum_{j=1}^n \operatorname{tr}(Q_jA_j)}{\sum_{j=1}^n \operatorname{tr}(Q_j)}}{\sum_{j=1}^n\left[\operatorname{tr}(P_j)+\operatorname{tr}(Q_j)\right]}\right) \\
&\le\frac{\sum_{j=1}^n \operatorname{tr}(P_j)}{\sum_{j=1}^n\left[\operatorname{tr}(P_j)+\operatorname{tr}(Q_j)\right]}f\left(\frac{\sum_{j=1}^n \operatorname{tr}(P_jA_j)}{\sum_{j=1}^n \operatorname{tr}(P_j)}\right) \\
&+\frac{\sum_{j=1}^n \operatorname{tr}(Q_j)}{\sum_{j=1}^n\left[\operatorname{tr}(P_j)+\operatorname{tr}(Q_j)\right]}f\left(\frac{\sum_{j=1}^n \operatorname{tr}(Q_jA_j)}{\sum_{j=1}^n \operatorname{tr}(Q_j)}\right).
\end{aligned}$$

Therefore

$$\begin{aligned}
&-\sum_{j=1}^n \left[\operatorname{tr}(P_j)+\operatorname{tr}(Q_j)\right]f\left(\frac{\sum_{j=1}^n \operatorname{tr}(P_jA_j)+\sum_{j=1}^n \operatorname{tr}(Q_jA_j)}{\sum_{j=1}^n\left[\operatorname{tr}(P_j)+\operatorname{tr}(Q_j)\right]}\right) \\
&\ge-\frac{\sum_{j=1}^n\left[\operatorname{tr}(P_j)+\operatorname{tr}(Q_j)\right]\sum_{j=1}^n \operatorname{tr}(P_j)}{\sum_{j=1}^n\left[\operatorname{tr}(P_j)+\operatorname{tr}(Q_j)\right]}f\left(\frac{\sum_{j=1}^n \operatorname{tr}(P_jA_j)}{\sum_{j=1}^n \operatorname{tr}(P_j)}\right) \\
&-\frac{\sum_{j=1}^n\left[\operatorname{tr}(P_j)+\operatorname{tr}(Q_j)\right]\sum_{j=1}^n \operatorname{tr}(Q_j)}{\sum_{j=1}^n\left[\operatorname{tr}(P_j)+\operatorname{tr}(Q_j)\right]}f\left(\frac{\sum_{j=1}^n \operatorname{tr}(Q_jA_j)}{\sum_{j=1}^n \operatorname{tr}(Q_j)}\right) \\
&=-\sum_{j=1}^n \operatorname{tr}(P_j)f\left(\frac{\sum_{j=1}^n \operatorname{tr}(P_jA_j)}{\sum_{j=1}^n \operatorname{tr}(P_j)}\right)-\sum_{j=1}^n \operatorname{tr}(Q_j)f\left(\frac{\sum_{j=1}^n \operatorname{tr}(Q_jA_j)}{\sum_{j=1}^n \operatorname{tr}(Q_j)}\right)
\end{aligned}$$

and by (2.4) we derive

$$\begin{aligned}
&J_n(\mathbf{P}+\mathbf{Q},\mathbf{A},f)\\
&\geq \sum_{j=1}^{n}\operatorname{tr}[P_j f(A_j)]+\sum_{j=1}^{n}\operatorname{tr}[Q_j f(A_j)]\\
&-\sum_{j=1}^{n}\operatorname{tr}(P_j) f\left(\frac{\sum_{j=1}^{n}\operatorname{tr}(P_jA_j)}{\sum_{j=1}^{n}\operatorname{tr}(P_j)}\right)-\sum_{j=1}^{n}\operatorname{tr}(Q_j)\left(\frac{\sum_{j=1}^{n}\operatorname{tr}(Q_jA_j)}{\sum_{j=1}^{n}\operatorname{tr}(Q_j)}\right)\\
&=\sum_{j=1}^{n}\operatorname{tr}[P_j f(A_j)]-\sum_{j=1}^{n}\operatorname{tr}(P_j) f\left(\frac{\sum_{j=1}^{n}\operatorname{tr}(P_jA_j)}{\sum_{j=1}^{n}\operatorname{tr}(P_j)}\right)\\
&+\sum_{j=1}^{n}\operatorname{tr}[Q_j f(A_j)]-\sum_{j=1}^{n}\operatorname{tr}(Q_j)\left(\frac{\sum_{j=1}^{n}\operatorname{tr}(Q_jA_j)}{\sum_{j=1}^{n}\operatorname{tr}(Q_j)}\right)\\
&=J_n(\mathbf{P},\mathbf{A},f)+J_n(\mathbf{Q},\mathbf{A},f)\geq 0
\end{aligned}$$

and the inequality (2.2) is proved.

(ii) If $\mathbf{P}\geq\mathbf{Q}$, then $\mathbf{P}=\mathbf{P}-\mathbf{Q}+\mathbf{Q}$ and if we use the property (2.2), then we get

$$J_n(\mathbf{P},\mathbf{A},f)=J_n(\mathbf{P}-\mathbf{Q}+\mathbf{Q},\mathbf{A},f)\geq J_n(\mathbf{P}-\mathbf{Q},\mathbf{A},f)+J_n(\mathbf{Q},\mathbf{A},f),$$

which gives

$$J_n(\mathbf{P},\mathbf{A},f)-J_n(\mathbf{Q},\mathbf{A},f)\geq J_n(\mathbf{P}-\mathbf{Q},\mathbf{A},f)\geq 0$$

and the inequality (2.3) is proved. □

Corollary 1. *With the assumptions of Theorem 6 and if we assume that there exists the positive constants* $m<M$ *such that*

$$m\mathbf{Q}\leq\mathbf{P}\leq M\mathbf{Q}, \tag{2.5}$$

then

$$mJ_n(\mathbf{Q},\mathbf{A},f)\leq J_n(\mathbf{P},\mathbf{A},f)\leq MJ_n(\mathbf{Q},\mathbf{A},f). \tag{2.6}$$

Proof. Observe that for $\alpha>0$ we have $J_n(\alpha\mathbf{Q},\mathbf{A})=\alpha J_n(\mathbf{Q},\mathbf{A})$. Utilizing the monotonicity property (2.3) we have

$$J_n(m\mathbf{Q},\mathbf{A},f)\leq J_n(\mathbf{P},\mathbf{A},f)\leq J_n(M\mathbf{Q},\mathbf{A},f),$$

which imply the desired result (2.6). □

We denote by $\mathcal{P}_n^+$ the set of all n-tuples $q=(q_1,...,q_n)$, $q_j\geq 0$ with $j\in\{1,...,n\}$ and $Q_n:=\sum_{j=1}^{n}q_j>0$. For p, $\mathbf{q}\in\mathcal{P}_n^+$ we denote $\mathbf{p}\geq\mathbf{q}$ if $p_j\geq q_j$ for any $j\in\{1,...,n\}$.

For $Q\in\mathcal{B}_1^{++}(H)$ with $\operatorname{tr}Q=1$, we define the functional

$$J_n(\mathbf{q};Q,\mathbf{A},f):=\sum_{j=1}^{n}q_j\operatorname{tr}[Qf(A_j)]-Q_nf\left(\frac{1}{Q_n}\sum_{j=1}^{n}q_j\operatorname{tr}(QA_j)\right),$$

where $Q_n:=\sum_{j=1}^{n}q_j>0$.

We observe that if we put $Q_j=q_jQ$, $j\in\{1,...,n\}$ then $J_n(\mathbf{q},Q,\mathbf{A},f)=J_n(\mathbf{Q},\mathbf{A},f)$ and we can state the following result:

Theorem 7. *Assume that f is convex on the interval I, $Q \in \mathcal{B}_1^{++}(H)$ with $\operatorname{tr} Q = 1$ and $\mathbf{A} := (A_1, ..., A_n)$ with $Sp(A_j) \subseteq I$, $j \in \{1, ..., n\}$.*

(i) *For all p, $\mathbf{q} \in \mathcal{P}_n^+$ we have*

$$J_n(\mathbf{p}+\mathbf{q}; Q, \mathbf{A}, f) \geq J_n(\mathbf{p}; Q; \mathbf{A}, f) + J_n(\mathbf{p}; Q, \mathbf{A}, f) \geq 0, \tag{2.7}$$

i.e., the functional $J_n(\cdot, Q, \mathbf{A}, f)$ is superadditive on $\mathcal{P}_n^+$;

(ii) *For p, $\mathbf{q} \in \mathcal{P}_n^+$ with $\mathbf{p} \geq \mathbf{q}$*

$$J_n(\mathbf{p}; Q, \mathbf{A}, f) \geq J_n(\mathbf{q}; Q, \mathbf{A}, f) \geq 0, \tag{2.8}$$

i.e., the functional $J_n(\cdot, Q, \mathbf{A}, f)$ is monotonic non-decreasing on $\mathcal{P}_n^+$.

Remark 1. *We observe that if all $q_j > 0$ then we have the inequality*

$$\min_{j\in\{1,...,n\}} \left\{\frac{p_j}{q_j}\right\} J_n(\mathbf{q}; Q, \mathbf{A}, f) \leq J_n(\mathbf{p}; Q, \mathbf{A}, f) \leq \max_{j\in\{1,...,n\}} \left\{\frac{p_j}{q_j}\right\} J_n(\mathbf{p}; Q, \mathbf{A}, f). \tag{2.9}$$

In particular, if $\mathbf{q}$ is the uniform distribution, i.e., $q_j = \frac{1}{n}, j \in \{1, ..., n\}$, then we have the inequalities

$$n \min_{j\in\{1,...,n\}} \{p_j\} J_n(Q, \mathbf{A}, f) \leq J_n(\mathbf{p}; Q, \mathbf{A}, f) \leq n \max_{j\in\{1,...,n\}} \{p_j\} J_n(Q, \mathbf{A}, f), \tag{2.10}$$

where

$$J_n(Q, \mathbf{A}, f) := \frac{1}{n}\sum_{j=1}^{n} \operatorname{tr}[Qf(A_j)] - f\left(\frac{1}{n}\sum_{j=1}^{n} \operatorname{tr}(QA_j)\right). \tag{2.11}$$

For $n = 2$ and by choosing $p_1 = \alpha$, $p_2 = 1-\alpha$ with $\alpha \in [0, 1]$, we get from (2.10) the inequality

$$\begin{aligned} &2\min\{\alpha, 1-\alpha\} \\ &\times \left[\frac{\operatorname{tr}[Qf(A)] + \operatorname{tr}[Qf(B)]}{2} - f\left(\operatorname{tr}\left[Q\left(\frac{A+B}{2}\right)\right]\right)\right] \\ &\leq (1-\alpha)\operatorname{tr}[Qf(A)] + \alpha \operatorname{tr}[Qf(B)] - f(\operatorname{tr}(Q[(1-\alpha)A + \alpha B])) \\ &\leq 2\max\{\alpha, 1-\alpha\} \\ &\times \left[\frac{\operatorname{tr}[Qf(A)] + \operatorname{tr}[Qf(B)]}{2} - f\left(\operatorname{tr}\left[Q\left(\frac{A+B}{2}\right)\right]\right)\right], \end{aligned} \tag{2.12}$$

where $f : I \to \mathbb{R}$ is a convex function and A and B are two bounded selfadjoint operators on the complex Hilbert space H with $\operatorname{Sp}(A), \operatorname{Sp}(B) \subseteq I$.

Let $\mathcal{P}_f(\mathbb{N})$ be the family of finite parts of the set of natural numbers $\mathbb{N}$, $\mathcal{A}(H)$ the linear space of all sequences of selfadjoint operators defined on the complex Hilbert space, i.e.,

$$\mathcal{A}(H) = \left\{\mathbf{A} = (A_k)_{k\in\mathbb{N}} \mid A_k \text{ are selfadjoint operators on } H \text{ for all } k \in \mathbb{N}\right\}$$

and $\mathcal{S}_+\left(\mathcal{B}_1^{++}(H)\right)$ the family of positive sequences from $\mathcal{B}_1(H)$.

Let $f: I \to \mathbb{R}$ is a convex function on the interval I. We consider the functional

$$J_K(\mathbf{Q}, \mathbf{A}, f) := \sum_{j\in K} \operatorname{tr}[Q_j f(A_j)] - \sum_{j\in K} \operatorname{tr}(Q_j) f\left(\frac{\sum_{j\in K} \operatorname{tr}(Q_j A_j)}{\sum_{j\in K} \operatorname{tr}(Q_j)}\right),$$

where $K \in \mathcal{P}_f(\mathbb{N})$, $\mathbf{Q} \in \mathcal{S}_+\left(\mathcal{B}_1^{++}(H)\right)$, $\mathbf{A} \in \mathcal{A}(H)$ and $\operatorname{Sp}(A_j) \subseteq I$, $j \in \mathbb{N}$.

Theorem 8. *Let $f: I \to \mathbb{R}$ be a convex function on the interval I and $\mathbf{Q} \in \mathcal{S}_+\left(\mathcal{B}_1^{++}(H)\right)$, $\mathbf{A} \in \mathcal{A}(H)$ with $\operatorname{Sp}(A_j) \subseteq I$, $j \in \mathbb{N}$.*

(i) *If $K, L \in \mathcal{P}_f(\mathbb{N}) \setminus \{\emptyset\}$ with $K \cap L = \emptyset$, then we have the inequality*

$$J_{K\cup L}(\mathbf{Q}, \mathbf{A}, f) \geq J_K(\mathbf{Q}, \mathbf{A}, f) + J_L(\mathbf{Q}, \mathbf{A}, f) \geq 0, \tag{2.13}$$

i.e., $J_\cdot(\mathbf{Q}, \mathbf{A}, f)$ is super-additive as an index set functional.

(ii) *If $\emptyset \neq K \subset L$, then we have*

$$J_L(\mathbf{Q}, \mathbf{A}, f) \geq J_K(\mathbf{Q}, \mathbf{A}, f) \geq 0, \tag{2.14}$$

i.e., $J_\cdot(\mathbf{Q}, \mathbf{A}, f)$ is monotonic non-decreasing as an index set functional.

Proof. (i). If $K, L \in \mathcal{P}_f(\mathbb{N}) \setminus \{\emptyset\}$ with $K \cap L = \emptyset$, then we have

$$\begin{aligned} & J_{K\cup L}(\mathbf{Q}, \mathbf{A}, f) \\ &= \sum_{j\in K\cup L} \operatorname{tr}[Q_j f(A_j)] - \sum_{j\in K\cup L} \operatorname{tr}(Q_j) f\left(\frac{\sum_{j\in K\cup L} \operatorname{tr}(Q_j A_j)}{\sum_{j\in K\cup L} \operatorname{tr}(Q_j)}\right) \\ &= \sum_{j\in K} \operatorname{tr}[Q_j f(A_j)] + \sum_{j\in L} \operatorname{tr}[Q_j f(A_j)] \\ &- \sum_{j\in K\cup L} \operatorname{tr}(Q_j) \\ &\times f\left(\frac{\sum_{j\in K} \operatorname{tr}(Q_j) \frac{\sum_{j\in K} \operatorname{tr}(Q_j A_j)}{\sum_{j\in K} \operatorname{tr}(Q_j)} + \sum_{j\in L} \operatorname{tr}(Q_j) \frac{\sum_{j\in L} \operatorname{tr}(Q_j A_j)}{\sum_{j\in L} \operatorname{tr}(Q_j)}}{\sum_{j\in K\cup L} \operatorname{tr}(Q_j)}\right) \\ &\geq \sum_{j\in K} \operatorname{tr}[Q_j f(A_j)] + \sum_{j\in L} \operatorname{tr}[Q_j f(A_j)] \\ &- \sum_{j\in K\cup L} \operatorname{tr}(Q_j) \left[\frac{\sum_{j\in K} \operatorname{tr}(Q_j)}{\sum_{j\in K\cup L} \operatorname{tr}(Q_j)} f\left(\frac{\sum_{j\in K} \operatorname{tr}(Q_j A_j)}{\sum_{j\in K} \operatorname{tr}(Q_j)}\right)\right. \\ &+ \left.\frac{\sum_{j\in L} \operatorname{tr}(Q_j)}{\sum_{j\in K\cup L} \operatorname{tr}(Q_j)} f\left(\frac{\sum_{j\in L} \operatorname{tr}(Q_j A_j)}{\sum_{j\in L} \operatorname{tr}(Q_j)}\right)\right] \\ &= \sum_{j\in K} \operatorname{tr}[Q_j f(A_j)] - \sum_{j\in K} \operatorname{tr}(Q_j) \left(\frac{\sum_{j\in K} \operatorname{tr}(Q_j A_j)}{\sum_{j\in K} \operatorname{tr}(Q_j)}\right) \\ &+ \sum_{j\in L} \operatorname{tr}[Q_j f(A_j)] - \sum_{j\in L} \operatorname{tr}(Q_j) f\left(\frac{\sum_{j\in L} \operatorname{tr}(Q_j A_j)}{\sum_{j\in L} \operatorname{tr}(Q_j)}\right) \\ &= J_K(\mathbf{Q}, \mathbf{A}, f) + J_L(\mathbf{Q}, \mathbf{A}, f) \geq 0, \end{aligned} \tag{2.15}$$

which proves (2.13).

(ii). If $\emptyset \neq K \subset L$ with $L \diagdown K \neq \emptyset$, then we have by (2.13) that

$$\begin{aligned} J_L(\mathbf{Q},\mathbf{A},f) &= J_{K\cup(L\diagdown K)}(\mathbf{Q},\mathbf{A},f) \\ &\geq J_L(\mathbf{Q},\mathbf{A},f) + J_{L\diagdown K}(\mathbf{Q},\mathbf{A},f) \geq J_K(\mathbf{Q},\mathbf{A},f) \end{aligned}$$

and the inequality (2.14) is thus proved. □

Corollary 2. *Assume that f is convex on the interval I and $\mathbf{A} := (A_1,...,A_n)$ with $Sp(A_j) \subseteq I$, $j \in \{1,...,n\}$. Then we have the inequality*

$$J_k(\mathbf{Q},\mathbf{A},f) \geq J_{k-1}(\mathbf{Q},\mathbf{A},f) \geq 0 \tag{2.16}$$

for any $k \in \{1,...,n\}$ with $n \geq k \geq 2$.

We also have that

$$\begin{aligned} J_n(\mathbf{Q},\mathbf{A},f) &\geq \max_{j,k\in\{1,...,n\}} \Big\{ \operatorname{tr}[Q_j f(A_j)] + \operatorname{tr}[Q_k f(A_k)] \\ &\quad - \operatorname{tr}(Q_j+Q_k) f\left(\frac{\operatorname{tr}(Q_jA_j)+\operatorname{tr}(Q_kA_k)}{\operatorname{tr}(Q_j+Q_k)}\right)\Big\} \\ &\geq 0. \end{aligned} \tag{2.17}$$

Now, consider the weighted functional

$$J_K(\mathbf{q};Q,\mathbf{A},f) := \sum_{j\in K} q_j \operatorname{tr}[Qf(A_j)] - Q_K f\left(\frac{1}{Q_K}\sum_{j\in K} q_j \operatorname{tr}(QA_j)\right),$$

where $K \in \mathcal{P}_f(\mathbb{N})$, $Q \in \mathcal{B}_1^{++}(H)$ with $\operatorname{tr} Q = 1$ and $\mathbf{q} \in \mathcal{P}_n^+$ with $Q_K := \sum_{j\in K} q_j > 0$.

Proposition 2. *If $K, L \in \mathcal{P}_f(\mathbb{N}) \diagdown \{\emptyset\}$ with $K \cap L = \emptyset$, then we have the inequality*

$$J_{K\cup L}(\mathbf{q};Q,\mathbf{A},f) \geq J_K(\mathbf{q};Q,\mathbf{A},f) + J_L(\mathbf{q};Q,\mathbf{A},f) \geq 0,$$

i.e., $J_\cdot(\mathbf{q},Q,\mathbf{A},f)$ is super-additive as an index set functional.

If $\emptyset \neq K \subset L$, then we have

$$J_L(\mathbf{q};Q,\mathbf{A},f) \geq J_K(\mathbf{q};Q,\mathbf{A},f) \geq 0,$$

i.e., $J_\cdot(\mathbf{Q},\mathbf{A},f)$ is monotonic non-decreasing as an index set functional.

We have the inequality

$$J_k(\mathbf{q};Q,\mathbf{A},f) \geq J_{k-1}(\mathbf{q};Q,\mathbf{A},f) \geq 0$$

for any $k \in \{1,...,n\}$ with $n \geq k \geq 2$.

We also have the lower bound:

$$\begin{aligned} J_n(\mathbf{q};Q,\mathbf{A},f) &\geq \max_{j,k\in\{1,...,n\}} \Big\{ p_j \operatorname{tr}[Qf(A_j)] + p_k \operatorname{tr}[Qf(A_k)] \\ &\quad - (p_j+p_k) f\left(\frac{p_j \operatorname{tr}(QA_j)+p_k \operatorname{tr}(QA_k)}{p_j+p_k}\right)\Big\} \\ &\geq 0. \end{aligned}$$

3. Determinant Inequalities

We define the determinant functional

$$D_n(\mathbf{q};\mathbf{A},Q) := \frac{\left[\operatorname{tr}\left(\frac{1}{Q_n}\sum_{j=1}^n q_j Q A_j\right)\right]^{Q_n}}{\prod_{i=1}^n \left[\Delta_Q(A_j)\right]^{q_i}}, \tag{3.1}$$

where $\mathbf{A} = (A_1, ..., A_n)$ is an n-tuple of selfadjoint positive operators $\mathbf{p} \in \mathcal{P}_n^+$ and $Q \in \mathcal{B}_1^{++}(H)$ with $\operatorname{tr} Q = 1$.

Theorem 9. *Let $Q \in \mathcal{B}_1^{++}(H)$ with $\operatorname{tr} Q = 1$ and $\mathbf{A} := (A_1, ..., A_n)$ with $Sp(A_j) \subseteq I$, $j \in \{1, ..., n\}$.*

(i) *For all p, $\mathbf{q} \in \mathcal{P}_n^+$ we have*

$$D_n(\mathbf{p}+\mathbf{q};\mathbf{A},Q) \geq D_n(\mathbf{p};\mathbf{A},Q)\, D_n(\mathbf{q};\mathbf{A},Q) \geq 1 \tag{3.2}$$

i.e., the functional $J_n(\cdot, Q, \mathbf{A})$ is super-multiplicative on $\mathcal{P}_n^+$;

(ii) *For p, $\mathbf{q} \in \mathcal{P}_n^+$ with $\mathbf{p} \geq \mathbf{q}$*

$$D_n(\mathbf{p};Q,\mathbf{A}) \geq D_n(\mathbf{q};Q,\mathbf{A}) \geq 1, \tag{3.3}$$

i.e., the functional $J_n(\cdot, Q, \mathbf{A})$ is monotonic non-decreasing on $\mathcal{P}_n^+$.

Proof. (i) Consider the convex function $f(t) = -\ln t$, $t > 0$. Observe that for $\mathbf{A} = (A_1, ..., A_n)$ an n-tuple of selfadjoint positive operators $\mathbf{p} \in \mathcal{P}_n^+$ and $Q \in \mathcal{B}_1^{++}(H)$ with $\operatorname{tr} Q = 1$, then

$$\begin{aligned} J_n(\mathbf{q}; Q, \mathbf{A}, -\ln) &= Q_n \ln\left(\frac{1}{Q_n}\sum_{j=1}^n q_j \operatorname{tr}(QA_j)\right) - \sum_{j=1}^n q_j \operatorname{tr}\left[Q\ln(A_j)\right] \\ &= \ln\left(\frac{1}{Q_n}\sum_{j=1}^n q_j \operatorname{tr}(QA_j)\right)^{Q_n} - \sum_{j=1}^n q_j \operatorname{tr}\left[Q\ln(A_j)\right]. \end{aligned}$$

If we take the exponential, then we get

$$\begin{aligned} &\exp J_n(\mathbf{q}; Q, \mathbf{A}, -\ln) \\ &= \exp\left[\ln\left(\frac{1}{Q_n}\sum_{j=1}^n q_j \operatorname{tr}(QA_j)\right)^{Q_n} - \sum_{j=1}^n q_j \operatorname{tr}\left[Q\ln(A_j)\right]\right] \\ &= \frac{\exp\ln\left(\frac{1}{Q_n}\sum_{j=1}^n q_j \operatorname{tr}(QA_j)\right)^{Q_n}}{\exp\left(\sum_{j=1}^n q_j \operatorname{tr}\left[Q\ln(A_j)\right]\right)} = \frac{\left(\frac{1}{Q_n}\sum_{j=1}^n q_j \operatorname{tr}(QA_j)\right)^{Q_n}}{\prod_{i=1}^n \left[\Delta_Q(A_j)\right]^{q_i}} \\ &= D_n(\mathbf{q};\mathbf{A},Q). \end{aligned}$$

Therefore, by the properties of $J_n(\cdot, Q, \mathbf{A}, -\ln)$,

$$\begin{aligned} D_n(\mathbf{p}+\mathbf{q};\mathbf{A},Q) &= \exp J_n(\mathbf{p}+\mathbf{q}, Q, \mathbf{A}, -\ln) \\ &\geq \exp[J_n(\mathbf{p}, Q, \mathbf{A}, -\ln) + J_n(\mathbf{q}, Q, \mathbf{A}, -\ln)] \\ &= \exp J_n(\mathbf{p}, Q, \mathbf{A}, -\ln) \exp J_n(\mathbf{q}, Q, \mathbf{A}, -\ln) \\ &= D_n(\mathbf{p};\mathbf{A},Q)\, D_n(\mathbf{q};\mathbf{A},Q). \end{aligned}$$

(ii) The monotonicity of $D_n(\cdot;\mathbf{A},Q)$ follows by the monotonicity of $J_n(\cdot, Q, \mathbf{A}, -\ln)$. □

Corollary 3. *Let $Q \in \mathcal{B}_1^{++}(H)$ with $\operatorname{tr} Q = 1$ and $\mathbf{A} := (A_1, ..., A_n)$ with $Sp(A_j) \subseteq I$, $j \in \{1, ..., n\}$. Then*

$$[D_n(\mathbf{q};Q,\mathbf{A})]^{\min_{j\in\{1,\dots,n\}}\left\{\frac{p_j}{q_j}\right\}} \leq D_n(\mathbf{p};\mathbf{A},Q) \leq [D_n(\mathbf{q};Q,\mathbf{A})]^{\max_{j\in\{1,\dots,n\}}\left\{\frac{p_j}{q_j}\right\}} \tag{3.4}$$

and

$$D_n(Q,\mathbf{A})^{n\min_{j\in\{1,\dots,n\}}\{p_j\}} \leq D_n(\mathbf{p};\mathbf{A},Q) \leq [D_n(Q,\mathbf{A})]^{n\max_{j\in\{1,\dots,n\}}\{p_j\}}, \tag{3.5}$$

where

$$D_n(\mathbf{A},Q) := \frac{\operatorname{tr}\left(\frac{1}{n}\sum_{j=1}^n QA_j\right)}{\prod_{j=1}^n [\Delta_Q(A_j)]^{1/n}}.$$

For $n = 2$ and by choosing $p_1 = \alpha$, $p_2 = 1-\alpha$ with $\alpha \in [0, 1]$, we get from (3.8) the inequality for two positive operators A, B

$$\begin{aligned} 1 &\leq \left(\frac{\operatorname{tr}\left(Q\frac{A+B}{2}\right)}{[\Delta_Q(A)]^{1/2}[\Delta_Q(B)]^{1/2}}\right)^{2\min\{\alpha,1-\alpha\}} \\ &\leq \frac{\operatorname{tr}(Q((1-\alpha)A+\alpha B))}{[\Delta_Q(A)]^{1-\alpha}[\Delta_Q(B)]^{\alpha}} \leq \left(\frac{\operatorname{tr}\left(Q\frac{A+B}{2}\right)}{[\Delta_Q(A)]^{1/2}[\Delta_Q(B)]^{1/2}}\right)^{2\max\{\alpha,1-\alpha\}}. \end{aligned} \tag{3.6}$$

We also consider

$$D_K(\mathbf{q};Q,\mathbf{A}) := \frac{\left[\operatorname{tr}\left(\frac{1}{Q_K}\sum_{j\in K} q_j QA_j\right)\right]^{Q_K}}{\prod_{j\in K}[\Delta_Q(A_j)]^{q_i}}$$

where $K \in \mathcal{P}_f(\mathbb{N})$.

Proposition 3. *If K, $L \in \mathcal{P}_f(\mathbb{N}) \setminus \{\emptyset\}$ with $K \cap L = \emptyset$, then we have the inequality*

$$D_{K\cup L}(\mathbf{p}+\mathbf{q};\mathbf{A},Q) \geq D_K(\mathbf{p};\mathbf{A},Q)\, D_L(\mathbf{q};\mathbf{A},Q) \geq 1 \tag{3.7}$$

i.e., $D_\cdot(\mathbf{q};Q,\mathbf{A})$ is super-multiplicative as an index set functional.

If $\emptyset \neq K \subset L$, then we have

$$D_L(\mathbf{q};\mathbf{A},Q) \geq D_K(\mathbf{p};\mathbf{A},Q) \geq 1, \tag{3.8}$$

i.e., $D_\cdot(\mathbf{q};Q,\mathbf{A})$ is monotonic non-decreasing as an index set functional.

We have the inequality

$$D_k(\mathbf{q};\mathbf{A},Q) \geq D_{k-1}(\mathbf{p};\mathbf{A},Q) \geq 1 \tag{3.9}$$

for any $k \in \{1,...,n\}$ *with* $n \geq k \geq 2$.

We also have that

$$D_n(\mathbf{q};Q,\mathbf{A}) \geq \max_{j,k\in\{1,\dots,n\}} \frac{\left[\operatorname{tr}\left(Q\frac{q_jA_j+q_kA_k}{q_j+q_k}\right)\right]^{q_j+q_k}}{\left[\Delta_Q(A_j)\right]^{q_j}\left[\Delta_Q(A_k)\right]^{q_k}} \geq 1. \tag{3.10}$$

4. Other Properties

We define $\mathcal{C}_1\left(\mathcal{B}_1^+(H)\right)$ the class of non-negative operators Q from $\mathcal{B}_1(H)$ with $\operatorname{tr}(Q) = 1$. We observe that, if $Q_1,\ Q_2 \in \mathcal{C}_1\left(\mathcal{B}_1^+(H)\right)$ then for all $t \in [0,1]$, $(1-t)Q_1 + tQ_2 \in \mathcal{C}_1\left(\mathcal{B}_1^+(H)\right)$ showing that $\mathcal{C}_1\left(\mathcal{B}_1^+(H)\right)$ is a convex subset of $\mathcal{B}_1(H)$. Also, if $Q_n \in \mathcal{C}_1\left(\mathcal{B}_1^+(H)\right)$ and $Q_n \to Q$ in the operator norm topology, then also $Q \in \mathcal{C}_1\left(\mathcal{B}_1^+(H)\right)$.

Proposition 4. *The mapping* $\Delta_\cdot(A)$ *is convex on* $\mathcal{C}_1\left(\mathcal{B}_1^+(H)\right)$ *for all positive invertible operator* A.

Proof. Let $Q_1,\ Q_2 \in \mathcal{C}_1\left(\mathcal{B}_1^+(H)\right)$ then for all $t \in [0,1]$,

$$\begin{aligned}
\Delta_{(1-t)Q_1+tQ_2}(A) &= \exp\operatorname{tr}(((1-t)Q_1+tQ_2)\ln A) \\
&= \exp\left[(1-t)\operatorname{tr}(Q_1\ln A) + t\operatorname{tr}(Q_2\ln A)\right] \\
&= (\exp\operatorname{tr}(Q_1\ln A))^{(1-t)}(\exp\operatorname{tr}(Q_2\ln A))^t \\
&\leq (1-t)\exp\operatorname{tr}(Q_1\ln A) + t\exp\operatorname{tr}(Q_2\ln A) \\
&= (1-t)\Delta_{Q_1}(A) + t\Delta_{Q_2}(A),
\end{aligned}$$

which proves the convexity of $\Delta_\cdot(A)$. □

Using Jensen's inequality we have

$$\Delta_{\sum_{k=1}^n p_kQ_k}(A) \leq \sum_{k=1}^n p_k\Delta_{Q_k}(A) \tag{4.1}$$

for all $Q_k \in \mathcal{C}_1\left(\mathcal{B}_1^+(H)\right)$, $p_k \geq 0$, $k \in \{1,...,n\}$ with $\sum_{k=1}^n p_k = 1$.

By Hermite-Hadamard integral inequalities we also have

$$\Delta_{\frac{P+Q}{2}}(A) \leq \int_0^1 \Delta_{(1-t)P+tQ}(A)\,dt \leq \frac{1}{2}\left[\Delta_P(A) + \Delta_Q(A)\right] \tag{4.2}$$

for all $P,\ Q \in \mathcal{C}_1\left(\mathcal{B}_1^+(H)\right)$.

Since

$$\begin{aligned}
\int_0^1 \Delta_{(1-t)P+tQ}(A)\,dt &= \int_0^1 \exp\left[(1-t)\operatorname{tr}(P\ln A) + t\operatorname{tr}(Q\ln A)\right]dt \\
&= \begin{cases} \frac{\Delta_Q(A)-\Delta_P(A)}{\operatorname{tr}((Q-P)\ln A)} \text{ if } \operatorname{tr}((Q-P)\ln A) \neq 0 \\ \\ \exp\operatorname{tr}(P\ln A) \text{ if } \operatorname{tr}((Q-P)\ln A) \neq 0, \end{cases}
\end{aligned}$$

hence

$$\Delta_{\frac{P+Q}{2}}(A) \leq \frac{\Delta_Q(A) - \Delta_P(A)}{\operatorname{tr}((Q-P)\ln A)} \leq \frac{1}{2}\left[\Delta_P(A) + \Delta_Q(A)\right] \tag{4.3}$$

provided $\operatorname{tr}((Q-P)\ln A)\neq 0$.

Theorem 10. *For all* $P, Q\in\mathcal{C}_1\left(\mathcal{B}_1^+(H)\right)$ *and positive invertible operator* A *such that* $\operatorname{tr}((Q-P)\ln A)\neq 0$ *we have*

$$
\begin{aligned}
0 &\leq \frac{\Delta_Q(A)-\Delta_P(A)}{\operatorname{tr}((Q-P)\ln A)}-\Delta_{\frac{P+Q}{2}}(A) \\
&\leq \frac{1}{8}\operatorname{tr}((Q-P)\ln A)\left[\Delta_Q(A)-\Delta_P(A)\right]
\end{aligned}
\tag{4.4}
$$

and

$$
\begin{aligned}
0 &\leq \frac{1}{2}\left[\Delta_P(A)+\Delta_Q(A)\right]-\frac{\Delta_Q(A)-\Delta_P(A)}{\operatorname{tr}((Q-P)\ln A)} \\
&\leq \frac{1}{8}\operatorname{tr}((Q-P)\ln A)\left[\Delta_Q(A)-\Delta_P(A)\right].
\end{aligned}
\tag{4.5}
$$

Proof. For $P, Q\in\mathcal{C}_1\left(\mathcal{B}_1^+(H)\right)$, we consider the function $\varphi_{P,Q}:[0,1]\to(0,\infty)$,

$$\varphi_{P,Q}(t):=\Delta_{(1-t)P+tQ}(A),\ t\in[0,1].$$

Observe that

$$\varphi_{P,Q}(t)=\exp\left[(1-t)\operatorname{tr}(P\ln A)+t\operatorname{tr}(Q\ln A)\right],$$

for all $t\in[0,1]$. Obviously, the function $\varphi_{P,Q}$ is also a convex function on $[0,1]$ .In

The function $\varphi_{P,Q}$ is differentiable on $(0,1)$ and

$$\varphi'_{P,Q}(t)=\operatorname{tr}((Q-P)\ln A)\exp\left[(1-t)\operatorname{tr}(P\ln A)+t\operatorname{tr}(Q\ln A)\right].$$

The lateral derivatives $\varphi'_{+P,Q}(0)$ and $\varphi'_{-P,Q}(1)$ also exist,

$$\varphi'_{+P,Q}(0)=\operatorname{tr}((Q-P)\ln A)\,\Delta_P(A)$$

and

$$\varphi'_{-P,Q}(1)=\operatorname{tr}((Q-P)\ln A)\,\Delta_Q(A).$$

In [1] we obtained the following reverse of first Hermite-Hadamard inequality for the convex function $f:[a,b]\to\mathbb{R}$,

$$0\leq\frac{1}{b-a}\int_a^b f(s)\,ds-f\left(\frac{a+b}{2}\right)\leq\frac{1}{8}(b-a)\left[f'_-(b)-f'_+(a)\right],$$

with $\frac{1}{8}$ the best possible constant.

Therefore

$$0\leq\int_0^1\varphi_{P,Q}(t)\,dt-\varphi_{P,Q}\left(\frac{1}{2}\right)\leq\frac{1}{8}\left[\varphi'_{-P,Q}(1)-\varphi'_{+P,Q}(0)\right],$$

namely

$$
\begin{aligned}
0 &\leq \int_0^1\Delta_{(1-t)P+tQ}(A)\,dt-\Delta_{\frac{P+Q}{2}}(A) \\
&\leq \frac{1}{8}\operatorname{tr}((Q-P)\ln A)\left[\Delta_Q(A)-\Delta_P(A)\right],
\end{aligned}
$$

which gives (4.4).

In [2] we also obtained the revese of the second Hermite-Hadamard inequality for the convex function $f:[a,b]\to\mathbb{R}$,

$$0\leq\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(s)\,ds\leq\frac{1}{8}(b-a)\left[f'_-(b)-f'_+(a)\right],$$

with $\frac{1}{8}$ the best possible constant. Applying this inequality, we derive the inequality (4.5). □

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