Generalized symmetrical sigmoid function activated Banach space valued ordinary and fractional neural network approximation

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Abstract

Here we research the univariate quantitative approximation, ordinary and fractional, of Banach space valued continuous functions on a compact interval or all the real line by quasi-interpolation Banach space valued neural network operators. These approximations are derived by establishing Jackson type inequalities involving the modulus of continuity of the engaged function or its Banach space valued high order derivative or fractional derivatives. Our operators are defined by using a density function generated by a generalized symmetrical sigmoid function. The approximations are pointwise and of the uniform norm. The related Banach space valued feed-forward neural networks are with one hidden layer.

2020 AMS Mathematics Subject Classification : 26A33, 41A17, 41A25, 41A30, 46B25.
Keywords and Phrases: generalized symmetrical sigmoid function, Banach space valued neural network approximation, Banach space valued quasi-interpolation operator, modulus of continuity, Banach space valued Caputo fractional derivative, Banach space valued fractional approximation, complex approximation.

1 Introduction

The author in [1] and [2], see Chapters 2-5, was the first to establish neural network approximation to continuous functions with rates by very specifically defined neural network operators of Cardaliagnet-Euvrard and "Squashing" types, by employing the modulus of continuity of the engaged function or its high order derivative, and producing very tight Jackson type inequalities. He treats there

RGMIA Res. Rep. Coll. **25** (2022), Art. 27, 23 pp. Received 27/02/22

both the univariate and multivariate cases. The defining these operators "bell-shaped "and "squashing "functions are assumed to be of compact suport. Also in [2] he gives the Nth order asymptotic expansion for the error of weak approximation of these two operators to a special natural class of smooth functions, see Chapters 4-5 there.

The author inspired by [15], continued his studies on neural networks approximation by introducing and using the proper quasi-interpolation operators of sigmoidal and hyperbolic tangent type which resulted into [3], [4], [5], [6], [7], by treating both the univariate and multivariate cases. He did also the corresponding fractional cases [8], [9], [13].

The author here performs generalized symmetrical sigmoid function based neural network approximations to continuous functions over compact intervals of the real line or over the whole $\mathbb{R}$ with valued to an arbitrary Banach space $(X, \|\cdot\|)$. Finally he treats completely the related X-valued fractional approximation. All convergences here are with rates expressed via the modulus of continuity of the involved function or its X-valued high order derivative, or X-valued fractional derivatives and given by very tight Jackson type inequalities.

Our compact intervals are not necessarily symmetric to the origin. Some of our upper bounds to error quantity are very flexible and general. In preparation to prove our results we establish important properties of the basic density function defining our operators which is induced by generalized symmetrical sigmoid function.

Feed-forward X-valued neural networks (FNNs) with one hidden layer, the only type of networks we deal with in this article, are mathematically expressed as

$$N_n(x) = \sum_{=0}^{n} c_j \sigma(\langle a_j \cdot x \rangle + b_j), \quad x \in \mathbb{R}^s, \ \ s \in \mathbb{N},$$

where for $0 \leq j \leq n$, $b_j \in \mathbb{R}$ are the thresholds, $a_j \in \mathbb{R}^s$ are the connection weights, $c_j \in X$ are the coefficients, $\langle a_j \cdot x \rangle$ is the inner product of a_j and x, and σ is the activation function of the network. About neural networks in general read [17], [18], [20]. See also [9] for a complete study of real valued approximation by neural network operators.

2 Auxiliary Results

Here we consider the generalized symmetrical sigmoid function ([16])

$$f_1(x) = \frac{x}{(1 + |x|^{\mu})^{\frac{1}{\mu}}}, \quad \mu > 0, \, x \in \mathbb{R}. \tag{1}$$

This has applications in immunology and protection from disease together with probability theory. It is also called a symmetrical protection curve.

The parameter μ is a shape parameter controling how fast the curve approaches the asymptotes for a given slope at the inflection point. When $\mu = 1$ f_1 is the absolute sigmoid function, and when $\mu = 2$ f_1 is the square root sigmoid function. When $\mu = 1.5$ the function approximates the arctangent function, when $\mu = 2.9$ it approximates the logistic function, and when $\mu = 3.4$ it approximates the error function. Parameter μ is estimated in the likelihood maximization ([16]). For more see [16].

Next we study the particular generator sigmoid function

$$f_2(x) = \frac{x}{\left(1+|x|^\lambda\right)^{\frac{1}{\lambda}}}, \quad \lambda \text{ is an odd number, } x \in \mathbb{R}. \tag{2}$$

We have that $f_2(0) = 0$, and

$$f_2(-x) = -\frac{x}{\left(1+|x|^\lambda\right)^{\frac{1}{\lambda}}} = -f_2(x),$$

i.e.

$$f_2(-x) = -f_2(x), \tag{3}$$

so f_2 is symmetric with respect to zero.

Let $x \geq 0$, then $f_2(x) = \frac{x}{\left(1+|x|^\lambda\right)^{\frac{1}{\lambda}}}$, and

$$f_2{}'(x) = \frac{\left(1+x^\lambda\right)^{\frac{1}{\lambda}} - x\frac{1}{\lambda}\left(1+x^\lambda\right)^{\frac{1}{\lambda}-1}\lambda x^{\lambda-1}}{\left(1+x^\lambda\right)^{\frac{2}{\lambda}}} =$$

$$\frac{\left(1+x^\lambda\right)^{\frac{1}{\lambda}}\left[1-\left(1+x^\lambda\right)^{-1}x^\lambda\right]}{\left(1+x^\lambda\right)^{\frac{2}{\lambda}}} = \frac{\left(1+x^\lambda\right)^{\frac{1}{\lambda}}\left[1-\frac{x^\lambda}{1+x^\lambda}\right]}{\left(1+x^\lambda\right)^{\frac{2}{\lambda}}} =$$

$$\frac{\left(1+x^\lambda\right)^{\frac{1}{\lambda}}\left[1+x^\lambda-x^\lambda\right]}{\left(1+x^\lambda\right)\left(1+x^\lambda\right)^{\frac{2}{\lambda}}} = \frac{\left(1+x^\lambda\right)^{\frac{1}{\lambda}}}{\left(1+x^\lambda\right)\left(1+x^\lambda\right)^{\frac{2}{\lambda}}} = \frac{1}{\left(1+x^\lambda\right)^{\frac{\lambda+1}{\lambda}}}. \tag{4}$$

That is when $x \geq 0$, we get that

$$f_2'(x) = \frac{1}{\left(1+x^\lambda\right)^{\frac{\lambda+1}{\lambda}}} > 0, \tag{5}$$

and f_2 is strictly increasing on $[0, +\infty)$. Let $x_1 < x_2 \leq 0$, then $-x_1 > -x_2$ and $f_2(-x_1) > f_2(-x_2)$, hence $-f_2(x_1) > -f_2(x_2)$ and $f_2(x_1) < f_2(x_2)$. Therefore f_2 is strictly increasing on $(-\infty, 0]$. Consequently f_2 is strictly increasing on $\mathbb{R}$.

Let $x > 0$, then $f_2(x) = \frac{x}{\left(1+x^\lambda\right)^{\frac{1}{\lambda}}} = \frac{1}{\frac{\left(1+x^\lambda\right)^{\frac{1}{\lambda}}}{x^{\frac{\lambda}{\lambda}}}} = \frac{1}{\left(\frac{1}{x^\lambda}+1\right)^{\frac{1}{\lambda}}} \underset{x\to+\infty}{\rightarrow} 1$. I.e.

$$f_2(+\infty) = 1. \tag{6}$$

Let $x < 0$, then $f_2(x) = \frac{x}{\left(1+(-1)^{\lambda} x^{\lambda}\right)^{\frac{1}{\lambda}}} = \frac{x}{(1-x^{\lambda})^{\frac{1}{\lambda}}} = \frac{1}{\left(\frac{1}{x^{\lambda}}-1\right)^{\frac{1}{\lambda}}} \underset{x\to-\infty}{\to} \frac{1}{(-1)^{\frac{1}{\lambda}}} =$ -1. I.e.

$$f_2(-\infty) = 1. \tag{7}$$

Let us consider the activation function:

$$\chi(x) = \frac{1}{4}\left[f_2(x+1) - f_2(x-1)\right] = \tag{8}$$

$$\frac{1}{4}\left[\frac{(x+1)}{\left(1+|x+1|^{\lambda}\right)^{\frac{1}{\lambda}}} - \frac{(x-1)}{\left(1+|x-1|^{\lambda}\right)^{\frac{1}{\lambda}}}\right].$$

We have that

$$\chi(-x) = \frac{1}{4}\left[\frac{-x+1}{\left(1+|-x+1|^{\lambda}\right)^{\frac{1}{\lambda}}} - \frac{(-x-1)}{\left(1+|-x-1|^{\lambda}\right)^{\frac{1}{\lambda}}}\right] =$$

$$\frac{1}{4}\left[\frac{-(x-1)}{\left(1+|x-1|^{\lambda}\right)^{\frac{1}{\lambda}}} + \frac{(x+1)}{\left(1+|x+1|^{\lambda}\right)^{\frac{1}{\lambda}}}\right] =$$

$$\frac{1}{4}\left[\frac{(x+1)}{\left(1+|x+1|^{\lambda}\right)^{\frac{1}{\lambda}}} - \frac{(x-1)}{\left(1+|x-1|^{\lambda}\right)^{\frac{1}{\lambda}}}\right] = \chi(x).$$

I.e.

$$\chi(x) = \chi(-x), \quad \forall\, x \in \mathbb{R}. \tag{9}$$

We see that

$$\chi(0) = \frac{1}{4}\left[f_2(1) - f_2(-1)\right] = \frac{1}{4}\left[f_2(1) - (-f_2(1))\right] =$$

$$\frac{1}{4}\left[f_2(1) + f_2(1)\right] = \frac{1}{2} f_2(1) = \frac{1}{2}\frac{1}{2^{\frac{1}{k}}} = \frac{1}{2\sqrt[k]{2}}.$$

That is

$$\chi(0) = \frac{1}{2\sqrt[k]{2}}. \tag{10}$$

Since $x+1 > x-1$, we get $\chi(x) > 0$, $\forall\, x \in \mathbb{R}$.

Let $x \geq 1$, then $x-1 \geq 0$, hence $x+1 > x-1 (\geq 0)$. Thus

$$\chi'(x) = \frac{1}{4}\left[f_2'(x+1) - f_2'(x-1)\right] =$$

$$\frac{1}{4}\left[\frac{1}{\left(1+(x+1)^{\lambda}\right)^{\frac{\lambda+1}{\lambda}}}-\frac{1}{\left(1+(x-1)^{\lambda}\right)^{\frac{\lambda+1}{\lambda}}}\right]= \tag{11}$$

$$\frac{1}{4}\left[\frac{\left(1+(x-1)^{\lambda}\right)^{\frac{\lambda+1}{\lambda}}-\left(1+(x+1)^{\lambda}\right)^{\frac{\lambda+1}{\lambda}}}{\left(1+(x+1)^{\lambda}\right)^{\frac{\lambda+1}{\lambda}}\left(1+(x-1)^{\lambda}\right)^{\frac{\lambda+1}{\lambda}}}\right]=:(\xi).$$

(from $0 \leq x-1 < x+1$, then $(x-1)^{\lambda} < (x+1)^{\lambda}$ and $1+(x-1)^{\lambda} < 1+(x+1)^{\lambda}$, and $\left(1+(x-1)^{\lambda}\right)^{\frac{\lambda+1}{\lambda}} < \left(1+(x+1)^{\lambda}\right)^{\frac{\lambda+1}{\lambda}}$).

Hence $\chi'(x) = (\xi) < 0$.

Therefore χ is strictly decreasing over $[1,+\infty)$.

Let now $0 \leq x \leq 1$, then $1-x \geq 0$, and $f_2(x-1) = f_2(-(1-x)) = -f_2(1-x)$, and $(f_2(x-1))' = -(f_2(1-x))' = -f_2'(1-x)(1-x)' =$
$-f_2'(1-x)(-1) = f_2'(1-x)$.

That is

$$(f_2(x-1))' = f_2'(1-x). \tag{12}$$

Again we have

$$\chi'(x) = \frac{1}{4}\left[f_2'(x+1) - f_2'(x-1)\right] = \frac{1}{4}\left[f_2'(x+1) - f_2'(1-x)\right] =$$

$$\frac{1}{4}\left[\frac{1}{\left(1+(x+1)^{\lambda}\right)^{\frac{\lambda+1}{\lambda}}}-\frac{1}{\left(1+(1-x)^{\lambda}\right)^{\frac{\lambda+1}{\lambda}}}\right]=$$

$$\frac{1}{4}\left[\frac{\left(1+(1-x)^{\lambda}\right)^{\frac{\lambda+1}{\lambda}}-\left(1+(1+x)^{\lambda}\right)^{\frac{\lambda+1}{\lambda}}}{\left(1+(x+1)^{\lambda}\right)^{\frac{\lambda+1}{\lambda}}\left(1+(1-x)^{\lambda}\right)^{\frac{\lambda+1}{\lambda}}}\right]<0, \tag{13}$$

(by $-x < x$ and $0 \leq 1-x < 1+x$, $(1-x)^{\lambda} < (1+x)^{\lambda}$ and $\left(1+(1-x)^{\lambda}\right)^{\frac{\lambda+1}{\lambda}} <$
$\left(1+(1+x)^{\lambda}\right)^{\frac{\lambda+1}{\lambda}}$)

when $0 < x \leq 1$, therefore $\chi'(x) < 0$, when $0 < x \leq 1$.

That is χ is strictly decreasing on $(0,1]$.

By continuity of χ it is strictly decreasing on $[0,\infty)$.

Since χ is symmetric with respect to y-axis, χ is striclty increasing on $(-\infty,0]$.

Next we would like to find $\chi'(0)$:

Let now $-1 \leq x \leq 0(< 1)$, then $x+1 \geq 0$, and $1-x > 0$, and again $(f_2(x-1))' = f_2'(1-x)$.

Again it holds

$$\chi'(x) = \frac{1}{4}\left[f_2'(x+1) - (f_2(x-1))'\right] = \frac{1}{4}\left[f_2'(x+1) - f_2'(1-x)\right] =$$

$$\frac{1}{4}\left[\frac{\left(1+(1-x)^{\lambda}\right)^{\frac{\lambda+1}{\lambda}} - \left(1+(1+x)^{\lambda}\right)^{\frac{\lambda+1}{\lambda}}}{\left(1+(x+1)^{\lambda}\right)^{\frac{\lambda+1}{\lambda}}\left(1+(1-x)^{\lambda}\right)^{\frac{\lambda+1}{\lambda}}}\right], \tag{14}$$

true for $-1 \leq x \leq 0$.

In fact $\chi'(x)$ is given by the same formula (see (13), (14)) over $[-1,1]$, which is a continuous function.

We have that

$$\chi'(0+) = \lim_{\substack{x\to 0+ \\ (x>0)}} \chi'(x) = 0. \tag{15}$$

Clearly it holds

$$\chi'(0-) = \lim_{\substack{x\to 0- \\ (x<0)}} \chi'(x) = 0. \tag{16}$$

Therefore it is

$$\chi'(0) = 0,$$

by continuity of χ' over $[-1,1]$.

Clearly it is

$$\lim_{x\to+\infty} \chi(x) = 0, \quad \lim_{x\to-\infty} \chi(x) = 0, \tag{17}$$

therefore the x-axis is the horizontal asymptote of $\chi(x)$.

The value

$$\chi(0) = \frac{1}{2\sqrt[k]{2}}$$

is the maximum of χ, which is a bell shaped function.

We give

Theorem 1 *We have that*

$$\sum_{i=-\infty}^{\infty} \chi(x-i) = 1, \ \ \forall\, x \in \mathbb{R}. \tag{18}$$

Proof. We observe that

$$\sum_{i=-\infty}^{\infty} \left(f_2(x-i) - f_2(x-1-i)\right) =$$

$$\sum_{i=0}^{\infty} \left(f_2(x-i) - f_2(x-1-i)\right) + \sum_{i=-\infty}^{-1} \left(f_2(x-i) - f_2(x-1-i)\right).$$

Furthermore ($\overline{\lambda} \in \mathbb{Z}^{+}$) we have

$$\sum_{i=0}^{\infty} \left(f_2\left(x-i\right)-f_2\left(x-1-i\right)\right)=\lim_{\overline{\lambda}\to\infty}\sum_{i=0}^{\overline{\lambda}}\left(f_2\left(x-i\right)-f_2\left(x-1-i\right)\right) \tag{19}$$

(telescoping sum)

$$=\lim_{\overline{\lambda}\to\infty}\left(f_2\left(x\right)-f_2\left(x-\left(\overline{\lambda}+1\right)\right)\right)=1+f_2\left(x\right).$$

Similarly

$$\sum_{i=-\infty}^{-1}\left(f_2\left(x-i\right)-f_2\left(x-1-i\right)\right)=\lim_{\overline{\lambda}\to\infty}\sum_{i=-\overline{\lambda}}^{-1}\left(f_2\left(x-i\right)-f_2\left(x-1-i\right)\right)$$

$$=\lim_{\overline{\lambda}\to\infty}\left(f_2\left(x+\overline{\lambda}\right)-f_2\left(x\right)\right)=1-f_2\left(x\right). \tag{20}$$

So, by adding the last two limits we obtain

$$\sum_{i=-\infty}^{\infty}\left(f_2\left(x-i\right)-f_2\left(x-1-i\right)\right)=2,\ \ \forall\ x\in\mathbb{R}. \tag{21}$$

Therefore

$$\sum_{i=-\infty}^{\infty}\left(f_2\left(x+1-i\right)-f_2\left(x-i\right)\right)=2,\ \ \forall\ x\in\mathbb{R}. \tag{22}$$

Consequently, by adding (21), (22) we get

$$\sum_{i=-\infty}^{\infty}\left(f_2\left(x+1-i\right)-f_2\left(x-1-i\right)\right)=4,\ \ \forall\ x\in\mathbb{R}, \tag{23}$$

proving the claim. ∎

Furthemore we give:

Because χ is even it holds

$$\sum_{i=-\infty}^{\infty}\chi\left(i-x\right)=1,\ \ \forall\ x\in\mathbb{R}, \tag{24}$$

and

$$\sum_{i=-\infty}^{\infty}\chi\left(i+x\right)=1,\ \ \forall\ x\in\mathbb{R}. \tag{25}$$

That is

$$\sum_{i=-\infty}^{\infty}\chi\left(x+i\right)=1,\ \ \forall\ x\in\mathbb{R}. \tag{26}$$

We give

Theorem 2 *It holds*
$$\int_{-\infty}^{\infty} \chi(x)\, dx = 1.$$

Proof. We observe that
$$\int_{-\infty}^{\infty} \chi(x)\, dx = \sum_{j=-\infty}^{\infty} \int_{j}^{j+1} \chi(x)\, dx = \sum_{j=-\infty}^{\infty} \int_{0}^{1} \chi(x+j)\, dx = \tag{27}$$
$$\int_0^1 \left(\sum_{j=-\infty}^{\infty} \chi(x+j)\, dx \right) = \int_0^1 1 dx = 1.$$
■

So that $\chi(x)$ is a density function on $\mathbb{R}$.

We need

Theorem 3 *Let $0 < \alpha < 1$, and $n \in \mathbb{N}$ with $n^{1-\alpha} > 2$. It holds*
$$\sum_{\substack{j = -\infty \\ : |nx - j| \geq n^{1-\alpha}}}^{\infty} \chi(nx - j) < \frac{1}{2\lambda \left(n^{1-\alpha} - 2\right)^{\lambda}}, \tag{28}$$
where $\lambda \in \mathbb{N}$ is an odd number.

Proof. We have that
$$\chi(x) = \frac{1}{4}\left[f_2(x+1) - f_2(x-1)\right], \quad \forall\, x \in \mathbb{R}.$$
Let $x \geq 1$. That is $0 \leq x - 1 < x + 1$. Applying the mean value theorem we get
$$\chi(x) = \frac{1}{4} \cdot 2 \cdot f_2'(\xi) = \frac{1}{2\left(1+\xi^{\lambda}\right)^{\frac{\lambda+1}{\lambda}}} > 0, \tag{29}$$
where $0 \leq x - 1 < \xi < x + 1$.

Then,
$$(x-1)^{\lambda} < \xi^{\lambda} < (x+1)^{\lambda},$$
and
$$1 + (x-1)^{\lambda} < 1 + \xi^{\lambda} < 1 + (x+1)^{\lambda}, \tag{30}$$
and
$$2\left(1 + (x-1)^{\lambda}\right)^{\frac{\lambda+1}{\lambda}} < 2\left(1+\xi^{\lambda}\right)^{\frac{\lambda+1}{\lambda}} < 2\left(1 + (x+1)^{\lambda}\right)^{\frac{\lambda+1}{\lambda}}.$$
Hence it holds
$$\frac{1}{2\left(1+\xi^{\lambda}\right)^{\frac{\lambda+1}{\lambda}}} < \frac{1}{2\left(1 + (x-1)^{\lambda}\right)^{\frac{\lambda+1}{\lambda}}}, \quad \forall\, x \geq 1.$$

We have found that

$$\chi(x) < \frac{1}{2\left(1+(x-1)^{\lambda}\right)^{\frac{\lambda+1}{\lambda}}}, \quad \forall\, x \geq 1. \tag{31}$$

Thus, we have

$$\sum_{\begin{cases} j=-\infty \\ : |nx-j| \geq n^{1-\alpha} \end{cases}}^{\infty} \chi(nx-j) = \sum_{\begin{cases} j=-\infty \\ : |nx-j| \geq n^{1-\alpha} \end{cases}}^{\infty} \chi(|nx-j|) <$$

$$\sum_{\begin{cases} j=-\infty \\ : |nx-j| \geq n^{1-\alpha} \end{cases}}^{\infty} \frac{1}{2\left(1+(|nx-j|-1)^{\lambda}\right)^{\frac{\lambda+1}{\lambda}}} \leq$$

$$\frac{1}{2}\int_{(n^{1-\alpha}-1)}^{\infty} \frac{1}{\left(1+(x-1)^{\lambda}\right)^{\frac{\lambda+1}{\lambda}}} dx = \frac{1}{2}\int_{(n^{1-\alpha}-2)}^{\infty} \frac{1}{(1+z^{\lambda})^{\frac{\lambda+1}{\lambda}}} dz =: (*). \tag{32}$$

We see that

$$\begin{gathered} 1+z^{\lambda} > z^{\lambda}, \\ (1+z^{\lambda})^{\frac{\lambda+1}{\lambda}} > z^{\lambda+1}, \\ \frac{1}{z^{\lambda+1}} > \frac{1}{(1+z^{\lambda})^{\frac{\lambda+1}{\lambda}}}. \end{gathered} \tag{33}$$

Therefore it holds

$$(*) < \frac{1}{2}\int_{(n^{1-\alpha}-2)}^{\infty} \frac{1}{z^{\lambda+1}} dz = \frac{1}{2}\int_{(n^{1-\alpha}-2)}^{\infty} z^{-(\lambda+1)} dz =$$

$$\frac{1}{2}\left(\frac{z^{-(\lambda+1)+1}}{-(\lambda+1)+1}\right)\Bigg|_{n^{1-\alpha}-2}^{\infty} = \frac{1}{2}\left(-\frac{z^{-\lambda}}{\lambda}\right)\Bigg|_{n^{1-\alpha}-2}^{\infty} =$$

$$\frac{1}{2\lambda}\left(z^{-\lambda}\right)\Bigg|_{\infty}^{n^{1-\alpha}-2} = \frac{1}{2\lambda\left(n^{1-\alpha}-2\right)^{\lambda}}, \tag{34}$$

proving (28). ■

Denote by $\lfloor\cdot\rfloor$ the integral part of the number and by $\lceil\cdot\rceil$ the ceiling of the number.

Theorem 4 *Let* $[a,b] \subset \mathbb{R}$ *and* $n \in \mathbb{N}$ *so that* $\lceil na\rceil \leq \lfloor nb\rfloor$. *Then*

$$\frac{1}{\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} \chi(|nx-k|)} < 2\sqrt[\lambda]{1+2^{\lambda}}, \tag{35}$$

where λ *is an odd number,* $\forall\, x \in [a,b]$.

Proof. Let $x \in [a,b]$. We see that

$$1 = \sum_{k=-\infty}^{\infty} \chi(nx-k) > \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \chi(nx-k) =$$

$$\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \chi(|nx-k|) > \chi(|nx-k_0|), \tag{36}$$

$\forall\ k_0 \in [\lceil na \rceil, \lfloor nb \rfloor] \cap \mathbb{Z}$.

We can choose $k_0 \in [\lceil na \rceil, \lfloor nb \rfloor] \cap \mathbb{Z}$ such that $|nx-k_0| < 1$.

Therefore we get

$$\chi(|nx-k_0|) > \chi(1) = \frac{1}{4}(f_2(2) - f_2(0)) = \frac{1}{4} f_2(2) = \tag{37}$$

$$\frac{1}{4} \frac{2}{(1+2^{\lambda})^{\frac{1}{\lambda}}} = \frac{1}{2(1+2^{\lambda})^{\frac{1}{\lambda}}},$$

and

$$\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \chi(|nx-k|) > \frac{1}{2(1+2^{\lambda})^{\frac{1}{\lambda}}}. \tag{38}$$

That is

$$\frac{1}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \chi(|nx-k|)} < 2\left(1+2^{\lambda}\right)^{\frac{1}{\lambda}}, \tag{39}$$

proving the claim. ■

We make

Remark 5 *We also notice that*

$$1 - \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \chi(nb-k) = \sum_{k=-\infty}^{\lceil na \rceil - 1} \chi(nb-k) + \sum_{k=\lfloor nb \rfloor + 1}^{\infty} \chi(nb-k)$$

$$> \chi(nb - \lfloor nb \rfloor - 1) \tag{40}$$

(call $\varepsilon := nb - \lfloor nb \rfloor$, $0 \leq \varepsilon < 1$)

$$= \chi(\varepsilon - 1) = \chi(1-\varepsilon) \geq \chi(1) > 0.$$

Therefore

$$\lim_{n \to \to \infty} \left(1 - \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \chi(nb-k)\right) > 0. \tag{41}$$

Similarly, it holds

$$1-\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}\chi(na-k)=\sum_{k=-\infty}^{\lceil na\rceil-1}\chi(na-k)+\sum_{k=\lfloor nb\rfloor+1}^{\infty}\chi(na-k)$$

$$>\chi(na-\lceil na\rceil+1) \tag{42}$$

(call $\eta:=\lceil na\rceil-na,\ 0\leq\eta<1$*)*

$$=\chi(1-\eta)\geq\chi(1)>0.$$

Therefore again

$$\lim_{n\to\infty}\left(1-\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}\chi(na-k)\right)>0. \tag{43}$$

Here we find that

$$\lim_{n\to\infty}\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}\chi(nx-k)\neq 1,\quad \text{for at least some } x\in[a,b]. \tag{44}$$

Note 6 *Let* $[a,b]\subset\mathbb{R}$*. For large enough* n *we always obtain* $\lceil na\rceil\leq\lfloor nb\rfloor$*. Also* $a\leq\frac{k}{n}\leq b$*, iff* $\lceil na\rceil\leq k\leq\lfloor nb\rfloor$*.*

In general it holds (by $\sum_{i=-\infty}^{\infty}\chi(x-i)=1,\ \forall\ x\in\mathbb{R}$*) that*

$$\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}\chi(nx-k)\leq 1. \tag{45}$$

Let $(X,\|\cdot\|)$ be a Banach space.

Definition 7 *Let* $f\in C([a,b],X)$ *and* $n\in\mathbb{N}:\lceil na\rceil\leq\lfloor nb\rfloor$*. We introduce and define the* X*-valued linear neural network operators*

$$H_n(f,x):=\frac{\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}f\left(\frac{k}{n}\right)\chi(nx-k)}{\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}\chi(nx-k)},\quad x\in[a,b]. \tag{46}$$

Clearly here $H_n(f,x)\in C([a,b],X)$.

For convenience we use the same H_n for real valued functions when needed. We study here the pointwise and uniform convergence of $H_n(f,x)$ to $f(x)$ with rates.

For convenience, also we call

$$H_n^*(f,x) := \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} f\left(\frac{k}{n}\right)\chi(nx-k), \tag{47}$$

(similarly, H_n^* can be defined for real valued functions) that is

$$H_n(f,x) := \frac{H_n^*(f,x)}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \chi(nx-k)}. \tag{48}$$

So that

$$H_n(f,x) - f(x) = \frac{H_n^*(f,x)}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \chi(nx-k)} - f(x) =$$

$$\frac{H_n^*(f,x) - f(x)\left(\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \chi(nx-k)\right)}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \chi(nx-k)}. \tag{49}$$

Consequently, we derive that

$$\|H_n(f,x) - f(x)\| \leq 2\left(\sqrt[\lambda]{1+2^{\lambda}}\right)\left\|H_n^*(f,x) - f(x)\left(\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \chi(nx-k)\right)\right\| =$$

$$2\left(\sqrt[\lambda]{1+2^{\lambda}}\right)\left\|\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \left(f\left(\frac{k}{n}\right) - f(x)\right)\chi(nx-k)\right\|. \tag{50}$$

We will estimate the right and hand side of (50).

For that we need, for $f \in C([a,b], X)$ the first modulus of continuity

$$\omega_1(f,\delta) := \sup_{\substack{x,y \in [a,b] \\ |x-y| \leq \delta}} \|f(x) - f(y)\|, \quad \delta > 0. \tag{51}$$

Similarly, it is defined ω_1 for $f \in C_{uB}(\mathbb{R}, X)$ (uniformly continuous and bounded functions from $\mathbb{R}$ into X), for $f \in C_B(\mathbb{R}, X)$ (continuous and bounded X-valued), and for $f \in C_u(\mathbb{R}, X)$ (uniformly continuous).

The fact $f \in C([a,b], X)$ or $f \in C_u(\mathbb{R}, X)$, is equivalent to $\lim_{\delta \to 0} \omega_1(f,\delta) = 0$, see [11].

We make

Definition 8 *When $f \in C_{uB}(\mathbb{R}, X)$, or $f \in C_B(\mathbb{R}, X)$, we define*

$$\overline{H_n}(f, x) := \sum_{k=-\infty}^{\infty} f\left(\frac{k}{n}\right) \chi(nx - k), \tag{52}$$

$n \in \mathbb{N}$, $x \in \mathbb{R}$,

the X-valued quasi-interpolation neural network operator.

We make

Remark 9 *We have that*

$$\left\| f\left(\frac{k}{n}\right) \right\| \leq \|f\|_{\infty,\mathbb{R}} < +\infty, \tag{53}$$

and

$$\left\| f\left(\frac{k}{n}\right) \right\| \chi(nx - k) \leq \|f\|_{\infty,\mathbb{R}} \chi(nx - k) \tag{54}$$

and

$$\sum_{k=-\overline{\lambda}}^{\overline{\lambda}} \left\| f\left(\frac{k}{n}\right) \right\| \chi(nx - k) \leq \|f\|_{\infty,\mathbb{R}} \left(\sum_{k=-\overline{\lambda}}^{\overline{\lambda}} \chi(nx - k) \right), \tag{55}$$

and finally

$$\sum_{k=-\infty}^{\infty} \left\| f\left(\frac{k}{n}\right) \right\| \chi(nx - k) \leq \|f\|_{\infty,\mathbb{R}}, \tag{56}$$

a convergent in $\mathbb{R}$ series.

So, the series $\sum_{k=-\infty}^{\infty} \left\| f\left(\frac{k}{n}\right) \right\| \chi(nx - k)$ is absolutely convergent in X, hence it is convergent in X and $\overline{H_n}(f, x) \in X$. We denote by $\|f\|_\infty := \sup_{x \in [a,b]} \|f(x)\|$, for $f \in C([a, b], X)$, similarly it is defined for $f \in C_B(\mathbb{R}, X)$.

3 Main Results

We present a set of X-valued neural network approximations to a function given with rates.

Theorem 10 *Let $f \in C([a, b], X)$, $0 < \alpha < 1$, $n \in \mathbb{N} : n^{1-\alpha} > 2$, $x \in [a, b]$, $\lambda \in \mathbb{N}$ is odd. Then*

i)

$$\|H_n(f, x) - f(x)\| \leq 2 \sqrt[\lambda]{1 + 2^\lambda} \left[\omega_1\left(f, \frac{1}{n^\alpha}\right) + \frac{\|f\|_\infty}{\lambda (n^{1-\alpha} - 2)^\lambda} \right] =: \rho_1, \tag{57}$$

and

ii)

$$\|H_n(f) - f\|_\infty \leq \rho_1. \tag{58}$$

We get that $\lim_{n\to\infty} H_n(f) = f$*, pointwise and uniformly.*

Proof. We see that

$$\left\| \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \left(f\left(\frac{k}{n}\right) - f(x) \right) \chi(nx-k) \right\| \leq$$

$$\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \left\| f\left(\frac{k}{n}\right) - f(x) \right\| \chi(nx-k) =$$

$$\sum_{\begin{cases} k=\lceil na \rceil \\ : \left|\frac{k}{n}-x\right| \leq \frac{1}{n^\alpha} \end{cases}}^{\lfloor nb \rfloor} \left\| f\left(\frac{k}{n}\right) - f(x) \right\| \chi(nx-k) +$$

$$\sum_{\begin{cases} k=\lceil na \rceil \\ : \left|\frac{k}{n}-x\right| > \frac{1}{n^\alpha} \end{cases}}^{\lfloor nb \rfloor} \left\| f\left(\frac{k}{n}\right) - f(x) \right\| \chi(nx-k) \leq \tag{59}$$

$$\sum_{\begin{cases} k=\lceil na \rceil \\ : \left|\frac{k}{n}-x\right| \leq \frac{1}{n^\alpha} \end{cases}}^{\lfloor nb \rfloor} \omega_1\left(f, \left|\frac{k}{n}-x\right|\right) \chi(nx-k) +$$

$$2\|f\|_\infty \sum_{\begin{cases} k=-\infty \\ : |k-nx| > n^{1-\alpha} \end{cases}}^{\infty} \chi(nx-k) \leq$$

$$\omega_1\left(f, \frac{1}{n^\alpha}\right) \sum_{\begin{cases} k=-\infty \\ : \left|\frac{k}{n}-x\right| \leq \frac{1}{n^\alpha} \end{cases}}^{\infty} \chi(nx-k) +$$

$$2\|f\|_\infty \sum_{\begin{cases} k=-\infty \\ : |k-nx| > n^{1-\alpha} \end{cases}}^{\infty} \chi(nx-k) \underset{\text{(by Theorem 3)}}{\leq}$$

$$\omega_1\left(f, \frac{1}{n^\alpha}\right) + \frac{\|f\|_\infty}{\lambda\left(n^{1-\alpha}-2\right)^\lambda}.$$

That is

$$\left\| \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \left(f\left(\frac{k}{n}\right) - f(x) \right) \chi(nx-k) \right\| \leq$$

$$\omega_1\left(f, \frac{1}{n^\alpha}\right) + \frac{\|f\|_\infty}{\lambda\left(n^{1-\alpha}-2\right)^\lambda}. \tag{60}$$

Using (50) we derive (57). ■

It follows

Theorem 11 *Let $f \in C_B(\mathbb{R}, X)$, $0 < \alpha < 1$, $n \in \mathbb{N} : n^{1-\alpha} > 2$, $x \in \mathbb{R}$, $\lambda \in \mathbb{N}$ is odd. Then*

i)

$$\left\| \overline{H_n}(f,x) - f(x) \right\| \leq \omega_1\left(f, \frac{1}{n^\alpha}\right) + \frac{\|f\|_\infty}{\lambda\left(n^{1-\alpha}-2\right)^\lambda} =: \rho_2, \tag{61}$$

and

ii)

$$\left\| \overline{H_n}(f) - f \right\|_\infty \leq \rho_2. \tag{62}$$

For $f \in C_{uB}(\mathbb{R}, X)$ we get $\lim_{n\to\infty} \overline{H_n}(f) = f$, pointwise and uniformly.

Proof. We observe that

$$\left\| \overline{H_n}(f,x) - f(x) \right\| = \left\| \sum_{k=-\infty}^{\infty} f\left(\frac{k}{n}\right) \chi(nx-k) - f(x) \sum_{k=-\infty}^{\infty} \chi(nx-k) \right\| =$$

$$\left\| \sum_{k=-\infty}^{\infty} \left(f\left(\frac{k}{n}\right) - f(x) \right) \chi(nx-k) \right\| \leq$$

$$\sum_{k=-\infty}^{\infty} \left\| f\left(\frac{k}{n}\right) - f(x) \right\| \chi(nx-k) = \tag{63}$$

$$\sum_{\substack{\left\{ k=-\infty \right. \\ \left. : \left|\frac{k}{n}-x\right| \leq \frac{1}{n^\alpha} \right.}}^{\infty} \left\| f\left(\frac{k}{n}\right) - f(x) \right\| \chi(nx-k) +$$

$$\sum_{\substack{\left\{ k=-\infty \right. \\ \left. : \left|\frac{k}{n}-x\right| > \frac{1}{n^\alpha} \right.}}^{\infty} \left\| f\left(\frac{k}{n}\right) - f(x) \right\| \chi(nx-k) \leq$$

$$\sum_{\substack{\left\{ k=-\infty \right. \\ \left. : \left|\frac{k}{n}-x\right| \leq \frac{1}{n^\alpha} \right.}}^{\infty} \omega_1\left(f, \left|\frac{k}{n}-x\right|\right) \chi(nx-k) +$$

$$2\|f\|_{\infty} \sum_{\begin{cases} k=-\infty \\ : \left|\frac{k}{n}-x\right| > \frac{1}{n^{\alpha}} \end{cases}}^{\infty} \chi(nx-k) \leq$$

$$\omega_1\left(f, \frac{1}{n^{\alpha}}\right) \sum_{\begin{cases} k=-\infty \\ : \left|\frac{k}{n}-x\right| \leq \frac{1}{n^{\alpha}} \end{cases}}^{\infty} \chi(nx-k) + \frac{2\|f\|_{\infty}}{2\lambda\left(n^{1-\alpha}-2\right)^{\lambda}} \leq \tag{64}$$

$$\omega_1\left(f, \frac{1}{n^{\alpha}}\right) + \frac{\|f\|_{\infty}}{\lambda\left(n^{1-\alpha}-2\right)^{\lambda}},$$

proving the claim. ■

In the next we study the high order neural network X-valued approximation by using the smoothness of f. Derivarives are defined similar to numerical ones, see [21].

Theorem 12 *Let $f \in C^N([a,b], X)$, $n, N \in \mathbb{N}$, λ is odd, $0 < \alpha < 1$, $x \in [a,b]$ and $n^{1-\alpha} > 2$. Then*

i)

$$\|H_n(f,x) - f(x)\| \leq \sqrt[\lambda]{1+2^{\lambda}} \left\{ \sum_{j=1}^{N} \frac{\left\|f^{(j)}(x)\right\|}{j!} \left[\frac{2}{n^{\alpha j}} + \frac{(b-a)^j}{\lambda\left(n^{1-\alpha}-2\right)^{\lambda}}\right] + \right.$$

$$\left. \left[\omega_1\left(f^{(N)}, \frac{1}{n^{\alpha}}\right) \frac{2}{n^{\alpha N} N!} + \frac{2\left\|f^{(N)}\right\|_{\infty}(b-a)^N}{N!\lambda\left(n^{1-\alpha}-2\right)^{\lambda}}\right] \right\}, \tag{65}$$

ii) assume further $f^{(j)}(x_0) = 0$, $j = 1, ..., N$, for some $x_0 \in [a,b]$, it holds

$$\|H_n(f,x_0) - f(x_0)\| \leq \sqrt[\lambda]{1+2^{\lambda}}\cdot$$

$$\left[\omega_1\left(f^{(N)}, \frac{1}{n^{\alpha}}\right) \frac{2}{n^{\alpha N} N!} + \frac{2\left\|f^{(N)}\right\|_{\infty}(b-a)^N}{N!\lambda\left(n^{1-\alpha}-2\right)^{\lambda}}\right], \tag{66}$$

and

iii)

$$\|H_n(f) - f\|_{\infty} \leq \sqrt[\lambda]{1+2^{\lambda}} \left\{ \sum_{j=1}^{N} \frac{\left\|f^{(j)}\right\|_{\infty}}{j!} \left[\frac{2}{n^{\alpha j}} + \frac{(b-a)^j}{\lambda\left(n^{1-\alpha}-2\right)^{\lambda}}\right] + \right.$$

$$\left. \left[\omega_1\left(f^{(N)}, \frac{1}{n^{\alpha}}\right) \frac{2}{n^{\alpha N} N!} + \frac{2\left\|f^{(N)}\right\|_{\infty}(b-a)^N}{N!\lambda\left(n^{1-\alpha}-2\right)^{\lambda}}\right] \right\}. \tag{67}$$

We derive that $\lim_{n\to\infty} H_n(f) = f$, pointwise and uniformly.

Proof. The proof is lengthy and very similar to the proof of the corresponding result in [14]. As such is omitted. ■

The integrals from now on are of Bochner type [19].

We need

Definition 13 *([12]) Let $[a,b] \subset \mathbb{R}$, X be a Banach space, $\alpha > 0$; $m = \lceil \alpha \rceil \in \mathbb{N}$, ($\lceil \cdot \rceil$ is the ceiling of the number), $f : [a,b] \to X$. We assume that $f^{(m)} \in L_1([a,b], X)$. We call the Caputo-Bochner left fractional derivative of order α:*

$$\left(D_{*a}^{\alpha} f\right)(x) := \frac{1}{\Gamma(m-\alpha)} \int_a^x (x-t)^{m-\alpha-1} f^{(m)}(t)\, dt, \quad \forall\, x \in [a,b]. \tag{68}$$

If $\alpha \in \mathbb{N}$, we set $D_{*a}^{\alpha} f := f^{(m)}$ the ordinary X-valued derivative (defined similar to numerical one, see [21], p. 83), and also set $D_{*a}^{0} f := f$.

By [12], $\left(D_{*a}^{\alpha} f\right)(x)$ exists almost everywhere in $x \in [a,b]$ and $D_{*a}^{\alpha} f \in L_1([a,b], X)$.

If $\left\| f^{(m)} \right\|_{L_\infty([a,b],X)} < \infty$, then by [12], $D_{*a}^{\alpha} f \in C([a,b], X)$, hence $\left\| D_{*a}^{\alpha} f \right\| \in C([a,b])$.

We mention

Lemma 14 *([11]) Let $\alpha > 0$, $\alpha \notin \mathbb{N}$, $m = \lceil \alpha \rceil$, $f \in C^{m-1}([a,b], X)$ and $f^{(m)} \in L_\infty([a,b], X)$. Then $D_{*a}^{\alpha} f(a) = 0$.*

We mention

Definition 15 *([10]) Let $[a,b] \subset \mathbb{R}$, X be a Banach space, $\alpha > 0$, $m := \lceil \alpha \rceil$. We assume that $f^{(m)} \in L_1([a,b], X)$, where $f : [a,b] \to X$. We call the Caputo-Bochner right fractional derivative of order α:*

$$\left(D_{b-}^{\alpha} f\right)(x) := \frac{(-1)^m}{\Gamma(m-\alpha)} \int_x^b (z-x)^{m-\alpha-1} f^{(m)}(z)\, dz, \quad \forall\, x \in [a,b]. \tag{69}$$

We observe that $\left(D_{b-}^{m} f\right)(x) = (-1)^m f^{(m)}(x)$, for $m \in \mathbb{N}$, and $\left(D_{b-}^{0} f\right)(x) = f(x)$.

By [10], $\left(D_{b-}^{\alpha} f\right)(x)$ exists almost everywhere on $[a,b]$ and $\left(D_{b-}^{\alpha} f\right) \in L_1([a,b], X)$.

If $\left\| f^{(m)} \right\|_{L_\infty([a,b],X)} < \infty$, and $\alpha \notin \mathbb{N}$, by [10], $D_{b-}^{\alpha} f \in C([a,b], X)$, hence $\left\| D_{b-}^{\alpha} f \right\| \in C([a,b])$.

We need

Lemma 16 *([11]) Let $f \in C^{m-1}([a,b], X)$, $f^{(m)} \in L_\infty([a,b], X)$, $m = \lceil \alpha \rceil$, $\alpha > 0$, $\alpha \notin \mathbb{N}$. Then $D_{b-}^{\alpha} f(b) = 0$.*

Convention 17 *We assume that*

$$D^{\alpha}_{*x_0} f(x) = 0, \text{ for } x < x_0, \tag{70}$$

and

$$D^{\alpha}_{x_0-} f(x) = 0, \text{ for } x > x_0, \tag{71}$$

for all $x, x_0 \in [a,b]$.

Next we present the corresponding X-valued fractional approximation by neural networks.

Theorem 18 *Let* $\alpha > 0$, $N = \lceil \alpha \rceil$, $\alpha \notin \mathbb{N}$, $f \in C^N([a,b], X)$, $0 < \beta < 1$, λ *is odd,* $x \in [a,b]$, $n \in \mathbb{N} : n^{1-\beta} > 2$. *Then*

i)

$$\left\| H_n(f,x) - \sum_{j=1}^{N-1} \frac{f^{(j)}(x)}{j!} H_n\left((\cdot - x)^j\right)(x) - f(x) \right\| \leq$$

$$\frac{2\sqrt[\lambda]{1+2^{\lambda}}}{\Gamma(\alpha+1)} \left\{ \frac{\left(\omega_1\left(D^{\alpha}_{x-} f, \frac{1}{n^{\beta}}\right)_{[a,x]} + \omega_1\left(D^{\alpha}_{*x} f, \frac{1}{n^{\beta}}\right)_{[x,b]}\right)}{n^{\alpha\beta}} + \right.$$

$$\left. \frac{1}{2\lambda(n^{1-\beta}-2)^{\lambda}} \left(\left\| D^{\alpha}_{x-} f \right\|_{\infty,[a,x]} (x-a)^{\alpha} + \left\| D^{\alpha}_{*x} f \right\|_{\infty,[x,b]} (b-x)^{\alpha} \right) \right\}, \tag{72}$$

ii) if $f^{(j)}(x) = 0$, *for* $j = 1, ..., N-1$, *we have*

$$\| H_n(f,x) - f(x) \| \leq \frac{2\sqrt[\lambda]{1+2^{\lambda}}}{\Gamma(\alpha+1)}$$

$$\left\{ \frac{\left(\omega_1\left(D^{\alpha}_{x-} f, \frac{1}{n^{\beta}}\right)_{[a,x]} + \omega_1\left(D^{\alpha}_{*x} f, \frac{1}{n^{\beta}}\right)_{[x,b]}\right)}{n^{\alpha\beta}} + \right.$$

$$\left. \frac{1}{2\lambda(n^{1-\beta}-2)^{\lambda}} \left(\left\| D^{\alpha}_{x-} f \right\|_{\infty,[a,x]} (x-a)^{\alpha} + \left\| D^{\alpha}_{*x} f \right\|_{\infty,[x,b]} (b-x)^{\alpha} \right) \right\}, \tag{73}$$

iii)

$$\| H_n(f,x) - f(x) \| \leq 2\sqrt[\lambda]{1+2^{\lambda}} \cdot$$

$$\left\{ \sum_{j=1}^{N-1} \frac{\left\| f^{(j)}(x) \right\|}{j!} \left\{ \frac{1}{n^{\beta j}} + \frac{(b-a)^j}{2\lambda(n^{1-\beta}-2)^{\lambda}} \right\} + \right.$$

$$\frac{1}{\Gamma(\alpha+1)} \left\{ \frac{\left(\omega_1\left(D^{\alpha}_{x-} f, \frac{1}{n^{\beta}}\right)_{[a,x]} + \omega_1\left(D^{\alpha}_{*x} f, \frac{1}{n^{\beta}}\right)_{[x,b]}\right)}{n^{\alpha\beta}} + \right.$$

$$\frac{1}{2\lambda\left(n^{1-\beta}-2\right)^{\lambda}}\left(\left\|D_{x-}^{\alpha}f\right\|_{\infty,[a,x]}(x-a)^{\alpha}+\left\|D_{*x}^{\alpha}f\right\|_{\infty,[x,b]}(b-x)^{\alpha}\right)\Bigg\}\Bigg\}, \quad (74)$$

$\forall\ x\in[a,b]$,

and

iv)

$$\|H_nf-f\|_{\infty}\leq 2\sqrt[\lambda]{1+2^{\lambda}}\cdot$$

$$\left\{\sum_{j=1}^{N-1}\frac{\left\|f^{(j)}\right\|_{\infty}}{j!}\left\{\frac{1}{n^{\beta j}}+\frac{(b-a)^{j}}{2\lambda\left(n^{1-\beta}-2\right)^{\lambda}}\right\}+\right.$$

$$\frac{1}{\Gamma(\alpha+1)}\left\{\frac{\left(\sup_{x\in[a,b]}\omega_1\left(D_{x-}^{\alpha}f,\frac{1}{n^{\beta}}\right)_{[a,x]}+\sup_{x\in[a,b]}\omega_1\left(D_{*x}^{\alpha}f,\frac{1}{n^{\beta}}\right)_{[x,b]}\right)}{n^{\alpha\beta}}+\right.$$

$$\frac{(b-a)^{\alpha}}{2\lambda\left(n^{1-\beta}-2\right)^{\lambda}}\left(\sup_{x\in[a,b]}\left\|D_{x-}^{\alpha}f\right\|_{\infty,[a,x]}+\sup_{x\in[a,b]}\left\|D_{*x}^{\alpha}f\right\|_{\infty,[x,b]}\right)\Bigg\}\Bigg\}. \quad (75)$$

Above, when $N=1$ the sum $\sum_{j=1}^{N-1}\cdot=0$.

As we see here we obtain X-valued fractionally type pointwise and uniform convergence with rates of $H_n\to I$ the unit operator, as $n\to\infty$.

Proof. The proof is very lengthy and very similar to the proof of the corresponding result in [14]. As such is omitted. ■

Next we apply Theorem 18 for $N=1$.

Theorem 19 *Let $0<\alpha,\beta<1$, $f\in C^{1}([a,b],X)$, $x\in[a,b]$, $n\in\mathbb{N}:n^{1-\beta}>2$, λ is odd. Then*

i)

$$\|H_n(f,x)-f(x)\|\leq$$

$$\frac{2\sqrt[\lambda]{1+2^{\lambda}}}{\Gamma(\alpha+1)}\left\{\frac{\left(\omega_1\left(D_{x-}^{\alpha}f,\frac{1}{n^{\beta}}\right)_{[a,x]}+\omega_1\left(D_{*x}^{\alpha}f,\frac{1}{n^{\beta}}\right)_{[x,b]}\right)}{n^{\alpha\beta}}+\right.$$

$$\frac{1}{2\lambda\left(n^{1-\beta}-2\right)^{\lambda}}\left(\left\|D_{x-}^{\alpha}f\right\|_{\infty,[a,x]}(x-a)^{\alpha}+\left\|D_{*x}^{\alpha}f\right\|_{\infty,[x,b]}(b-x)^{\alpha}\right)\Bigg\}, \quad (76)$$

and

ii)

$$\|H_nf-f\|_{\infty}\leq\frac{2\sqrt[\lambda]{1+2^{\lambda}}}{\Gamma(\alpha+1)}\cdot$$

$$\left\{ \frac{\left(\sup_{x\in[a,b]} \omega_1 \left(D_{x-}^{\alpha} f, \frac{1}{n^\beta} \right)_{[a,x]} + \sup_{x\in[a,b]} \omega_1 \left(D_{*x}^{\alpha} f, \frac{1}{n^\beta} \right)_{[x,b]} \right)}{n^{\alpha\beta}} + \right.$$

$$\left. \frac{(b-a)^{\alpha}}{2\lambda \left(n^{1-\beta} - 2 \right)^{\lambda}} \left(\sup_{x\in[a,b]} \left\| D_{x-}^{\alpha} f \right\|_{\infty,[a,x]} + \sup_{x\in[a,b]} \left\| D_{*x}^{\alpha} f \right\|_{\infty,[x,b]} \right) \right\}. \tag{77}$$

When $\alpha = \frac{1}{2}$ we derive

Corollary 20 *Let $0 < \beta < 1$, $f \in C^1([a,b], X)$, $x \in [a,b]$, $n \in \mathbb{N} : n^{1-\beta} > 2$, λ is odd. Then*

i)

$$\| H_n(f,x) - f(x) \| \leq$$

$$\frac{4\sqrt[\lambda]{1+2^{\lambda}}}{\sqrt{\pi}} \left\{ \frac{\left(\omega_1 \left(D_{x-}^{\frac{1}{2}} f, \frac{1}{n^\beta} \right)_{[a,x]} + \omega_1 \left(D_{*x}^{\frac{1}{2}} f, \frac{1}{n^\beta} \right)_{[x,b]} \right)}{n^{\frac{\beta}{2}}} + \right.$$

$$\left. \frac{1}{2\lambda \left(n^{1-\beta} - 2 \right)^{\lambda}} \left(\left\| D_{x-}^{\frac{1}{2}} f \right\|_{\infty,[a,x]} \sqrt{(x-a)} + \left\| D_{*x}^{\frac{1}{2}} f \right\|_{\infty,[x,b]} \sqrt{(b-x)} \right) \right\}, \tag{78}$$

and

ii)

$$\| H_n f - f \|_{\infty} \leq \frac{4\sqrt[\lambda]{1+2^{\lambda}}}{\sqrt{\pi}} \cdot$$

$$\left\{ \frac{\left(\sup_{x\in[a,b]} \omega_1 \left(D_{x-}^{\frac{1}{2}} f, \frac{1}{n^\beta} \right)_{[a,x]} + \sup_{x\in[a,b]} \omega_1 \left(D_{*x}^{\frac{1}{2}} f, \frac{1}{n^\beta} \right)_{[x,b]} \right)}{n^{\frac{\beta}{2}}} + \right.$$

$$\left. \frac{\sqrt{(b-a)}}{2\lambda \left(n^{1-\beta} - 2 \right)^{\lambda}} \left(\sup_{x\in[a,b]} \left\| D_{x-}^{\frac{1}{2}} f \right\|_{\infty,[a,x]} + \sup_{x\in[a,b]} \left\| D_{*x}^{\frac{1}{2}} f \right\|_{\infty,[x,b]} \right) \right\} < \infty. \tag{79}$$

Next we make

Conclusion 21 *Some convergence analysis follows:*

Let $0 < \beta < 1$, $f \in C^1([a,b], X)$, $x \in [a,b]$, $n \in \mathbb{N} : n^{1-\beta} > 2$, λ is odd. We elaborate on (79). Assume that

$$\omega_1 \left(D_{x-}^{\frac{1}{2}} f, \frac{1}{n^\beta} \right)_{[a,x]} \leq \frac{K_1}{n^\beta}, \tag{80}$$

and

$$\omega_1\left(D_{*x}^{\frac{1}{2}}f, \frac{1}{n^{\beta}}\right)_{[x,b]} \leq \frac{K_2}{n^{\beta}}, \tag{81}$$

$\forall\, x \in [a,b]$, $\forall\, n \in \mathbb{N}$, *where* $K_1, K_2 > 0$.

Then it holds

$$\frac{\left[\sup\limits_{x\in[a,b]} \omega_1\left(D_{x-}^{\frac{1}{2}}f, \frac{1}{n^{\beta}}\right)_{[a,x]} + \sup\limits_{x\in[a,b]} \omega_1\left(D_{*x}^{\frac{1}{2}}f, \frac{1}{n^{\beta}}\right)_{[x,b]}\right]}{n^{\frac{\beta}{2}}} \leq$$

$$\frac{\frac{(K_1+K_2)}{n^{\beta}}}{n^{\frac{\beta}{2}}} = \frac{(K_1+K_2)}{n^{\frac{3\beta}{2}}} = \frac{K}{n^{\frac{3\beta}{2}}}, \tag{82}$$

where $K := K_1 + K_2 > 0$.

The other summand of the right hand side of (79), for large enough n*, converges to zero at the speed* $\frac{1}{n^{\lambda(1-\beta)}}$*, so it is about* $\frac{L}{n^{\lambda(1-\beta)}}$*, where* $L > 0$ *is a constant.*

Then, for large enough $n \in \mathbb{N}$*, by (79), (82) and the above comment, we obtain that*

$$\|H_n f - f\|_{\infty} \leq \frac{M}{\min\left(n^{\frac{3\beta}{2}}, n^{\lambda(1-\beta)}\right)}, \tag{83}$$

where $M > 0$.

Clearly we have two cases:

i)

$$\|H_n f - f\|_{\infty} \leq \frac{M}{n^{\lambda(1-\beta)}}, \text{ when } \frac{2\lambda}{3+2\lambda} \leq \beta < 1, \tag{84}$$

with speed of convergence $\frac{1}{n^{\lambda(1-\beta)}}$,

and

ii)

$$\|H_n f - f\|_{\infty} \leq \frac{M}{n^{\frac{3\beta}{2}}}, \text{ when } 0 < \beta \leq \frac{2\lambda}{3+2\lambda}, \tag{85}$$

with speed of convergence $\frac{1}{n^{\frac{3\beta}{2}}}$.

In Theorem 10, for $f \in C([a,b], X)$ *and for large enough* $n \in \mathbb{N}$*, when* $0 < \beta \leq \frac{\lambda}{1+\lambda}$*, the speed is* $\frac{1}{n^{\beta}}$*. So when* $0 < \beta \leq \frac{2\lambda}{3+2\lambda}$ $\left(< \frac{\lambda}{1+\lambda}\right)$*, we get by (85) that* $\|H_n f - f\|_{\infty}$ *converges much faster to zero. The last comes because we assumed differentiability of* f.

Notice that in Corollary 20 no initial condition is assumed.

Conclusion 22 *When* $(X, \|\cdot\|) = (\mathbb{C}, |\cdot|)$*, all of our main results give nice and great applications, etc.*

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