SOME IMPROVEMENTS OF THE MONOTONICITY PROPERTY FOR THE TRACE CLASS P-DETERMINANT OF POSITIVE OPERATORS IN HILBERT SPACES

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Abstract. Let H be a complex Hilbert space. For a given operator $P \geq 0$ with $P \in \mathcal{B}_1(H)$, the trace class associated to $\mathcal{B}(H)$ and $\operatorname{tr}(P) = 1$, we define the P-*determinant* of the positive invertible operator A by

$$\Delta_P(A) := \exp \operatorname{tr}(P \ln A).$$

In this paper we show among others that, if $0 < m \leq B - A \leq M$ and $0 < \gamma \leq A \leq \Gamma$, then

$$1 < \left(1 + \frac{M}{\Gamma}\right)^{\frac{m}{M}} \leq \frac{\Delta_P(B)}{\Delta_P(A)} \leq \left(1 + \frac{m}{\gamma}\right)^{\frac{M}{m}}$$

for all $P \geq 0$ with $P \in \mathcal{B}_1(H)$ and $\operatorname{tr}(P) = 1$. We also show that, if $B > A > 0$, then for all $P \geq 0$ with $P \in \mathcal{B}_1(H)$ and $\operatorname{tr}(P) = 1$,

$$1 < \left(1 + \frac{\|B - A\|}{\|A\|}\right)^{\frac{1}{\left\|(B-A)^{-1}\right\|\|B-A\|}} \leq \frac{\Delta_P(B)}{\Delta_P(A)}$$
$$\leq \left(1 + \frac{\left\|A^{-1}\right\|}{\left\|(B - A)^{-1}\right\|}\right)^{\|B-A\|\left\|(B-A)^{-1}\right\|}.$$

1. Introduction

In 1952, in the paper [7], B. Fuglede and R. V. Kadison introduced the determinant of a (invertible) operator and established its fundamental properties. The notion generalizes the usual determinant and can be considered for any operator in a finite von Neumann algebra (M, τ) with a faithful normal trace.

Let $T \in M$ be normal and $|T| := (T^*T)^{1/2}$ its modulus. By the spectral theorem one can represent T as an integral

$$T = \int_{\operatorname{Sp}(T)} \lambda dE(\lambda),$$

where $E(\lambda)$ is a projection valued measure and $\operatorname{Sp}(T)$ is the spectrum of T. The measure $\mu_T := \tau \circ E$ becomes a probability measure on the complex plane and has the support in the spectrum $\operatorname{Sp}(T)$.

For any $T \in M$ the Fuglede-Kadison determinant (*FK-determinant*) is defined by

$$\Delta_{FK}(T) := \exp\left(\int_0^\infty \ln t d\mu_{|T|}\right).$$

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If T is invertible, then

$$\Delta_{FK}(T) := \exp(\tau(\ln(|T|))),$$

where $\ln(|T|)$ is defined by the use of functional calculus.

Let $B(H)$ be the space of all bounded linear operators on a Hilbert space H, and I stands for the identity operator on H. An operator A in $B(H)$ is said to be positive (in symbol: $A \geq 0$) if $\langle Ax, x\rangle \geq 0$ for all $x \in H$. In particular, $A > 0$ means that A is positive and invertible. For a pair A, B of selfadjoint operators the order relation $A \geq B$ means as usual that $A - B$ is positive.

In 1998, Fujii et al. [8], [9], introduced the *normalized determinant* $\Delta_x(A)$ for positive invertible operators A on a Hilbert space H and a fixed unit vector $x \in H$, namely $\|x\| = 1$, defined by

$$\Delta_x(A) := \exp\langle \ln Ax, x\rangle$$

and discussed it as a continuous geometric mean and observed some inequalities around the determinant from this point of view. For some recent results, see [13].

We need now some preparations for trace of operators in Hilbert spaces.

Let $(H, \langle\cdot,\cdot\rangle)$ be a complex Hilbert space and $\{e_i\}_{i\in I}$ an *orthonormal basis* of H. We say that $A \in \mathcal{B}(H)$ is a *Hilbert-Schmidt operator* if

$$\sum_{i\in I} \|Ae_i\|^2 < \infty. \tag{1.1}$$

It is well know that, if $\{e_i\}_{i\in I}$ and $\{f_j\}_{j\in J}$ are orthonormal bases for H and $A \in \mathcal{B}(H)$ then

$$\sum_{i\in I} \|Ae_i\|^2 = \sum_{j\in I} \|Af_j\|^2 = \sum_{j\in I} \|A^* f_j\|^2 \tag{1.2}$$

showing that the definition (1.1) is independent of the orthonormal basis and A is a Hilbert-Schmidt operator iff A^* is a Hilbert-Schmidt operator.

Let $\mathcal{B}_2(H)$ the set of *Hilbert-Schmidt operators* in $\mathcal{B}(H)$. For $A \in \mathcal{B}_2(H)$ we define

$$\|A\|_2 := \left. \sum_{i\in I} \|Ae_i\|^2 \right)^{1/2} \tag{1.3}$$

for $\{e_i\}_{i\in I}$ an orthonormal basis of H.

Using the triangle inequality in $l^2(I)$, one checks that $\mathcal{B}_2(H)$ is a *vector space* and that $\|\cdot\|_2$ is a norm on $\mathcal{B}_2(H)$, which is usually called in the literature as the *Hilbert-Schmidt norm.*

Denote *the modulus* of an operator $A \in \mathcal{B}(H)$ by $|A| := (A^*A)^{1/2}$.

Because $\||A|x\| = \|Ax\|$ for all $x \in H$, A is Hilbert-Schmidt iff $|A|$ is Hilbert-Schmidt and $\|A\|_2 = \||A|\|_2$. From (1.2) we have that if $A \in \mathcal{B}_2(H)$, then $A^* \in \mathcal{B}_2(H)$ and $\|A\|_2 = \|A^*\|_2$.

The following theorem collects some of the most important properties of Hilbert-Schmidt operators:

Theorem 1. *We have:*

(i) $(\mathcal{B}_2(H), \|\cdot\|_2)$ *is a Hilbert space with inner product*

$$\langle A, B\rangle_2 := \sum_{i\in I} \langle Ae_i, Be_i\rangle = \sum_{i\in I} \langle B^*Ae_i, e_i\rangle \tag{1.4}$$

and the definition does not depend on the choice of the orthonormal basis $\{e_i\}_{i\in I}$;

(ii) We have the inequalities

$$\|A\| \leq \|A\|_2 \tag{1.5}$$

for any $A \in \mathcal{B}_2(H)$ *and, if* $A \in \mathcal{B}_2(H)$ *and* $T \in \mathcal{B}(H)$, *then* $AT, TA \in \mathcal{B}_2(H)$ *with*

$$\|AT\|_2, \ \ \|TA\|_2 \leq \|T\| \|A\|_2 \tag{1.6}$$

(iii) $\mathcal{B}_2(H)$ *is an operator ideal in* $\mathcal{B}(H)$, *i.e.*

$$\mathcal{B}(H)\mathcal{B}_2(H)\mathcal{B}(H) \subseteq \mathcal{B}_2(H).$$

If $\{e_i\}_{i\in I}$ an orthonormal basis of H, we say that $A \in \mathcal{B}(H)$ is *trace class* if

$$\|A\|_1 := \sum_{i\in I} \langle |A| e_i, e_i\rangle < \infty. \tag{1.7}$$

The definition of $\|A\|_1$ does not depend on the choice of the orthonormal basis $\{e_i\}_{i\in I}$. We denote by $\mathcal{B}_1(H)$ the set of trace class operators in $\mathcal{B}(H)$.

The following proposition holds:

Proposition 1. *If* $A \in \mathcal{B}(H)$, *then the following are equivalent:*

(i) $A \in \mathcal{B}_1(H)$;

(ii) $|A|^{1/2} \in \mathcal{B}_2(H)$.

The following properties are also well known:

Theorem 2. *With the above notations:*

(i) We have

$$\|A\|_1 = \|A^*\|_1 \ \text{ and } \ \|A\|_2 \leq \|A\|_1 \tag{1.8}$$

for any $A \in \mathcal{B}_1(H)$;

(ii) $\mathcal{B}_1(H)$ *is an operator ideal in* $\mathcal{B}(H)$, *i.e.*

$$\mathcal{B}(H)\mathcal{B}_1(H)\mathcal{B}(H) \subseteq \mathcal{B}_1(H);$$

(iii) We have

$$\mathcal{B}_2(H)\mathcal{B}_2(H) = \mathcal{B}_1(H);$$

(iv) We have

$$\|A\|_1 = \sup\{\langle A, B\rangle_2 \mid B \in \mathcal{B}_2(H), \ \|B\|_2 \leq 1\};$$

(v) $(\mathcal{B}_1(H), \|\cdot\|_1)$ *is a Banach space.*

We define the *trace* of a trace class operator $A \in \mathcal{B}_1(H)$ to be

$$\operatorname{tr}(A) := \sum_{i\in I} \langle Ae_i, e_i\rangle, \tag{1.9}$$

where $\{e_i\}_{i\in I}$ an orthonormal basis of H. Note that this coincides with the usual definition of the trace if H is finite-dimensional. We observe that the series (1.9) converges absolutely and it is independent from the choice of basis.

The following result collects some properties of the trace:

Theorem 3. *We have:*

(i) If $A \in \mathcal{B}_1(H)$ *then* $A^* \in \mathcal{B}_1(H)$ *and*

$$\operatorname{tr}(A^*) = \overline{\operatorname{tr}(A)}; \tag{1.10}$$

(ii) If $A \in \mathcal{B}_1(H)$ *and* $T \in \mathcal{B}(H)$, *then* $AT, TA \in \mathcal{B}_1(H)$,

$$\operatorname{tr}(AT) = \operatorname{tr}(TA) \text{ and } |\operatorname{tr}(AT)| \le \|A\|_1 \|T\|; \tag{1.11}$$

(iii) $\operatorname{tr}(\cdot)$ *is a bounded linear functional on* $\mathcal{B}_1(H)$ *with* $\|\operatorname{tr}\| = 1$;

(iv) If A, $B \in \mathcal{B}_2(H)$ *then* AB, $BA \in \mathcal{B}_1(H)$ *and* $\operatorname{tr}(AB) = \operatorname{tr}(BA)$.

Now, if we assume that $P \ge 0$ and $P \in \mathcal{B}_1(H)$, then for all $T \in \mathcal{B}(H)$, PT, $TP \in \mathcal{B}_1(H)$ and $\operatorname{tr}(PT) = \operatorname{tr}(TP)$. Also, since $P^{1/2} \in \mathcal{B}_2(H)$, $TP^{1/2} \in \mathcal{B}_2(H)$, hence $P^{1/2}TP^{1/2}$ and $TP^{1/2}P^{1/2} = TP \in \mathcal{B}_1(H)$ with $\operatorname{tr}\left(P^{1/2}TP^{1/2}\right) = \operatorname{tr}(TP)$. Therefore, if $P \ge 0$ and $P \in \mathcal{B}_1(H)$,

$$\operatorname{tr}(PT) = \operatorname{tr}(TP) = \operatorname{tr}\left(P^{1/2}TP^{1/2}\right)$$

for all $T \in \mathcal{B}(H)$.

If $T \ge 0$, then $P^{1/2}TP^{1/2} \ge 0$, which implies that $\operatorname{tr}(PT) \ge 0$ that shows that the functional $\mathcal{B}(H) \ni T \longmapsto \operatorname{tr}(PT)$ is linear and isotonic functional. Also, by (1.11), if $T_n \to T$ for $n \to \infty$ in $\mathcal{B}(H)$ then $\lim_{n\to\infty} \operatorname{tr}(PT_n) = \operatorname{tr}(PT)$, namely $\mathcal{B}(H) \ni T \longmapsto \operatorname{tr}(PT)$ is also continuous in the norm topology.

For a survey on recent trace inequalities see [5] and the references therein.

Now, for a given $P \ge 0$ with $P \in \mathcal{B}_1(H)$ and $\operatorname{tr}(P) = 1$, we define the P-*determinant* of the positive invertible operator A by

$$\Delta_P(A) := \exp \operatorname{tr}(P \ln A) = \exp \operatorname{tr}((\ln A) P) = \exp \operatorname{tr}\left(P^{1/2}(\ln A) P^{1/2}\right). \tag{1.12}$$

Assume that $P \ge 0$ with $P \in \mathcal{B}_1(H)$ and $\operatorname{tr}(P) = 1$. We observe that we have the following elementary properties:

(i) *continuity*: the map $A \to \Delta_P(A)$ is norm continuous;
(ii) *power equality:* $\Delta_P(A^t) = \Delta_P(A)^t$ for all $t > 0$;
(iii) *homogeneity*: $\Delta_P(tA) = t\Delta_x(A)$ and $\Delta_P(tI) = t$ for all $t > 0$;
(iv) *monotonicity*: $0 < A \le B$ implies $\Delta_P(A) \le \Delta_P(B)$.

In the recent paper [6] we obtained the following results:

Theorem 4. *Let* $P \ge 0$ *with* $P \in \mathcal{B}_1(H)$ *and* $\operatorname{tr}(P) = 1$, *then for all* A, $B > 0$ *and* $t \in [0,1]$ *we have the Ky Fan's type inequality*

$$\Delta_P((1-t)A + tB) \ge [\Delta_P(A)]^{1-t} [\Delta_P(B)]^t. \tag{1.13}$$

and

Theorem 5. *Let* $P \ge 0$ *with* $P \in \mathcal{B}_1(H)$ *and* $\operatorname{tr}(P) = 1$, *then for all* $A > 0$ *and* $a > 0$ *we have the double inequality*

$$a \exp\left[1 - a \operatorname{tr}\left(PA^{-1}\right)\right] \le \Delta_P(A) \le a \exp\left[a^{-1}\operatorname{tr}(PA) - 1\right]. \tag{1.14}$$

In particular

$$1 \le \frac{\operatorname{tr}(PA)}{\Delta_P(A)} \le \exp\left[\operatorname{tr}(PA)\operatorname{tr}\left(PA^{-1}\right) - 1\right] \tag{1.15}$$

and

$$1 \le \frac{\Delta_P(A)}{\left[\operatorname{tr}\left(PA^{-1}\right)\right]^{-1}} \le \exp\left[\operatorname{tr}\left(PA^{-1}\right)\operatorname{tr}(PA) - 1\right]. \tag{1.16}$$

The first inequalities in (1.15) and 1.16) are best possible from (1.14).

Motivated by the above results, in this paper we show among others that, if $0 < m \leq B - A \leq M$ and $0 < \gamma \leq A \leq \Gamma$, then

$$1 < \left(1 + \frac{M}{\Gamma}\right)^{\frac{m}{M}} \leq \frac{\Delta_P(B)}{\Delta_P(A)} \leq \left(1 + \frac{m}{\gamma}\right)^{\frac{M}{m}}$$

for all $P \geq 0$ with $P \in \mathcal{B}_1(H)$ and $\operatorname{tr}(P) = 1$. We also show that, if $B > A > 0$, then for all $P \geq 0$ with $P \in \mathcal{B}_1(H)$ and $\operatorname{tr}(P) = 1$,

$$1 < \left(1 + \frac{\|B - A\|}{\|A\|}\right)^{\frac{1}{\|(B-A)^{-1}\|\|B-A\|}} \leq \frac{\Delta_P(B)}{\Delta_P(A)}$$
$$\leq \left(1 + \frac{\|A^{-1}\|}{\left\|(B-A)^{-1}\right\|}\right)^{\|B-A\|\left\|(B-A)^{-1}\right\|}.$$

2. Main Results

We can state the following representation result that is of interest in itself:

Lemma 1. *For all A, $B > 0$ we have*

(2.1)
$$\begin{aligned}\ln B - \ln A &= \int_0^\infty \left[(\lambda + A)^{-1} - (\lambda + B)^{-1}\right] d\lambda \\ &= \int_0^\infty \left(\int_0^1 (\lambda + (1-t)A + tB)^{-1} (B - A) (\lambda + (1-t)A + tB)^{-1} dt\right) d\lambda.\end{aligned}$$

Proof. Observe that for $t > 0$, $t \neq 1$, we have

$$\int_0^u \frac{d\lambda}{(\lambda + t)(\lambda + 1)} = \frac{\ln t}{t - 1} + \frac{1}{1 - t} \ln\left(\frac{u + t}{u + 1}\right)$$

for all $u > 0$.

By taking the limit over $u \to \infty$ in this equality, we derive

$$\frac{\ln t}{t - 1} = \int_0^\infty \frac{d\lambda}{(\lambda + t)(\lambda + 1)},$$

which gives the representation for the logarithm

$$\ln t = (t - 1) \int_0^\infty \frac{d\lambda}{(\lambda + 1)(\lambda + t)} \tag{2.2}$$

for all $t > 0$.

If we use the continuous functional calculus for selfadjoint operators, we have

$$\ln T = \int_0^\infty \frac{1}{\lambda + 1} (T - 1)(\lambda + T)^{-1} d\lambda \tag{2.3}$$

for all operators $T > 0$.

We have from (2.3) for A, $B > 0$ that

$$\ln B - \ln A = \int_0^\infty \frac{1}{\lambda + 1} \left[(B - 1)(\lambda + B)^{-1} - (A - 1)(\lambda + A)^{-1}\right] d\lambda. \tag{2.4}$$

Since

$$\begin{aligned}&(B-1)(\lambda+B)^{-1}-(A-1)(\lambda+A)^{-1}\\&=B(\lambda+B)^{-1}-A(\lambda+A)^{-1}-\left((\lambda+B)^{-1}-(\lambda+A)^{-1}\right)\end{aligned}$$

and

$$\begin{aligned}&B(\lambda+B)^{-1}-A(\lambda+A)^{-1}\\&=(B+\lambda-\lambda)(\lambda+B)^{-1}-(A+\lambda-\lambda)(\lambda+A)^{-1}\\&=1-\lambda(\lambda+B)^{-1}-1+\lambda(\lambda+A)^{-1}=\lambda(\lambda+A)^{-1}-\lambda(\lambda+B)^{-1},\end{aligned}$$

hence

$$\begin{aligned}&(B-1)(\lambda+B)^{-1}-(A-1)(\lambda+A)^{-1}\\&=\lambda(\lambda+A)^{-1}-\lambda(\lambda+B)^{-1}-\left((\lambda+B)^{-1}-(\lambda+A)^{-1}\right)\\&=(\lambda+1)\left[(\lambda+A)^{-1}-(\lambda+B)^{-1}\right]\end{aligned}$$

and by (2.4) we get

$$\ln B-\ln A=\int_0^\infty\left[(\lambda+A)^{-1}-(\lambda+B)^{-1}\right]d\lambda, \tag{2.5}$$

we proves the first equality in (2.1).

Consider the continuous function g defined on an interval I for which the corresponding operator function is Gâteaux differentiable on the segment $[C,D]$: $\{(1-t)C+tD,\ t\in[0,1]\}$ for $C,\ D$ selfadjoint operators with spectra in I. We consider the auxiliary function defined on $[0,1]$ by

$$f_{C,D}(t):=f((1-t)C+tD),\ t\in[0,1].$$

Then we have, by the properties of the integral, that

$$f(D)-f(C)=\int_0^1\frac{d}{dt}(f_{C,D}(t))\,dt=\int_0^1\nabla f_{(1-t)C+tD}(D-C)\,dt.$$

If we write this equality for the function $f(t)=-t^{-1}$ and $C,\ D>0$, then we get the representation

$$C^{-1}-D^{-1}=\int_0^1((1-t)C+tD)^{-1}(D-C)((1-t)C+tD)^{-1}dt. \tag{2.6}$$

Now, if we take in (2.6) $C=\lambda+A$, $D=\lambda+B$, then

$$\begin{aligned}&(\lambda+A)^{-1}-(\lambda+B)^{-1}\\&=\int_0^1((1-t)(\lambda+A)+t(\lambda+B))^{-1}(B-A)\\&\times((1-t)(\lambda+A)+t(\lambda+B))^{-1}dt\\&=\int_0^1(\lambda+(1-t)A+tB)^{-1}(B-A)(\lambda+(1-t)A+tB)^{-1}dt.\end{aligned} \tag{2.7}$$

By employing (2.7) and (2.5) we derive the desired result (2.1). □

Theorem 6. *Assume that* A, $B>0$ *and* $P\geq 0$ *with* $P\in\mathcal{B}_1(H)$ *and* $\operatorname{tr}(P)=1$, *then*

$$\begin{aligned}
&\operatorname{tr}[P(\ln B)]-\operatorname{tr}[P(\ln A)] \\
&=\int_0^\infty\left(\operatorname{tr}\left[P(\lambda+A)^{-1}\right]-\operatorname{tr}\left[P(\lambda+B)^{-1}\right]\right)d\lambda \\
&=\int_0^\infty\left(\int_0^1\operatorname{tr}\left(P(\lambda+(1-t)A+tB)^{-1}(B-A)\right.\right. \\
&\left.\left.\times(\lambda+(1-t)A+tB)^{-1}\right)dt\right)d\lambda,
\end{aligned}\tag{2.8}$$

or, equivalently

$$\begin{aligned}
&\frac{\Delta_P(B)}{\Delta_P(A)} \\
&=\exp\left[\int_0^\infty\left(\operatorname{tr}\left[P(\lambda+A)^{-1}\right]-\operatorname{tr}\left[P(\lambda+B)^{-1}\right]\right)d\lambda\right] \\
&=\exp\left[\int_0^\infty\left(\int_0^1\operatorname{tr}\left(P(\lambda+(1-t)A+tB)^{-1}(B-A)\right.\right.\right. \\
&\left.\left.\left.\times(\lambda+(1-t)A+tB)^{-1}\right)dt\right)d\lambda\right].
\end{aligned}\tag{2.9}$$

Proof. If we multiply both sides of (2.1) by $P^{1/2}$, we get

$$\begin{aligned}
&P^{1/2}(\ln B)P^{1/2}-P^{1/2}(\ln A)P^{1/2} \\
&=\int_0^\infty\left[P^{1/2}(\lambda+A)^{-1}P^{1/2}-P^{1/2}(\lambda+B)^{-1}P^{1/2}\right]d\lambda \\
&=\int_0^\infty\left(\int_0^1P^{1/2}(\lambda+(1-t)A+tB)^{-1}(B-A)(\lambda+(1-t)A+tB)^{-1}P^{1/2}dt\right)d\lambda.
\end{aligned}$$

If we take the trace and use its properties, we obtain

$$\begin{aligned}
&\operatorname{tr}[P(\ln B)]-\operatorname{tr}[P(\ln A)] \\
&=\int_0^\infty\left[\operatorname{tr}\left[P(\lambda+A)^{-1}\right]-\operatorname{tr}\left[P(\lambda+B)^{-1}\right]\right]d\lambda \\
&=\int_0^\infty\left(\int_0^1\operatorname{tr}\left(P^{1/2}(\lambda+(1-t)A+tB)^{-1}(B-A)\right.\right. \\
&\left.\left.\times(\lambda+(1-t)A+tB)^{-1}P^{1/2}\right)dt\right)d\lambda \\
&=\int_0^\infty\left(\int_0^1\operatorname{tr}\left(P(\lambda+(1-t)A+tB)^{-1}(B-A)\right.\right. \\
&\left.\left.\times(\lambda+(1-t)A+tB)^{-1}\right)dt\right)d\lambda,
\end{aligned}$$

which gives the representation (2.8). □

Corollary 1. *Assume that* $A\geq m_1>0$, $B\geq m_2>0$ *and* $P\geq 0$ *with* $P\in\mathcal{B}_1(H)$ *and* $\operatorname{tr}(P)=1$, *then*

$$\exp\left[-\Phi(m_1,m_2)\|B-A\|\right]\leq\frac{\Delta_P(B)}{\Delta_P(A)}\leq\exp\left[\Phi(m_1,m_2)\|B-A\|\right],\tag{2.10}$$

where

$$\Phi(m_1,m_2):=\begin{cases}\frac{\ln m_2-\ln m_1}{m_2-m_1} & \text{if } m_2\neq m_1,\\ \\ \frac{1}{m} & \text{if } m_2=m_1=m.\end{cases}$$

Proof. If we take the modulus in (2.8), then we get

$$\begin{aligned}&\left|\operatorname{tr}\left[P(\ln B)\right]-\operatorname{tr}\left[P(\ln A)\right]\right|\\&\leq\int_0^\infty\left(\int_0^1\left|\operatorname{tr}\left(P(\lambda+(1-t)A+tB)^{-1}(B-A)\right.\right.\right.\\&\quad\left.\left.\left.\times(\lambda+(1-t)A+tB)^{-1}\right)\right|dt\right)d\lambda\\&\leq\|P\|_1\int_0^\infty\left(\int_0^1\left\|(\lambda+(1-t)A+tB)^{-1}(B-A)\right.\right.\\&\quad\times\left.\left.\times(\lambda+(1-t)A+tB)^{-1}\right\|dt\right)d\lambda\\&\leq\|B-A\|\int_0^\infty\left(\int_0^1\left\|(\lambda+(1-t)A+tB)^{-1}\right\|^2dt\right)d\lambda.\end{aligned}$$

Assume that $m_2>m_1$. Then

$$(1-t)A+tB+\lambda\geq(1-t)m_1+tm_2+\lambda,$$

which implies that

$$((1-t)A+tB+\lambda)^{-1}\leq((1-t)m_1+tm_2+\lambda)^{-1},$$

and

$$\left\|((1-t)A+tB+\lambda)^{-1}\right\|^2\leq((1-t)m_1+tm_2+\lambda)^{-2}$$

for all $t\in[0,1]$ and $\lambda\geq0$.

Therefore

$$\begin{aligned}&\left|\operatorname{tr}\left[P(\ln B)\right]-\operatorname{tr}\left[P(\ln A)\right]\right|\\&\leq\|B-A\|\int_0^\infty\left(\int_0^1((1-t)m_1+tm_2+\lambda)^{-2}dt\right)d\lambda\\&=\frac{\|B-A\|}{m_2-m_1}\int_0^\infty\left(\int_0^1((1-t)m_1+tm_2+\lambda)^{-1}\right.\\&\quad\left.\times(m_2-m_1)((1-t)m_1+tm_2+\lambda)^{-1}dt\right)d\lambda,\end{aligned}\tag{2.11}$$

for $P\geq0$ with $P\in\mathcal{B}_1(H)$ and $\operatorname{tr}(P)=1$.

If we use the identity (2.1) for $A=m_1$, $B=m_2$ we get the scalar identity

$$\begin{aligned}\ln m_2-\ln m_1=&\int_0^\infty\left(\int_0^1((1-t)m_1+tm_2+\lambda)^{-1}(m_2-m_1)\right.\\&\left.\times((1-t)m_1+tm_2+\lambda)^{-1}dt\right)d\lambda\end{aligned}$$

and by (2.11) we obtain

$$\left|\operatorname{tr}\left[P(\ln B)\right]-\operatorname{tr}\left[P(\ln A)\right]\right|\leq\frac{\ln m_2-\ln m_1}{m_2-m_1}\|B-A\|\tag{2.12}$$

for $P\geq0$ with $P\in\mathcal{B}_1(H)$ and $\operatorname{tr}(P)=1$.

The case $m_2<m_1$ goes in a similar way.

Now, assume that $A,\ B \geq m > 0$. Let $\epsilon > 0$, then $B + \epsilon \geq m + \epsilon$. Put $m_2 = m + \epsilon > m = m_1$. If we write the inequality (2.12) for $B + \epsilon$ and A, we get

$$\left|\operatorname{tr}\left[P\left(\ln\left(B+\epsilon\right)\right)\right] - \operatorname{tr}\left[P\left(\ln A\right)\right]\right| \leq \frac{\ln\left(m+\epsilon\right) - \ln m}{\epsilon}\left\|B-A\right\|$$

for $P \geq 0$ with $P \in \mathcal{B}_1\left(H\right)$ and $\operatorname{tr}\left(P\right) = 1$.

If we take the limit over $\epsilon \to 0+$ and observe that

$$\lim_{\epsilon\to 0+}\frac{\ln\left(m+\epsilon\right) - \ln m}{\epsilon} = \frac{1}{m},$$

then we get

$$\left|\operatorname{tr}\left[P\left(\ln B\right)\right] - \operatorname{tr}\left[P\left(\ln A\right)\right]\right| \leq \frac{1}{m}\left\|B-A\right\|$$

for $P \geq 0$ with $P \in \mathcal{B}_1\left(H\right)$ and $\operatorname{tr}\left(P\right) = 1$.

Therefore

$$-\Phi\left(m_1, m_2\right)\left\|B-A\right\| \leq \operatorname{tr}\left[P\left(\ln B\right)\right] - \operatorname{tr}\left[P\left(\ln A\right)\right] \leq \Phi\left(m_1, m_2\right)\left\|B-A\right\|$$

for $P \geq 0$ with $P \in \mathcal{B}_1\left(H\right)$ and $\operatorname{tr}\left(P\right) = 1$, which gives the desired result (2.10). $\square$

Theorem 7. *Assume that* $0 < m \leq B - A \leq M$ *and* $0 < \gamma \leq A \leq \Gamma$, *then*

$$1 < \left(1 + \frac{M}{\Gamma}\right)^{\frac{m}{M}} \leq \frac{\Delta_P(B)}{\Delta_P(A)} \leq \left(1 + \frac{m}{\gamma}\right)^{\frac{M}{m}} \tag{2.13}$$

for all $P \geq 0$ *with* $P \in \mathcal{B}_1\left(H\right)$ *and* $\operatorname{tr}\left(P\right) = 1$.

Proof. Since $m \leq B - A \leq M$ then by multiplying both sides by $\left(\lambda + \left(1-t\right)A + tB\right)^{-1} > 0$ we derive

$$\begin{aligned} & m\left(\lambda + \left(1-t\right)A + tB\right)^{-2} \\ & \leq \left(\lambda + \left(1-t\right)A + tB\right)^{-1}\left(B-A\right)\left(\lambda + \left(1-t\right)A + tB\right)^{-1} \\ & \leq M\left(\lambda + \left(1-t\right)A + tB\right)^{-2} \end{aligned} \tag{2.14}$$

for all $t \in [0,1]$ and $\lambda > 0$.

Observe that

$$\left(1-t\right)A + tB = A + t\left(B-A\right),$$

and since $\gamma \leq A \leq \Gamma$, hence

$$\lambda + \gamma + tm \leq \lambda + \left(1-t\right)A + tB \leq \lambda + \Gamma + tM,$$

namely,

$$\left(\lambda + \Gamma + tM\right)^{-1} \leq \left(\lambda + \left(1-t\right)A + tB\right)^{-1} \leq \left(\lambda + \gamma + tm\right)^{-1},$$

which gives that

$$\left(\lambda + \Gamma + tM\right)^{-2} \leq \left(\lambda + \left(1-t\right)A + tB\right)^{-2} \leq \left(\lambda + \gamma + tm\right)^{-2} \tag{2.15}$$

for all $t \in [0,1]$ and $\lambda > 0$.

By utilizing (2.14) and (2.15), we derive

$$\begin{aligned} & m\left(\lambda + \Gamma + tM\right)^{-2} \\ & \leq \left(\lambda + \left(1-t\right)A + tB\right)^{-1}\left(B-A\right)\left(\lambda + \left(1-t\right)A + tB\right)^{-1} \\ & \leq M\left(\lambda + \gamma + tm\right)^{-2} \end{aligned} \tag{2.16}$$

for all $t \in [0,1]$ and $\lambda > 0$.

If we multiply both sides by $P^{1/2}$ we get

$$\begin{aligned}
&mP^{1/2}\left(\lambda+\Gamma+tM\right)^{-2}P^{1/2}\\
&\leq P^{1/2}\left(\lambda+\left(1-t\right)A+tB\right)^{-1}\left(B-A\right)\left(\lambda+\left(1-t\right)A+tB\right)^{-1}P^{1/2}\\
&\leq MP^{1/2}\left(\lambda+\gamma+tm\right)^{-2}P^{1/2}
\end{aligned}$$

and by taking the trace, we derive

$$\begin{aligned}
&m\operatorname{tr}\left[P\left(\lambda+\Gamma+tM\right)^{-2}\right]\\
&\leq \operatorname{tr}\left[P\left(\lambda+\left(1-t\right)A+tB\right)^{-1}\left(B-A\right)\left(\lambda+\left(1-t\right)A+tB\right)^{-1}\right]\\
&\leq M\operatorname{tr}\left[P\left(\lambda+\gamma+tm\right)^{-2}\right]
\end{aligned}\tag{2.17}$$

for all $t\in[0,1]$ and $\lambda>0$.

This is equivalent to

$$\begin{aligned}
&m\left(\lambda+\Gamma+tM\right)^{-2}\\
&\leq \operatorname{tr}\left[P\left(\lambda+\left(1-t\right)A+tB\right)^{-1}\left(B-A\right)\left(\lambda+\left(1-t\right)A+tB\right)^{-1}\right]\\
&\leq M\left(\lambda+\gamma+tm\right)^{-2}
\end{aligned}\tag{2.18}$$

for all $t\in[0,1]$ and $\lambda>0$.

If we take the integrals in (2.18), then we get

$$\begin{aligned}
&m\int_0^\infty\left(\int_0^1\left(\lambda+\Gamma+tM\right)^{-2}dt\right)d\lambda\\
&\leq \int_0^1\operatorname{tr}\left[P\left(\lambda+\left(1-t\right)A+tB\right)^{-1}\left(B-A\right)\left(\lambda+\left(1-t\right)A+tB\right)^{-1}\right]\\
&\leq M\int_0^\infty\left(\int_0^1\left(\lambda+\gamma+tm\right)^{-2}dt\right)d\lambda,
\end{aligned}$$

namely, by (2.8)

$$\begin{aligned}
m\int_0^\infty\left(\int_0^1\left(\lambda+\Gamma+tM\right)^{-2}dt\right)d\lambda &\leq \operatorname{tr}\left[P\left(\ln B\right)\right]-\operatorname{tr}\left[P\left(\ln A\right)\right]\\
&\leq M\int_0^\infty\left(\int_0^1\left(\lambda+\gamma+tm\right)^{-2}dt\right)d\lambda.
\end{aligned}\tag{2.19}$$

Observe that

$$\begin{aligned}
\int_0^1\left(\lambda+\gamma+tm\right)^{-2}dt &= -\frac{1}{m}\left(\lambda+\gamma+m\right)^{-1}+\frac{1}{m}\left(\lambda+\gamma\right)^{-1}\\
&=\frac{1}{m}\left(\left(\lambda+\gamma\right)^{-1}-\left(\lambda+\gamma+m\right)^{-1}\right),
\end{aligned}$$

which gives

$$M\int_0^\infty\left(\int_0^1\left(\lambda+\gamma+tm\right)^{-2}dt\right)d\lambda=\frac{M}{m}\int_0^\infty\left(\left(\lambda+\gamma\right)^{-1}-\left(\lambda+\gamma+m\right)^{-1}\right)d\lambda.$$

By the first identity in (2.1) in the scalar case, we have

$$\ln\left(\gamma+m\right)-\ln\gamma=\int_0^\infty\left[\left(\lambda+\gamma\right)^{-1}-\left(\lambda+\gamma+m\right)^{-1}\right]d\lambda$$

and then

$$M\int_0^\infty\left(\int_0^1(\lambda+\gamma+tm)^{-2}dt\right)d\lambda=M\frac{\ln(\gamma+m)-\ln\gamma}{m}=\ln\left(1+\frac{m}{\gamma}\right)^{\frac{M}{m}}.$$

Similarly,

$$m\int_0^\infty\left(\int_0^1(\lambda+\Gamma+tM)^{-2}dt\right)d\lambda=m\frac{\ln(\Gamma+M)-\ln\Gamma}{M}=\ln\left(1+\frac{M}{\Gamma}\right)^{\frac{m}{M}}$$

and by (2.19) we get

$$\ln\left(1+\frac{M}{\Gamma}\right)^{\frac{m}{M}}\le\ln B-\ln A\le\left(1+\frac{m}{\gamma}\right)^{\frac{M}{m}}. \tag{2.20}$$

If $P\ge0$ with $P\in\mathcal{B}_1(H)$ and $\operatorname{tr}(P)=1$, then by (2.20) we obtain

$$\ln\left(1+\frac{M}{\Gamma}\right)^{\frac{m}{M}}\le\operatorname{tr}[P(\ln B)]-\operatorname{tr}[P(\ln A)]\le\ln\left(1+\frac{m}{\gamma}\right)^{\frac{M}{m}}$$

and by taking the exponential, we derive

$$1<\left(1+\frac{M}{\Gamma}\right)^{\frac{m}{M}}\le\frac{\exp\operatorname{tr}[P(\ln B)]}{\exp\operatorname{tr}[P(\ln A)]}\le\left(1+\frac{m}{\gamma}\right)^{\frac{M}{m}}$$

for all $P\ge0$ with $P\in\mathcal{B}_1(H)$ and $\operatorname{tr}(P)=1$ and the inequality (2.13) is obtained. □

3. Related Results

Let A and B be strictly positive operators on a Hilbert space H such that $B-A\ge m1_H>0$. In 2015, [10], T. Furuta obtained the following result for any non-constant operator monotone function f on $[0,\infty)$

$$f(B)-f(A)\ge f(\|A\|+m)-f(\|A\|)\ge f(\|B\|)-f(\|B\|-m)>0. \tag{3.1}$$

If $B>A>0$, then

$$\begin{aligned} f(B)-f(A)&\ge f\left(\|A\|+\frac{1}{\left\|(B-A)^{-1}\right\|}\right)-f(\|A\|)\\ &\ge f(\|B\|)-f\left(\|B\|-\frac{1}{\left\|(B-A)^{-1}\right\|}\right)>0. \end{aligned} \tag{3.2}$$

The inequality between the first and third term in (3.2) was obtained earlier by H. Zuo and G. Duan in [14].

If we write the inequality (3.1) for $f(t)=\ln t$, then we get for $B-A\ge m>0$

$$\ln B-\ln A\ge\ln\left(\frac{\|A\|+m}{\|A\|}\right)\ge\ln\left(\frac{\|B\|}{\|B\|-m}\right)>0.$$

By multiplying both sides with $P^{1/2}$ and taking the trace, we get

$$\operatorname{tr}\left[P\left(\ln B\right)\right]-\operatorname{tr}\left[P\left(\ln A\right)\right]\geq \ln\left(\frac{\|A\|+m}{\|A\|}\right)\geq \ln\left(\frac{\|B\|}{\|B\|-m}\right)>0.$$

If we take the exponential, we can state that

$$\frac{\operatorname{tr}\left[P\left(\ln B\right)\right]}{\operatorname{tr}\left[P\left(\ln A\right)\right]}\geq \frac{\|A\|+m}{\|A\|}\geq \frac{\|B\|}{\|B\|-m}>1,$$

namely

$$\frac{\Delta_P(B)}{\Delta_P(A)}\geq \frac{\|A\|+m}{\|A\|}\geq \frac{\|B\|}{\|B\|-m}>1 \tag{3.3}$$

provided that $B-A\geq m>0$ and $P\geq 0$ with $P\in\mathcal{B}_1\left(H\right)$ and $\operatorname{tr}\left(P\right)=1$.

If $B>A>0$, then by (3.2) written for for $f\left(t\right)=\ln t$, we get that

$$\begin{aligned}\ln B-\ln A &\geq \ln\left(1+\frac{1}{\|A\|\left\|\left(B-A\right)^{-1}\right\|}\right)\\ &\geq \ln\left(\frac{\|B\|\left\|\left(B-A\right)^{-1}\right\|}{\|B\|\left\|\left(B-A\right)^{-1}\right\|-1}\right)>0.\end{aligned}$$

By utilizing a similar argument as above, we obtain

$$\frac{\Delta_P(B)}{\Delta_P(A)}\geq 1+\frac{1}{\|A\|\left\|\left(B-A\right)^{-1}\right\|}\geq \frac{\|B\|\left\|\left(B-A\right)^{-1}\right\|}{\|B\|\left\|\left(B-A\right)^{-1}\right\|-1}\geq 1 \tag{3.4}$$

provided that $B>A>0$ and $P\geq 0$ with $P\in\mathcal{B}_1\left(H\right)$ and $\operatorname{tr}\left(P\right)=1$.

Its is well known that, if $P\geq 0$, then

$$\left|\left\langle Px,y\right\rangle\right|^2\leq \left\langle Px,x\right\rangle\left\langle Py,y\right\rangle$$

for all x, $y\in H$.

Therefore, if $T>0$, then

$$\begin{aligned}0&\leq \left\langle x,x\right\rangle^2=\left\langle T^{-1}Tx,x\right\rangle^2=\left\langle Tx,T^{-1}x\right\rangle^2\\ &\leq \left\langle Tx,x\right\rangle\left\langle TT^{-1}x,T^{-1}x\right\rangle=\left\langle Tx,x\right\rangle\left\langle x,T^{-1}x\right\rangle\end{aligned}$$

for all $x\in H$.

If $x\in H$, $\|x\|=1$, then

$$1\leq \left\langle Tx,x\right\rangle\left\langle x,T^{-1}x\right\rangle\leq \left\langle Tx,x\right\rangle\sup_{\|x\|=1}\left\langle x,T^{-1}x\right\rangle=\left\langle Tx,x\right\rangle\left\|T^{-1}\right\|,$$

which implies the following operator inequality

$$\left\|T^{-1}\right\|^{-1}\leq T. \tag{3.5}$$

Proposition 2. *If B, $A>0$, then for all $P\geq 0$ with $P\in\mathcal{B}_1\left(H\right)$ and $\operatorname{tr}\left(P\right)=1$,*

$$\begin{aligned}\exp\left[-\Psi\left(\left\|A^{-1}\right\|,\left\|B^{-1}\right\|\right)\|B-A\|\right]&\leq \frac{\Delta_P(B)}{\Delta_P(A)}\\ &\leq \exp\left[\Psi\left(\left\|A^{-1}\right\|,\left\|B^{-1}\right\|\right)\|B-A\|\right],\end{aligned} \tag{3.6}$$

where

$$\Psi\left(\left\|A^{-1}\right\|,\left\|B^{-1}\right\|\right) := \begin{cases} \frac{\ln\left\|B^{-1}\right\|-\ln\left\|A^{-1}\right\|}{\left\|B^{-1}\right\|-\left\|A^{-1}\right\|}\left\|A^{-1}\right\|\left\|B^{-1}\right\| & \text{if } \left\|A^{-1}\right\| \neq \left\|B^{-1}\right\|, \\ \left\|B^{-1}\right\| & \text{if } \left\|A^{-1}\right\| = \left\|B^{-1}\right\|. \end{cases}$$

Proof. Since $A \geq \left\|A^{-1}\right\|^{-1}$ and $B \geq \left\|B^{-1}\right\|^{-1}$, then by (2.10) for $m_1 = \left\|A^{-1}\right\|^{-1}$ and $m_2 = \left\|B^{-1}\right\|^{-1}$ we get

$$\begin{aligned} &\exp\left[-\Phi\left(\left\|A^{-1}\right\|^{-1},\left\|B^{-1}\right\|^{-1}\right)\|B-A\|\right] \\ &\leq \frac{\Delta_P(B)}{\Delta_P(A)} \\ &\leq \exp\left[\Phi\left(\left\|A^{-1}\right\|^{-1},\left\|B^{-1}\right\|^{-1}\right)\|B-A\|\right], \end{aligned} \tag{3.7}$$

where

$$\begin{aligned} &\Phi\left(\left\|A^{-1}\right\|^{-1},\left\|B^{-1}\right\|^{-1}\right) \\ &= \begin{cases} \frac{\ln\left\|B^{-1}\right\|^{-1}-\ln\left\|A^{-1}\right\|^{-1}}{\left\|B^{-1}\right\|^{-1}-\left\|A^{-1}\right\|^{-1}} & \text{if } \left\|A^{-1}\right\| \neq \left\|B^{-1}\right\|, \\ \left\|B^{-1}\right\| & \text{if } \left\|A^{-1}\right\| = \left\|B^{-1}\right\|. \end{cases} \\ &= \Psi\left(\left\|A^{-1}\right\|,\left\|B^{-1}\right\|\right) \end{aligned}$$

and the inequality (3.6) is proved. □

Finally, we can also state:

Proposition 3. *If $B > A > 0$, then for all $P \geq 0$ with $P \in \mathcal{B}_1(H)$ and $\operatorname{tr}(P) = 1$,*

$$\begin{aligned} 1 &< \left(1+\frac{\|B-A\|}{\|A\|}\right)^{\frac{1}{\left\|(B-A)^{-1}\right\|\|B-A\|}} \leq \frac{\Delta_P(B)}{\Delta_P(A)} \\ &\leq \left(1+\frac{\left\|A^{-1}\right\|}{\left\|(B-A)^{-1}\right\|}\right)^{\|B-A\|\left\|(B-A)^{-1}\right\|}. \end{aligned} \tag{3.8}$$

Proof. We have $\left\|(B-A)^{-1}\right\|^{-1} \leq B-A \leq \|B-A\|$ and $\left\|A^{-1}\right\|^{-1} \leq A \leq \|A\|$. By taking $m = \left\|(B-A)^{-1}\right\|^{-1}$, $M = \|B-A\|$, $\gamma = \left\|A^{-1}\right\|^{-1}$ and $\Gamma = \|A\|$ in (2.13) we get (3.8) for all $P \geq 0$ with $P \in \mathcal{B}_1(H)$ and $\operatorname{tr}(P) = 1$. □

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