SOME IMPROVEMENTS OF THE MONOTONICITY PROPERTY FOR THE NORMALIZED DETERMINANT OF POSITIVE OPERATORS IN HILBERT SPACES

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Abstract. For positive invertible operators A on a Hilbert space H and a fixed unit vector $x \in H$, define the normalized determinant by $\Delta_x(A) := \exp\langle \ln Ax, x\rangle$. In this paper we show among others that, if $0 < m \le B - A \le M$ and $0 < \gamma \le A \le \Gamma$, then

$$1 < \left(1 + \frac{M}{\Gamma}\right)^{\frac{m}{M}} \le \frac{\Delta_x(B)}{\Delta_x(A)} \le \left(1 + \frac{m}{\gamma}\right)^{\frac{M}{m}}$$

for all $x \in H$ with $\|x\| = 1$.

If $B > A > 0$, then for all $x \in H$ with $\|x\| = 1$,

$$1 < \left(1 + \frac{\|B-A\|}{\|A\|}\right)^{\frac{1}{\left\|(B-A)^{-1}\right\|\|B-A\|}} \le \frac{\Delta_x(B)}{\Delta_x(A)}$$
$$\le \left(1 + \frac{\left\|A^{-1}\right\|}{\left\|(B-A)^{-1}\right\|}\right)^{\|B-A\|\left\|(B-A)^{-1}\right\|}.$$

1. Introduction

Let $B(H)$ be the space of all bounded linear operators on a Hilbert space H, and I stands for the identity operator on H. An operator A in $B(H)$ is said to be positive (in symbol: $A \ge 0$) if $\langle Ax, x\rangle \ge 0$ for all $x \in H$. In particular, $A > 0$ means that A is positive and invertible. For a pair A, B of selfadjoint operators the order relation $A \ge B$ means as usual that $A - B$ is positive.

In 1998, Fujii et al. [2], [3], introduced the *normalized determinant* $\Delta_x(A)$ for positive invertible operators A on a Hilbert space H and a fixed unit vector $x \in H$, namely $\|x\| = 1$, defined by $\Delta_x(A) := \exp\langle \ln Ax, x\rangle$ and discussed it as a continuous geometric mean and observed some inequalities around the determinant from this point of view.

Some of the fundamental properties of normalized determinant are as follows, [2].

For each unit vector $x \in H$, see also [5], we have:

(i) *continuity*: the map $A \to \Delta_x(A)$ is norm continuous;
(ii) *bounds*: $\left\langle A^{-1}x, x\right\rangle^{-1} \le \Delta_x(A) \le \langle Ax, x\rangle$;
(iii) *continuous mean*: $\langle A^p x, x\rangle^{1/p} \downarrow \Delta_x(A)$ for $p \downarrow 0$ and $\langle A^p x, x\rangle^{1/p} \uparrow \Delta_x(A)$ for $p \uparrow 0$;
(iv) *power equality:* $\Delta_x(A^t) = \Delta_x(A)^t$ for all $t > 0$;

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(v) *homogeneity*: $\Delta_x(tA) = t\Delta_x(A)$ and $\Delta_x(tI) = t$ for all $t > 0$;
(vi) *monotonicity*: $0 < A \leq B$ implies $\Delta_x(A) \leq \Delta_x(B)$;
(vii) *multiplicativity*: $\Delta_x(AB) = \Delta_x(A)\Delta_x(B)$ for commuting A and B;
(viii) *Ky Fan type inequality*: $\Delta_x((1-\alpha)A + \alpha B) \geq \Delta_x(A)^{1-\alpha}\Delta_x(B)^{\alpha}$ for $0 < \alpha < 1$.

We define the logarithmic mean of two positive numbers a, b by

$$L(a,b) := \begin{cases} \frac{b-a}{\ln b - \ln a} \text{ if } b \neq a, \\ \\ a \text{ if } b = a. \end{cases} \tag{1.1}$$

In [2] the authors obtained the following additive reverse inequality for the operator A which satisfy the condition $0 < mI \leq A \leq MI$, where m, M are positive numbers,

$$0 \leq \langle Ax, x\rangle - \Delta_x(A) \leq L(m,M)\left[\ln L(m,M) + \frac{M\ln m - m\ln M}{M-m} - 1\right] \tag{1.2}$$

for all $x \in H$, $\|x\| = 1$.

We recall that *Specht's ratio* is defined by [9]

$$S(h) := \begin{cases} \frac{h^{\frac{1}{h-1}}}{e\ln\left(h^{\frac{1}{h-1}}\right)} \text{ if } h \in (0,1) \cup (1,\infty), \\ \\ 1 \text{ if } h = 1. \end{cases} \tag{1.3}$$

It is well known that $\lim_{h\to 1} S(h) = 1$, $S(h) = S\left(\frac{1}{h}\right) > 1$ for $h > 0$, $h \neq 1$. The function is decreasing on $(0,1)$ and increasing on $(1,\infty)$.

In [3], the authors obtained the following multiplicative reverse inequality as well

$$1 \leq \frac{\langle Ax, x\rangle}{\Delta_x(A)} \leq S\left(\frac{M}{m}\right) \tag{1.4}$$

for $0 < mI \leq A \leq MI$ and $x \in H$, $\|x\| = 1$.

Motivated by the above results, in this paper we show among others that, if $0 < m \leq B - A \leq M$ and $0 < \gamma \leq A \leq \Gamma$, then

$$1 < \left(1 + \frac{M}{\Gamma}\right)^{\frac{m}{M}} \leq \frac{\Delta_x(B)}{\Delta_x(A)} \leq \left(1 + \frac{m}{\gamma}\right)^{\frac{M}{m}}$$

for all $x \in H$ with $\|x\| = 1$. If $B > A > 0$, then for all $x \in H$ with $\|x\| = 1$,

$$1 < \left(1 + \frac{\|B-A\|}{\|A\|}\right)^{\frac{1}{\|(B-A)^{-1}\|\|B-A\|}} \leq \frac{\Delta_x(B)}{\Delta_x(A)}$$
$$\leq \left(1 + \frac{\|A^{-1}\|}{\left\|(B-A)^{-1}\right\|}\right)^{\|B-A\|\left\|(B-A)^{-1}\right\|}.$$

2. Main Results

We can state the following representation result that is of interest in itself:

Lemma 1. *For all A, $B > 0$ we have*

$$\ln B - \ln A \tag{2.1}$$
$$= \int_0^\infty \left[(\lambda + A)^{-1} - (\lambda + B)^{-1} \right] d\lambda$$
$$= \int_0^\infty \left(\int_0^1 (\lambda + (1-t) A + tB)^{-1} (B - A) (\lambda + (1-t) A + tB)^{-1} dt \right) d\lambda.$$

Proof. Observe that for $t > 0$, $t \neq 1$, we have

$$\int_0^u \frac{d\lambda}{(\lambda + t)(\lambda + 1)} = \frac{\ln t}{t-1} + \frac{1}{1-t} \ln \left(\frac{u+t}{u+1} \right)$$

for all $u > 0$.

By taking the limit over $u \to \infty$ in this equality, we derive

$$\frac{\ln t}{t-1} = \int_0^\infty \frac{d\lambda}{(\lambda + t)(\lambda + 1)},$$

which gives the representation for the logarithm

$$\ln t = (t-1) \int_0^\infty \frac{d\lambda}{(\lambda + 1)(\lambda + t)} \tag{2.2}$$

for all $t > 0$.

If we use the continuous functional calculus for selfadjoint operators, we have

$$\ln T = \int_0^\infty \frac{1}{\lambda + 1} (T - 1) (\lambda + T)^{-1} d\lambda \tag{2.3}$$

for all operators $T > 0$.

We have from (2.3) for A, $B > 0$ that

$$\ln B - \ln A = \int_0^\infty \frac{1}{\lambda + 1} \left[(B-1)(\lambda + B)^{-1} - (A-1)(\lambda + A)^{-1} \right] d\lambda. \tag{2.4}$$

Since

$$(B-1)(\lambda + B)^{-1} - (A-1)(\lambda + A)^{-1}$$
$$= B(\lambda + B)^{-1} - A(\lambda + A)^{-1} - \left((\lambda + B)^{-1} - (\lambda + A)^{-1} \right)$$

and

$$B(\lambda + B)^{-1} - A(\lambda + A)^{-1}$$
$$= (B + \lambda - \lambda)(\lambda + B)^{-1} - (A + \lambda - \lambda)(\lambda + A)^{-1}$$
$$= 1 - \lambda(\lambda + B)^{-1} - 1 + \lambda(\lambda + A)^{-1} = \lambda(\lambda + A)^{-1} - \lambda(\lambda + B)^{-1},$$

hence

$$(B-1)(\lambda + B)^{-1} - (A-1)(\lambda + A)^{-1}$$
$$= \lambda(\lambda + A)^{-1} - \lambda(\lambda + B)^{-1} - \left((\lambda + B)^{-1} - (\lambda + A)^{-1} \right)$$
$$= (\lambda + 1) \left[(\lambda + A)^{-1} - (\lambda + B)^{-1} \right]$$

and by (2.4) we get

$$\ln B - \ln A = \int_0^\infty \left[(\lambda + A)^{-1} - (\lambda + B)^{-1} \right] d\lambda, \tag{2.5}$$

we proves the first equality in (2.1).

Consider the continuous function g defined on an interval I for which the corresponding operator function is Gâteaux differentiable on the segment $[C,D]$: $\{(1-t)C+tD,\ t\in[0,1]\}$ for $C,\ D$ selfadjoint operators with spectra in I. We consider the auxiliary function defined on $[0,1]$ by

$$f_{C,D}(t) := f((1-t)C+tD),\ t\in[0,1].$$

Then we have, by the properties of the integral, that

$$f(D)-f(C)=\int_0^1 \frac{d}{dt}(f_{C,D}(t))\,dt=\int_0^1 \nabla f_{(1-t)C+tD}(D-C)\,dt.$$

If we write this equality for the function $f(t)=-t^{-1}$ and $C,\ D>0$, then we get the representation

$$C^{-1}-D^{-1}=\int_0^1 ((1-t)C+tD)^{-1}(D-C)((1-t)C+tD)^{-1}\,dt. \tag{2.6}$$

Now, if we take in (2.6) $C=\lambda+A$, $D=\lambda+B$, then

$$\begin{aligned}
&(\lambda+A)^{-1}-(\lambda+B)^{-1} \\
&=\int_0^1 ((1-t)(\lambda+A)+t(\lambda+B))^{-1}(B-A) \\
&\times((1-t)(\lambda+A)+t(\lambda+B))^{-1}\,dt \\
&=\int_0^1 (\lambda+(1-t)A+tB)^{-1}(B-A)(\lambda+(1-t)A+tB)^{-1}\,dt.
\end{aligned} \tag{2.7}$$

By employing (2.7) and (2.5) we derive the desired result (2.1). □

Theorem 1. *For all $A,\ B>0$ and $x\in H$ with $\|x\|=1$, we have*

$$\begin{aligned}
\frac{\Delta_x(B)}{\Delta_x(A)}=\exp\Bigg[\int_0^\infty\Bigg(\int_0^1\Big\langle(B-A)(\lambda+(1-t)A+tB)^{-1}x, \\
(\lambda+(1-t)A+tB)^{-1}x\Big\rangle dt\Bigg)d\lambda\Bigg].
\end{aligned} \tag{2.8}$$

Proof. From (2.1) we get

$$\begin{aligned}
&\langle \ln Bx,x\rangle-\langle \ln Ax,x\rangle \\
&=\int_0^\infty\left[\left\langle(\lambda+A)^{-1}x,x\right\rangle-\left\langle(\lambda+B)^{-1}x,x\right\rangle\right]d\lambda \\
&=\int_0^\infty\Bigg(\int_0^1\Big\langle(\lambda+(1-t)A+tB)^{-1}(B-A) \\
&\times(\lambda+(1-t)A+tB)^{-1}x,x\Big\rangle dt\Bigg)d\lambda \\
&=\int_0^\infty\Bigg(\int_0^1\Big\langle(B-A)(\lambda+(1-t)A+tB)^{-1}x, \\
&(\lambda+(1-t)A+tB)^{-1}x\Big\rangle dt\Bigg)d\lambda
\end{aligned} \tag{2.9}$$

for $x\in H$ with $\|x\|=1$.

If we take the exponential, then we get

$$\begin{aligned}&\frac{\exp\langle\ln Bx,x\rangle}{\exp\langle\ln Ax,x\rangle}\\&=\exp\left[\int_0^\infty\left(\int_0^1\left\langle(B-A)\left(\lambda+(1-t)A+tB\right)^{-1}x,\right.\right.\right.\\&\quad\left.\left.\left.\left(\lambda+(1-t)A+tB\right)^{-1}x\right\rangle dt\right)d\lambda\right]\end{aligned}$$

for $x\in H$ with $\|x\|=1$, which proves (2.8). □

Corollary 1. *Assume that* $A\geq m_1>0$, $B\geq m_2>0$, *then*

$$\exp\left[-\Phi\left(m_1,m_2\right)\|B-A\|\right]\leq\frac{\Delta_x(B)}{\Delta_x(A)}\leq\exp\left[\Phi\left(m_1,m_2\right)\|B-A\|\right],\tag{2.10}$$

where

$$\Phi\left(m_1,m_2\right):=\begin{cases}\frac{\ln m_2-\ln m_1}{m_2-m_1} & \text{if } m_2\neq m_1,\\[2ex] \frac{1}{m} & \text{if } m_2=m_1=m.\end{cases}$$

Proof. If we take the modulus in (2.9), then we get for $x\in H$ with $\|x\|=1$ that

$$\begin{aligned}&\left|\langle\ln Bx,x\rangle-\langle\ln Ax,x\rangle\right|\\&\leq\int_0^\infty\left(\int_0^1\left|\left\langle(B-A)\left(\lambda+(1-t)A+tB\right)^{-1}x,\right.\right.\right.\\&\quad\left.\left.\left.\left(\lambda+(1-t)A+tB\right)^{-1}x\right\rangle\right|dt\right)d\lambda\\&\leq\|B-A\|\int_0^\infty\left(\int_0^1\left\|\left(\lambda+(1-t)A+tB\right)^{-1}x\right\|^2dt\right)d\lambda\\&\leq\|B-A\|\|x\|\int_0^\infty\left(\int_0^1\left\|\left(\lambda+(1-t)A+tB\right)^{-1}\right\|^2dt\right)d\lambda\\&=\|B-A\|\int_0^\infty\left(\int_0^1\left\|\left(\lambda+(1-t)A+tB\right)^{-1}\right\|^2dt\right)d\lambda.\end{aligned}$$

Assume that $m_2>m_1$. Then

$$(1-t)A+tB+\lambda\geq(1-t)m_1+tm_2+\lambda,$$

which implies that

$$\left((1-t)A+tB+\lambda\right)^{-1}\leq\left((1-t)m_1+tm_2+\lambda\right)^{-1},$$

and

$$\left\|\left((1-t)A+tB+\lambda\right)^{-1}\right\|^2\leq\left((1-t)m_1+tm_2+\lambda\right)^{-2}$$

for all $t\in[0,1]$ and $\lambda\geq0$.

Therefore

$$\begin{aligned} (2.11) \qquad & |\langle \ln Bx, x\rangle - \langle \ln Ax, x\rangle| \\ & \leq \|B-A\| \int_0^\infty \left(\int_0^1 ((1-t)\, m_1 + t m_2 + \lambda)^{-2}\, dt \right) d\lambda \\ & = \frac{\|B-A\|}{m_2 - m_1} \int_0^\infty \left(\int_0^1 ((1-t)\, m_1 + t m_2 + \lambda)^{-1} \right. \\ & \quad \left. \times (m_2 - m_1) ((1-t)\, m_1 + t m_2 + \lambda)^{-1}\, dt \right) d\lambda, \end{aligned}$$

for $x \in H$ with $\|x\| = 1$.

If we use the identity (2.1) for $A = m_1$, $B = m_2$ we get the scalar identity

$$\begin{aligned} \ln m_2 - \ln m_1 = & \int_0^\infty \left(\int_0^1 ((1-t)\, m_1 + t m_2 + \lambda)^{-1} (m_2 - m_1) \right. \\ & \left. \times ((1-t)\, m_1 + t m_2 + \lambda)^{-1}\, dt \right) d\lambda \end{aligned}$$

and by (2.11) we obtain

$$(2.12) \qquad |\langle \ln Bx, x\rangle - \langle \ln Ax, x\rangle| \leq \frac{\ln m_2 - \ln m_1}{m_2 - m_1} \|B - A\|$$

for $x \in H$ with $\|x\| = 1$.

The case $m_2 < m_1$ goes in a similar way.

Now, assume that A, $B \geq m > 0$. Let $\epsilon > 0$, then $B + \epsilon \geq m + \epsilon$. Put $m_2 = m + \epsilon > m = m_1$. If we write the inequality (2.12) for $B + \epsilon$ and A, we get

$$|\langle \ln (B + \epsilon)\, x, x\rangle - \langle \ln Ax, x\rangle| \leq \frac{\ln (m + \epsilon) - \ln m}{\epsilon} \|B - A\|$$

for $x \in H$ with $\|x\| = 1$.

If we take the limit over $\epsilon \to 0+$ and observe that

$$\lim_{\epsilon \to 0+} \frac{\ln (m + \epsilon) - \ln m}{\epsilon} = \frac{1}{m},$$

then we get

$$|\langle \ln Bx, x\rangle - \langle \ln Ax, x\rangle| \leq \frac{1}{m} \|B - A\|$$

for $x \in H$ with $\|x\| = 1$.

Therefore

$$-\Phi (m_1, m_2) \|B - A\| \leq \langle \ln Bx, x\rangle - \langle \ln Ax, x\rangle \leq \Phi (m_1, m_2) \|B - A\|$$

for $x \in H$ with $\|x\| = 1$, which gives the desired result (2.10). □

Theorem 2. *Assume that* $0 < m \leq B - A \leq M$ *and* $0 < \gamma \leq A \leq \Gamma$, *then*

$$(2.13) \qquad 1 < \left(1 + \frac{M}{\Gamma}\right)^{\frac{m}{M}} \leq \frac{\Delta_x(B)}{\Delta_x(A)} \leq \left(1 + \frac{m}{\gamma}\right)^{\frac{M}{m}}$$

for all $x \in H$ *with* $\|x\| = 1$.

Proof. Since $m \leq B-A \leq M$ then by multiplying both sides by $(\lambda + (1-t)A + tB)^{-1} > 0$ we derive

$$m(\lambda + (1-t)A + tB)^{-2} \leq (\lambda + (1-t)A + tB)^{-1}(B-A)(\lambda + (1-t)A + tB)^{-1} \leq M(\lambda + (1-t)A + tB)^{-2} \tag{2.14}$$

for all $t \in [0,1]$ and $\lambda > 0$.

Observe that

$$(1-t)A + tB = A + t(B-A),$$

and since $\gamma \leq A \leq \Gamma$, hence

$$\lambda + \gamma + tm \leq \lambda + (1-t)A + tB \leq \lambda + \Gamma + tM,$$

namely,

$$(\lambda + \Gamma + tM)^{-1} \leq (\lambda + (1-t)A + tB)^{-1} \leq (\lambda + \gamma + tm)^{-1},$$

which gives that

$$(\lambda + \Gamma + tM)^{-2} \leq (\lambda + (1-t)A + tB)^{-2} \leq (\lambda + \gamma + tm)^{-2} \tag{2.15}$$

for all $t \in [0,1]$ and $\lambda > 0$.

By utilizing (2.14) and (2.15), we derive

$$m(\lambda + \Gamma + tM)^{-2} \leq (\lambda + (1-t)A + tB)^{-1}(B-A)(\lambda + (1-t)A + tB)^{-1} \leq M(\lambda + \gamma + tm)^{-2} \tag{2.16}$$

for all $t \in [0,1]$ and $\lambda > 0$.

If we take the integrals in (2.16), then we get

$$m\int_0^\infty \left(\int_0^1 (\lambda + \Gamma + tM)^{-2} dt\right) d\lambda \leq \int_0^1 (\lambda + (1-t)A + tB)^{-1}(B-A)(\lambda + (1-t)A + tB)^{-1} \leq M\int_0^\infty \left(\int_0^1 (\lambda + \gamma + tm)^{-2} dt\right) d\lambda,$$

namely, by (2.1)

$$m\int_0^\infty \left(\int_0^1 (\lambda + \Gamma + tM)^{-2} dt\right) d\lambda \leq \ln B - \ln A \leq M\int_0^\infty \left(\int_0^1 (\lambda + \gamma + tm)^{-2} dt\right) d\lambda. \tag{2.17}$$

Observe that

$$\int_0^1 (\lambda + \gamma + tm)^{-2} dt = -\frac{1}{m}(\lambda + \gamma + m)^{-1} + \frac{1}{m}(\lambda + \gamma)^{-1} = \frac{1}{m}\left((\lambda + \gamma)^{-1} - (\lambda + \gamma + m)^{-1}\right),$$

which gives

$$M\int_0^\infty\left(\int_0^1(\lambda+\gamma+tm)^{-2}dt\right)d\lambda=\frac{M}{m}\int_0^\infty\left((\lambda+\gamma)^{-1}-(\lambda+\gamma+m)^{-1}\right)d\lambda.$$

By the first identity in (2.1) in the scalar case, we have

$$\ln(\gamma+m)-\ln\gamma=\int_0^\infty\left[(\lambda+\gamma)^{-1}-(\lambda+\gamma+m)^{-1}\right]d\lambda$$

and then

$$\begin{aligned}M\int_0^\infty\left(\int_0^1(\lambda+\gamma+tm)^{-2}dt\right)d\lambda&=M\frac{\ln(\gamma+m)-\ln\gamma}{m}\\&=\ln\left(1+\frac{m}{\gamma}\right)^{\frac{M}{m}}.\end{aligned}$$

Similarly,

$$\begin{aligned}m\int_0^\infty\left(\int_0^1(\lambda+\Gamma+tM)^{-2}dt\right)d\lambda&=m\frac{\ln(\Gamma+M)-\ln\Gamma}{M}\\&=\ln\left(1+\frac{M}{\Gamma}\right)^{\frac{m}{M}}\end{aligned}$$

and by (2.17) we get

$$\ln\left(1+\frac{M}{\Gamma}\right)^{\frac{m}{M}}\le\ln B-\ln A\le\left(1+\frac{m}{\gamma}\right)^{\frac{M}{m}}. \tag{2.18}$$

If $x\in H$ with $\|x\|=1$, then by (2.18) we obtain

$$\ln\left(1+\frac{M}{\Gamma}\right)^{\frac{m}{M}}\le\langle\ln Bx,x\rangle-\langle\ln Ax,x\rangle\le\ln\left(1+\frac{m}{\gamma}\right)^{\frac{M}{m}}$$

and by taking the exponential, we derive

$$1<\left(1+\frac{M}{\Gamma}\right)^{\frac{m}{M}}\le\frac{\exp\langle\ln Bx,x\rangle}{\exp\langle\ln Ax,x\rangle}\le\left(1+\frac{m}{\gamma}\right)^{\frac{M}{m}}$$

for $x\in H$ with $\|x\|=1$ and the inequality (2.13) is obtained. □

3. Related Results

Let A and B be strictly positive operators on a Hilbert space H such that $B-A\ge m1_H>0$. In 2015, [4], T. Furuta obtained the following result for any non-constant operator monotone function f on $[0,\infty)$

$$f(B)-f(A)\ge f(\|A\|+m)-f(\|A\|)\ge f(\|B\|)-f(\|B\|-m)>0. \tag{3.1}$$

If $B>A>0$, then

$$\begin{aligned}f(B)-f(A)&\ge f\left(\|A\|+\frac{1}{\left\|(B-A)^{-1}\right\|}\right)-f(\|A\|)\\&\ge f(\|B\|)-f\left(\|B\|-\frac{1}{\left\|(B-A)^{-1}\right\|}\right)>0.\end{aligned} \tag{3.2}$$

The inequality between the first and third term in (3.2) was obtained earlier by H. Zuo and G. Duan in [10].

If we write the inequality (3.1) for $f(t) = \ln t$, then we get for $B - A \geq m > 0$

$$\ln B - \ln A \geq \ln\left(\frac{\|A\| + m}{\|A\|}\right) \geq \ln\left(\frac{\|B\|}{\|B\| - m}\right) > 0.$$

By taking the inner product over $x \in H$ with $\|x\| = 1$, we get

$$\langle \ln Bx, x\rangle - \langle \ln Ax, x\rangle \geq \ln\left(\frac{\|A\| + m}{\|A\|}\right) \geq \ln\left(\frac{\|B\|}{\|B\| - m}\right) > 0.$$

If we take the exponential, we can state

$$\frac{\exp\langle \ln Bx, x\rangle}{\exp\langle \ln Ax, x\rangle} \geq \frac{\|A\| + m}{\|A\|} \geq \frac{\|B\|}{\|B\| - m} > 1,$$

namely

$$\frac{\Delta_x(B)}{\Delta_x(A)} \geq \frac{\|A\| + m}{\|A\|} \geq \frac{\|B\|}{\|B\| - m} > 1 \tag{3.3}$$

provided that $B - A \geq m > 0$ and $x \in H$ with $\|x\| = 1$.

If $B > A > 0$, then by (3.2) written for for $f(t) = \ln t$, we get that

$$\begin{aligned}\ln B - \ln A &\geq \ln\left(1 + \frac{1}{\|A\|\left\|(B-A)^{-1}\right\|}\right)\\ &\geq \ln\left(\frac{\|B\|\left\|(B-A)^{-1}\right\|}{\|B\|\left\|(B-A)^{-1}\right\| - 1}\right) > 0.\end{aligned}$$

By utilizing a similar argument as above, we get

$$\frac{\Delta_x(B)}{\Delta_x(A)} \geq 1 + \frac{1}{\|A\|\left\|(B-A)^{-1}\right\|} \geq \frac{\|B\|\left\|(B-A)^{-1}\right\|}{\|B\|\left\|(B-A)^{-1}\right\| - 1} \geq 1 \tag{3.4}$$

provided that $B > A > 0$ and $x \in H$ with $\|x\| = 1$.

Its is well known that, if $P \geq 0$, then

$$|\langle Px, y\rangle|^2 \leq \langle Px, x\rangle \langle Py, y\rangle$$

for all x, $y \in H$.

Therefore, if $T > 0$, then

$$\begin{aligned}0 &\leq \langle x, x\rangle^2 = \left\langle T^{-1}Tx, x\right\rangle^2 = \left\langle Tx, T^{-1}x\right\rangle^2\\ &\leq \langle Tx, x\rangle \left\langle TT^{-1}x, T^{-1}x\right\rangle = \langle Tx, x\rangle \left\langle x, T^{-1}x\right\rangle\end{aligned}$$

for all $x \in H$.

If $x \in H$, $\|x\| = 1$, then

$$1 \leq \langle Tx, x\rangle \left\langle x, T^{-1}x\right\rangle \leq \langle Tx, x\rangle \sup_{\|x\|=1} \left\langle x, T^{-1}x\right\rangle = \langle Tx, x\rangle \left\|T^{-1}\right\|,$$

which implies the following operator inequality

$$\left\|T^{-1}\right\|^{-1} \leq T. \tag{3.5}$$

Proposition 1. *If B, $A>0$, then for all $x\in H$ with $\|x\|=1$,*

$$\exp\left[-\Psi\left(\left\|A^{-1}\right\|,\left\|B^{-1}\right\|\right)\|B-A\|\right]\leq\frac{\Delta_x(B)}{\Delta_x(A)}\leq\exp\left[\Psi\left(\left\|A^{-1}\right\|,\left\|B^{-1}\right\|\right)\|B-A\|\right], \tag{3.6}$$

where

$$\Psi\left(\left\|A^{-1}\right\|,\left\|B^{-1}\right\|\right):=\begin{cases}\frac{\ln\left\|B^{-1}\right\|-\ln\left\|A^{-1}\right\|}{\left\|B^{-1}\right\|-\left\|A^{-1}\right\|}\left\|A^{-1}\right\|\left\|B^{-1}\right\| & \text{if } \left\|A^{-1}\right\|\neq\left\|B^{-1}\right\|,\\ \left\|B^{-1}\right\| & \text{if } \left\|A^{-1}\right\|=\left\|B^{-1}\right\|.\end{cases}$$

Proof. Since $A\geq\left\|A^{-1}\right\|^{-1}$ and $B\geq\left\|B^{-1}\right\|^{-1}$, then by (2.10) for $m_1=\left\|A^{-1}\right\|^{-1}$ and $m_2=\left\|B^{-1}\right\|^{-1}$ we get

$$\begin{aligned}&\exp\left[-\Phi\left(\left\|A^{-1}\right\|^{-1},\left\|B^{-1}\right\|^{-1}\right)\|B-A\|\right]\\ &\leq\frac{\Delta_x(B)}{\Delta_x(A)}\\ &\leq\exp\left[\Phi\left(\left\|A^{-1}\right\|^{-1},\left\|B^{-1}\right\|^{-1}\right)\|B-A\|\right],\end{aligned} \tag{3.7}$$

where

$$\begin{aligned}&\Phi\left(\left\|A^{-1}\right\|^{-1},\left\|B^{-1}\right\|^{-1}\right)\\ &=\begin{cases}\frac{\ln\left\|B^{-1}\right\|^{-1}-\ln\left\|A^{-1}\right\|^{-1}}{\left\|B^{-1}\right\|^{-1}-\left\|A^{-1}\right\|^{-1}} & \text{if } \left\|A^{-1}\right\|\neq\left\|B^{-1}\right\|,\\ \left\|B^{-1}\right\| & \text{if } \left\|A^{-1}\right\|=\left\|B^{-1}\right\|.\end{cases}\\ &=\Psi\left(\left\|A^{-1}\right\|,\left\|B^{-1}\right\|\right)\end{aligned}$$

and the inequality (3.6) is proved. □

Finally, we can also state:

Proposition 2. *If $B>A>0$, then for all $x\in H$ with $\|x\|=1$,*

$$\begin{aligned}1<\left(1+\frac{\|B-A\|}{\|A\|}\right)^{\frac{1}{\left\|(B-A)^{-1}\right\|\|B-A\|}}&\leq\frac{\Delta_x(B)}{\Delta_x(A)}\\ \leq\left(1+\frac{\left\|A^{-1}\right\|}{\left\|(B-A)^{-1}\right\|}\right)^{\|B-A\|\left\|(B-A)^{-1}\right\|}&.\end{aligned} \tag{3.8}$$

Proof. We have $\left\|(B-A)^{-1}\right\|^{-1}\leq B-A\leq\|B-A\|$ and $\left\|A^{-1}\right\|^{-1}\leq A\leq\|A\|$. By taking $m=\left\|(B-A)^{-1}\right\|^{-1}$, $M=\|B-A\|$, $\gamma=\left\|A^{-1}\right\|^{-1}$ and $\Gamma=\|A\|$ in (2.13), we get (3.8) for all $x\in H$ with $\|x\|=1$. □

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