# A SUB-MULTIPLICATIVE PROPERTY FOR THE TRACE CLASS $P$-DETERMINANT OF POSITIVE OPERATORS IN HILBERT SPACES

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Abstract. Let $H$ be a complex Hilbert space. For a given operator $P \geq 0$ with $P \in \mathcal{B}_1(H)$, the trace class associated to $\mathcal{B}(H)$ and $\operatorname{tr}(P) = 1$, we define the $P$-*determinant* of the positive invertible operator $A$ by

$$\Delta_P(A) := \exp \operatorname{tr}(P \ln A).$$

In this paper we show among others that, if $AB + BA \geq 0$, then

$$\Delta_P(A + B + 1) \leq \Delta_P(A + 1) \Delta_P(B + 1)$$

for all $A$, $B \geq 0$, which is a sub-multiplicative property for the map $\Delta_P(\cdot + 1)$.

## 1. Introduction

In 1952, in the paper [3], B. Fuglede and R. V. Kadison introduced the determinant of a (invertible) operator and established its fundamental properties. The notion generalizes the usual determinant and can be considered for any operator in a finite von Neumann algebra $(M, \tau)$ with a faithful normal trace.

Let $T \in M$ be normal and $|T| := (T^*T)^{1/2}$ its modulus. By the spectral theorem one can represent $T$ as an integral

$$T = \int_{\operatorname{Sp}(T)} \lambda dE(\lambda),$$

where $E(\lambda)$ is a projection valued measure and $\operatorname{Sp}(T)$ is the spectrum of $T$. The measure $\mu_T := \tau \circ E$ becomes a probability measure on the complex plane and has the support in the spectrum $\operatorname{Sp}(T)$.

For any $T \in M$ the Fuglede-Kadison determinant (*FK-determinant*) is defined by

$$\Delta_{FK}(T) := \exp\left(\int_0^\infty \ln t d\mu_{|T|}\right).$$

If $T$ is invertible, then

$$\Delta_{FK}(T) := \exp(\tau(\ln(|T|))),$$

where $\ln(|T|)$ is defined by the use of functional calculus.

Let $B(H)$ be the space of all bounded linear operators on a Hilbert space $H$, and $I$ stands for the identity operator on $H$. An operator $A$ in $B(H)$ is said to be positive (in symbol: $A \geq 0$) if $\langle Ax, x\rangle \geq 0$ for all $x \in H$. In particular, $A > 0$ means that $A$ is positive and invertible. For a pair $A$, $B$ of selfadjoint operators the order relation $A \geq B$ means as usual that $A - B$ is positive.

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In 1998, Fujii et al. [4], [5], introduced the *normalized determinant* $\Delta_x(A)$ for positive invertible operators $A$ on a Hilbert space $H$ and a fixed unit vector $x \in H$, namely $\|x\| = 1$, defined by

$$\Delta_x(A) := \exp \langle \ln Ax, x\rangle$$

and discussed it as a continuous geometric mean and observed some inequalities around the determinant from this point of view. For some recent results, see [9].

We need now some preparations for trace of operators in Hilbert spaces.

Let $(H, \langle \cdot, \cdot \rangle)$ be a complex Hilbert space and $\{e_i\}_{i\in I}$ an *orthonormal basis* of $H$. We say that $A \in \mathcal{B}(H)$ is a *Hilbert-Schmidt operator* if

$$\sum_{i\in I} \|Ae_i\|^2 < \infty. \tag{1.1}$$

It is well know that, if $\{e_i\}_{i\in I}$ and $\{f_j\}_{j\in J}$ are orthonormal bases for $H$ and $A \in \mathcal{B}(H)$ then

$$\sum_{i\in I} \|Ae_i\|^2 = \sum_{j\in I} \|Af_j\|^2 = \sum_{j\in I} \|A^* f_j\|^2 \tag{1.2}$$

showing that the definition (1.1) is independent of the orthonormal basis and $A$ is a Hilbert-Schmidt operator iff $A^*$ is a Hilbert-Schmidt operator.

Let $\mathcal{B}_2(H)$ the set of *Hilbert-Schmidt operators* in $\mathcal{B}(H)$. For $A \in \mathcal{B}_2(H)$ we define

$$\|A\|_2 := \left(\sum_{i\in I} \|Ae_i\|^2\right)^{1/2} \tag{1.3}$$

for $\{e_i\}_{i\in I}$ an orthonormal basis of $H$.

Using the triangle inequality in $l^2(I)$, one checks that $\mathcal{B}_2(H)$ is a *vector space* and that $\|\cdot\|_2$ is a norm on $\mathcal{B}_2(H)$, which is usually called in the literature as the *Hilbert-Schmidt norm.*

Denote *the modulus* of an operator $A \in \mathcal{B}(H)$ by $|A| := (A^*A)^{1/2}$.

Because $\||A|x\| = \|Ax\|$ for all $x \in H$, $A$ is Hilbert-Schmidt iff $|A|$ is Hilbert-Schmidt and $\|A\|_2 = \||A|\|_2$. From (1.2) we have that if $A \in \mathcal{B}_2(H)$, then $A^* \in \mathcal{B}_2(H)$ and $\|A\|_2 = \|A^*\|_2$.

The following theorem collects some of the most important properties of Hilbert-Schmidt operators:

**Theorem 1.** *We have:*

*(i)* $(\mathcal{B}_2(H), \|\cdot\|_2)$ *is a Hilbert space with inner product*

$$\langle A, B\rangle_2 := \sum_{i\in I} \langle Ae_i, Be_i\rangle = \sum_{i\in I} \langle B^*Ae_i, e_i\rangle \tag{1.4}$$

*and the definition does not depend on the choice of the orthonormal basis* $\{e_i\}_{i\in I}$*;*

*(ii) We have the inequalities*

$$\|A\| \le \|A\|_2 \tag{1.5}$$

*for any* $A \in \mathcal{B}_2(H)$ *and, if* $A \in \mathcal{B}_2(H)$ *and* $T \in \mathcal{B}(H)$, *then* $AT, TA \in \mathcal{B}_2(H)$ *with*

$$\|AT\|_2,\ \|TA\|_2 \le \|T\| \|A\|_2 \tag{1.6}$$

*(iii)* $\mathcal{B}_2(H)$ *is an operator ideal in* $\mathcal{B}(H)$, *i.e.*

$$\mathcal{B}(H)\mathcal{B}_2(H)\mathcal{B}(H)\subseteq\mathcal{B}_2(H).$$

If $\{e_i\}_{i\in I}$ an orthonormal basis of $H$, we say that $A\in\mathcal{B}(H)$ is *trace class* if

$$\|A\|_1:=\sum_{i\in I}\langle |A|e_i,e_i\rangle<\infty. \tag{1.7}$$

The definition of $\|A\|_1$ does not depend on the choice of the orthonormal basis $\{e_i\}_{i\in I}$. We denote by $\mathcal{B}_1(H)$ the set of trace class operators in $\mathcal{B}(H)$.

The following proposition holds:

**Proposition 1.** *If* $A\in\mathcal{B}(H)$, *then the following are equivalent:*
*(i)* $A\in\mathcal{B}_1(H)$;
*(ii)* $|A|^{1/2}\in\mathcal{B}_2(H)$.

The following properties are also well known:

**Theorem 2.** *With the above notations:*
*(i) We have*

$$\|A\|_1=\|A^*\|_1 \text{ and } \|A\|_2\le\|A\|_1 \tag{1.8}$$

*for any* $A\in\mathcal{B}_1(H)$;
*(ii)* $\mathcal{B}_1(H)$ *is an operator ideal in* $\mathcal{B}(H)$, *i.e.*

$$\mathcal{B}(H)\mathcal{B}_1(H)\mathcal{B}(H)\subseteq\mathcal{B}_1(H);$$

*(iii) We have*

$$\mathcal{B}_2(H)\mathcal{B}_2(H)=\mathcal{B}_1(H);$$

*(iv) We have*

$$\|A\|_1=\sup\{\langle A,B\rangle_2 \mid B\in\mathcal{B}_2(H),\ \|B\|_2\le1\};$$

*(v)* $(\mathcal{B}_1(H),\|\cdot\|_1)$ *is a Banach space.*

We define the *trace* of a trace class operator $A\in\mathcal{B}_1(H)$ to be

$$\operatorname{tr}(A):=\sum_{i\in I}\langle Ae_i,e_i\rangle, \tag{1.9}$$

where $\{e_i\}_{i\in I}$ an orthonormal basis of $H$. Note that this coincides with the usual definition of the trace if $H$ is finite-dimensional. We observe that the series (1.9) converges absolutely and it is independent from the choice of basis.

The following result collects some properties of the trace:

**Theorem 3.** *We have:*
*(i) If* $A\in\mathcal{B}_1(H)$ *then* $A^*\in\mathcal{B}_1(H)$ *and*

$$\operatorname{tr}(A^*)=\overline{\operatorname{tr}(A)}; \tag{1.10}$$

*(ii) If* $A\in\mathcal{B}_1(H)$ *and* $T\in\mathcal{B}(H)$, *then* $AT,\ TA\in\mathcal{B}_1(H)$,

$$\operatorname{tr}(AT)=\operatorname{tr}(TA) \text{ and } |\operatorname{tr}(AT)|\le\|A\|_1\|T\|; \tag{1.11}$$

*(iii)* $\operatorname{tr}(\cdot)$ *is a bounded linear functional on* $\mathcal{B}_1(H)$ *with* $\|\operatorname{tr}\|=1$;
*(iv) If* $A,\ B\in\mathcal{B}_2(H)$ *then* $AB,\ BA\in\mathcal{B}_1(H)$ *and* $\operatorname{tr}(AB)=\operatorname{tr}(BA)$.

Now, if we assume that $P \geq 0$ and $P \in \mathcal{B}_1(H)$, then for all $T \in \mathcal{B}(H)$, $PT$, $TP \in \mathcal{B}_1(H)$ and $\operatorname{tr}(PT) = \operatorname{tr}(TP)$. Also, since $P^{1/2} \in \mathcal{B}_2(H)$, $TP^{1/2} \in \mathcal{B}_2(H)$, hence $P^{1/2}TP^{1/2}$ and $TP^{1/2}P^{1/2} = TP \in \mathcal{B}_1(H)$ with $\operatorname{tr}\left(P^{1/2}TP^{1/2}\right) = \operatorname{tr}(TP)$. Therefore, if $P \geq 0$ and $P \in \mathcal{B}_1(H)$,

$$\operatorname{tr}(PT) = \operatorname{tr}(TP) = \operatorname{tr}\left(P^{1/2}TP^{1/2}\right)$$

for all $T \in \mathcal{B}(H)$.

If $T \geq 0$, then $P^{1/2}TP^{1/2} \geq 0$, which implies that $\operatorname{tr}(PT) \geq 0$ that shows that the functional $\mathcal{B}(H) \ni T \longmapsto \operatorname{tr}(PT)$ is linear and isotonic functional. Also, by (1.11), if $T_n \to T$ for $n \to \infty$ in $\mathcal{B}(H)$ then $\lim_{n\to\infty} \operatorname{tr}(PT_n) = \operatorname{tr}(PT)$, namely $\mathcal{B}(H) \ni T \longmapsto \operatorname{tr}(PT)$ is also continuous in the norm topology.

For a survey on recent trace inequalities see [1] and the references therein.

Now, for a given $P \geq 0$ with $P \in \mathcal{B}_1(H)$ and $\operatorname{tr}(P) = 1$, we define the *$P$-determinant* of the positive invertible operator $A$ by

$$\Delta_P(A) := \exp\operatorname{tr}(P\ln A) = \exp\operatorname{tr}((\ln A)P) = \exp\operatorname{tr}\left(P^{1/2}(\ln A)P^{1/2}\right). \tag{1.12}$$

Assume that $P \geq 0$ with $P \in \mathcal{B}_1(H)$ and $\operatorname{tr}(P) = 1$. We observe that we have the following elementary properties:

(i) *continuity*: the map $A \to \Delta_P(A)$ is norm continuous;
(ii) *power equality:* $\Delta_P(A^t) = \Delta_P(A)^t$ for all $t > 0$;
(iii) *homogeneity*: $\Delta_P(tA) = t\Delta_x(A)$ and $\Delta_P(tI) = t$ for all $t > 0$;
(iv) *monotonicity*: $0 < A \leq B$ implies $\Delta_P(A) \leq \Delta_P(B)$.

In the recent paper [2] we obtained the following results:

**Theorem 4.** *Let $P \geq 0$ with $P \in \mathcal{B}_1(H)$ and $\operatorname{tr}(P) = 1$, then for all $A$, $B > 0$ and $t \in [0,1]$ we have the Ky Fan's type inequality*

$$\Delta_P((1-t)A + tB) \geq [\Delta_P(A)]^{1-t}[\Delta_P(B)]^t. \tag{1.13}$$

and

**Theorem 5.** *Let $P \geq 0$ with $P \in \mathcal{B}_1(H)$ and $\operatorname{tr}(P) = 1$, then for all $A > 0$ and $a > 0$ we have the double inequality*

$$a\exp\left[1 - a\operatorname{tr}\left(PA^{-1}\right)\right] \leq \Delta_P(A) \leq a\exp\left[a^{-1}\operatorname{tr}(PA) - 1\right]. \tag{1.14}$$

*In particular*

$$1 \leq \frac{\operatorname{tr}(PA)}{\Delta_P(A)} \leq \exp\left[\operatorname{tr}(PA)\operatorname{tr}\left(PA^{-1}\right) - 1\right] \tag{1.15}$$

*and*

$$1 \leq \frac{\Delta_P(A)}{\left[\operatorname{tr}\left(PA^{-1}\right)\right]^{-1}} \leq \exp\left[\operatorname{tr}\left(PA^{-1}\right)\operatorname{tr}(PA) - 1\right]. \tag{1.16}$$

*The first inequalities in (1.15) and 1.16) are best possible from (1.14).*

Motivated by the above results, in this paper we show among others that, if $AB + BA \geq 0$, then

$$\Delta_P(A + B + 1) \leq \Delta_P(A + 1)\Delta_P(B + 1)$$

for all $A$, $B \geq 0$, which is a sub-multiplicative property for the map $\Delta_P(\cdot + 1)$.

## 2. Main Results

The following representation result holds:

**Lemma 1.** *For all $A$, $B \geq 0$ and $a > 0$ we have*

$$\ln(A+a)+\ln(B+a)-\ln(A+B+a)-\ln a = \int_0^\infty (a+\lambda)^{-1} S(\lambda,a,A,B)\,d\lambda + \int_0^\infty (a+\lambda)^{-1} Q(\lambda,a,A,B)\,d\lambda, \tag{2.1}$$

*where*

$$S(\lambda,a,A,B) := (A+B+a+\lambda)^{-1}(AB+BA)(A+B+a+\lambda)^{-1}$$

*and*

$$\begin{aligned} Q(\lambda,a,A,B) := {} & (A+B+a+\lambda)^{-1} \\ & \times \left[B(A+a+\lambda)^{-1}AB + A(B+a+\lambda)^{-1}BA\right] \\ & \times (A+B+a+\lambda)^{-1} \end{aligned}$$

*for $\lambda > 0$.*

*Proof.* Observe that for $t > 0$, $t \neq 1$, we have

$$\int_0^u \frac{d\lambda}{(\lambda+t)(\lambda+1)} = \frac{\ln t}{t-1} + \frac{1}{1-t}\ln\left(\frac{u+t}{u+1}\right)$$

for all $u > 0$.

By taking the limit over $u \to \infty$ in this equality, we derive

$$\frac{\ln t}{t-1} = \int_0^\infty \frac{d\lambda}{(\lambda+t)(\lambda+1)},$$

which gives the representation for the logarithm

$$\ln t = (t-1)\int_0^\infty \frac{d\lambda}{(\lambda+1)(\lambda+t)} \tag{2.2}$$

for all $t > 0$.

If we use the continuous functional calculus for selfadjoint operators, we have

$$\ln T = \int_0^\infty \frac{1}{\lambda+1}(T-1)(\lambda+T)^{-1}\,d\lambda \tag{2.3}$$

for all operators $T > 0$.

Observe that

$$\begin{aligned} \int_0^\infty \frac{1}{\lambda+1}(T-1)(\lambda+T)^{-1}\,d\lambda &= \int_0^\infty \frac{1}{\lambda+1}(T+\lambda-\lambda-1)(\lambda+T)^{-1}\,d\lambda \\ &= \int_0^\infty \left[(\lambda+1)^{-1}-(\lambda+T)^{-1}\right]d\lambda \end{aligned}$$

and then

$$\ln T = \int_0^\infty \left[(\lambda+1)^{-1}-(\lambda+T)^{-1}\right]d\lambda.$$

Therefore

$$\ln(A+a)+\ln(B+a)-\ln(A+B+a)-\ln a = \int_0^\infty K_\lambda d\lambda \tag{2.4}$$

where

$$K_\lambda := (A+B+a+\lambda)^{-1} + (a+\lambda)^{-1} - (A+a+\lambda)^{-1} - (B+a+\lambda)^{-1}.$$

To simplify calculations, consider $\delta := a+\lambda$ and set

$$L_\delta := (A+B+\delta)^{-1} + \delta^{-1} - (A+\delta)^{-1} - (B+\delta)^{-1}.$$

If we multiply both sides by $A+B+\delta$ we get

$$\begin{aligned}
W_\delta &:= (A+B+\delta) L_\delta (A+B+\delta) \\
&= (A+B+\delta) + \delta^{-1}(A+B+\delta)^2 \\
&- (A+B+\delta)(A+\delta)^{-1}(A+B+\delta) \\
&- (A+B+\delta)(B+\delta)^{-1}(A+B+\delta) \\
&= (A+B+\delta) + \delta^{-1}(A+B+\delta)^2 \\
&- (A+B+\delta) - B(A+\delta)^{-1}(A+B+\delta) \\
&- A(B+\delta)^{-1}(A+B+\delta) - (A+B+\delta) \\
&= \delta^{-1}(A+B+\delta)^2 - B(A+\delta)^{-1}B - B \\
&- A(B+\delta)^{-1}A - A - (A+B+\delta)
\end{aligned}$$

$$\begin{aligned}
&= \delta^{-1}\left(A^2 + AB + \delta A + BA + B^2 + \delta B + \delta A + \delta B + \delta^2\right) \\
&- B(A+\delta)^{-1}B - 2B - A(B+\delta)^{-1}A - 2A - \delta \\
&= \delta^{-1}\left(A^2 + AB + BA + B^2\right) + 2B + 2A + \delta \\
&- B(A+\delta)^{-1}B - A(B+\delta)^{-1}A - 2A - 2B - \delta \\
&= \delta^{-1}\left(A^2 + AB + BA + B^2\right) - B(A+\delta)^{-1}B - A(B+\delta)^{-1}A \\
&= \delta^{-1}\left[A^2 + AB + BA + B^2 - \delta B(A+\delta)^{-1}B - \delta A(B+\delta)^{-1}A\right] \\
&= \delta^{-1}\left[A^2 + AB + BA + B^2 - B\left(\delta^{-1}A+1\right)^{-1}B - A\left(\delta^{-1}B+1\right)^{-1}A\right].
\end{aligned}$$

Observe that

$$\begin{aligned}
&B^2 - B\left(\delta^{-1}A+1\right)^{-1}B \\
&= B\left(\delta^{-1}A+1\right)^{-1}\left(\delta^{-1}A+1\right)B - B\left(\delta^{-1}A+1\right)^{-1}B \\
&= B\left(\delta^{-1}A+1\right)^{-1}\left(\delta^{-1}A+1-1\right)B \\
&= \delta^{-1}B\left(\delta^{-1}A+1\right)^{-1}AB = B(A+\delta)^{-1}AB
\end{aligned}$$

and

$$\begin{aligned}
&A^2 - A\left(\delta^{-1}B+1\right)^{-1}A \\
&= A\left(\delta^{-1}B+1\right)^{-1}\left(\delta^{-1}B+1\right)A - A\left(\delta^{-1}B+1\right)^{-1}A \\
&= A\left(\delta^{-1}B+1\right)^{-1}\left(\delta^{-1}B+1-1\right)A \\
&= \delta^{-1}A\left(\delta^{-1}B+1\right)^{-1}BA = A(B+\delta)^{-1}BA.
\end{aligned}$$

Therefore

$$W_\delta = \delta^{-1}\left[AB + BA + B(A+\delta)^{-1}AB + A(B+\delta)^{-1}BA\right]$$

which gives that

$$L_\delta := (A+B+\delta)^{-1}W_\delta(A+B+\delta)^{-1}.$$

We obtain then the following representation

$$\begin{aligned}K_\lambda &= (a+\lambda)^{-1}(A+B+a+\lambda)^{-1}(AB+BA)(A+B+a+\lambda)^{-1} \\ &+ (a+\lambda)^{-1}(A+B+a+\lambda)^{-1} \\ &\times\left[B(A+a+\lambda)^{-1}AB + A(B+a+\lambda)^{-1}BA\right](A+B+a+\lambda)^{-1} \\ &= (a+\lambda)^{-1}S(\lambda,a,A,B) + (a+\lambda)^{-1}P(\lambda,a,A,B)\end{aligned} \tag{2.5}$$

for $a,\ \lambda > 0$.

By utilizing (2.4) and (2.5) we derive the representation (2.1). □

**Corollary 1.** *For all $A,\ B \geq 0$ we have*

$$\begin{aligned}&\ln(A+1) + \ln(B+1) - \ln(A+B+1) \\ &= \int_0^\infty (1+\lambda)^{-1}S(\lambda,A,B)\,d\lambda + \int_0^\infty (1+\lambda)^{-1}Q(\lambda,A,B)\,d\lambda,\end{aligned} \tag{2.6}$$

*where*

$$S(\lambda,A,B) := (A+B+1+\lambda)^{-1}(AB+BA)(A+B+1+\lambda)^{-1}$$

*and*

$$\begin{aligned}Q(\lambda,a,A,B) &:= (A+B+1+\lambda)^{-1} \\ &\times\left[B(A+1+\lambda)^{-1}AB + A(B+1+\lambda)^{-1}BA\right] \\ &\times (A+B+1+\lambda)^{-1}\end{aligned}$$

*for $\lambda > 0$.*

**Theorem 6.** *Let $P \geq 0$ with $P \in \mathcal{B}_1(H)$ and $\operatorname{tr}(P) = 1$, then for all $A,\ B \geq 0$ and $a > 0$ we have the representation*

$$\begin{aligned}\frac{\Delta_P(A+a)\Delta_P(B+a)}{a\Delta_P(A+B+a)} &= \exp\left(\int_0^\infty (a+\lambda)^{-1}\operatorname{tr}[PS(\lambda,a,A,B)]\,d\lambda\right) \\ &\times\exp\left(\int_0^\infty (a+\lambda)^{-1}\operatorname{tr}[PQ(\lambda,a,A,B)]\,d\lambda\right).\end{aligned} \tag{2.7}$$

*Proof.* If we multiply both sides of (2.1) by $P^{1/2}$, then we get

$$\begin{aligned}&P^{1/2}\ln(A+a)P^{1/2} + P^{1/2}\ln(B+a)P^{1/2} \\ &- P^{1/2}\ln(A+B+a)P^{1/2} - (\ln a)P \\ &= \int_0^\infty (a+\lambda)^{-1}P^{1/2}S(\lambda,a,A,B)P^{1/2}d\lambda \\ &+ \int_0^\infty (a+\lambda)^{-1}P^{1/2}P(\lambda,a,A,B)P^{1/2}d\lambda.\end{aligned}$$

If we take the trace and use its properties, we get

$$\begin{aligned}\text{tr}\left[P\ln\left(A+a\right)\right]+\text{tr}\left[P\ln\left(B+a\right)\right]-\text{tr}\left[P\ln\left(A+B+a\right)\right]-\ln a \\ =\int_0^\infty\left(a+\lambda\right)^{-1}\text{tr}\left[PS\left(\lambda,a,A,B\right)\right]d\lambda \\ +\int_0^\infty\left(a+\lambda\right)^{-1}\text{tr}\left[PQ\left(\lambda,a,A,B\right)\right]d\lambda.\end{aligned}\tag{2.8}$$

Further, if we take the exponential in (2.8), then we get the desired result (2.7). □

**Corollary 2.** *Let $P\geq 0$ with $P\in\mathcal{B}_1\left(H\right)$ and $\text{tr}\left(P\right)=1$, then for all $A$, $B\geq 0$ and $a>0$ we have the bounds*

$$\begin{aligned}&\exp\left(\int_0^\infty\left(a+\lambda\right)^{-1}\text{tr}\left[PS\left(\lambda,a,A,B\right)\right]d\lambda\right) \\ &\leq\frac{\Delta_P\left(A+a\right)\Delta_P\left(B+a\right)}{a\Delta_P\left(A+B+a\right)} \\ &\leq\exp\left(\int_0^\infty\left(a+\lambda\right)^{-1}\text{tr}\left[PR\left(\lambda,a,A,B\right)\right]d\lambda\right),\end{aligned}\tag{2.9}$$

*where*

$$R\left(\lambda,a,A,B\right)=\left(A+B+a+\lambda\right)^{-1}\left(A+B\right)^2\left(A+B+a+\lambda\right)^{-1}$$

*for $a$, $\lambda>0$.*

*In particular,*

$$\begin{aligned}&\exp\left(\int_0^\infty\left(1+\lambda\right)^{-1}\text{tr}\left[PS\left(\lambda,A,B\right)\right]d\lambda\right) \\ &\leq\frac{\Delta_P\left(A+1\right)\Delta_P\left(B+1\right)}{\Delta_P\left(A+B+1\right)} \\ &\leq\exp\left(\int_0^\infty\left(1+\lambda\right)^{-1}\text{tr}\left[PR\left(\lambda,A,B\right)\right]d\lambda\right),\end{aligned}\tag{2.10}$$

*where*

$$R\left(\lambda,A,B\right)=\left(A+B+1+\lambda\right)^{-1}\left(A+B\right)^2\left(A+B+1+\lambda\right)^{-1}$$

*for $\lambda>0$.*

*Proof.* Assume that$A$, $B\geq 0$. Observe that for $a$, $\lambda>0$

$$\begin{aligned}\left(A+a+\lambda\right)^{-1}A&=\left(A+a+\lambda\right)^{-1}\left(A+a+\lambda-a-\lambda\right) \\ &=1-\left(a+\lambda\right)\left(A+a+\lambda\right)^{-1},\end{aligned}$$

which shows that

$$0\leq\left(A+a+\lambda\right)^{-1}A\leq 1.$$

If we multiply this inequality both sides by $B$, then we get

$$0\leq B\left(A+a+\lambda\right)^{-1}AB\leq B^2.$$

Similarly,

$$0\leq A\left(B+a+\lambda\right)^{-1}BA\leq A^2.$$

Therefore

$$0\leq B\left(A+a+\lambda\right)^{-1}AB+A\left(B+a+\lambda\right)^{-1}BA\leq A^2+B^2$$

and by multiplying both sides by $(A+B+1+\lambda)^{-1}$ we deduce

$$0 \leq Q(\lambda, a, A, B) \leq (A+B+a+\lambda)^{-1}\left(A^2+B^2\right)(A+B+a+\lambda)^{-1}$$

for $a,\ \lambda > 0$.

Now, if to this inequality we add $S(\lambda, a, A, B)$, then we obtain

$$\begin{aligned} S(\lambda, a, A, B) &\leq Q(\lambda, a, A, B) + S(\lambda, a, A, B) \\ &\leq (A+B+a+\lambda)^{-1}\left(A^2+B^2\right)(A+B+a+\lambda)^{-1} \\ &+ (A+B+a+\lambda)^{-1}(AB+BA)(A+B+a+\lambda)^{-1} \\ &= (A+B+a+\lambda)^{-1}(A+B)^2(A+B+a+\lambda)^{-1} \\ &= R(\lambda, a, A, B) \end{aligned} \tag{2.11}$$

for $a,\ \lambda > 0$.

Further if we multiply both sides of (2.11) by $P^{1/2}$ and take the trace, then we get

$$\begin{aligned} \operatorname{tr}[PS(\lambda, a, A, B)] &\leq \operatorname{tr}[PQ(\lambda, a, A, B)] + \operatorname{tr}[PS(\lambda, a, A, B)] \\ &\leq \operatorname{tr}[PR(\lambda, a, A, B)]. \end{aligned}$$

If we multiply these inequality by $(a+\lambda)^{-1}$, integrate over $\lambda$ and employ the representation (2.7) for the middle term, then we obtain the desired result (2.9). □

**Corollary 3.** *Let $P \geq 0$ with $P \in \mathcal{B}_1(H)$ and $\operatorname{tr}(P) = 1$, and $A,\ B \geq 0,\ a > 0$.*

(i) *If $AB + BA \geq 0$, then*

$$1 \leq \frac{\Delta_P(A+a)\,\Delta_P(B+a)}{a\Delta_P(A+B+a)}. \tag{2.12}$$

*In particular,*

$$\Delta_P(A+B+1) \leq \Delta_P(A+1)\,\Delta_P(B+1), \tag{2.13}$$

*which is a sub-multiplicative property for the map $\Delta_P(\cdot+1)$.*

(ii) *If $A + B \leq K$, with $K$ a positive constant, then*

$$\frac{\Delta_P(A+a)\,\Delta_P(B+a)}{a\Delta_P(A+B+a)} \leq \exp\left(\frac{K^2}{a}\operatorname{tr}\left[P(A+B+a)^{-1}\right]\right). \tag{2.14}$$

*In particular,*

$$\frac{\Delta_P(A+1)\,\Delta_P(B+1)}{\Delta_P(A+B+1)} \leq \exp\left(K^2 \operatorname{tr}\left[P(A+B+1)^{-1}\right]\right). \tag{2.15}$$

*Proof.* (i). If $AB + BA \geq 0$, then by multiplying both sides by $(A+B+a+\lambda)^{-1}$ we get

$$(A+B+a+\lambda)^{-1}(AB+BA)(A+B+a+\lambda)^{-1} \geq 0$$

for $a,\ \lambda > 0$, which implies that $\operatorname{tr}[PS(\lambda, a, A, B)] \geq 0$ giving that

$$\int_0^\infty (1+\lambda)^{-1}\operatorname{tr}[PS(\lambda, A, B)]\,d\lambda \geq 0$$

and the inequality (2.12) is proved.

(ii). If $A+B\leq K$, then

$$\begin{aligned}&(A+B+a+\lambda)^{-1}(A+B)^{2}(A+B+a+\lambda)^{-1}\\&\leq K^{2}(A+B+a+\lambda)^{-2}\end{aligned}$$

for $a,\ \lambda>0$. This implies that

$$\begin{aligned}&\int_{0}^{\infty}(a+\lambda)^{-1}(A+B+a+\lambda)^{-1}(A+B)^{2}(A+B+a+\lambda)^{-1}d\lambda\\&\leq K^{2}\int_{0}^{\infty}(a+\lambda)^{-1}(A+B+a+\lambda)^{-2}d\lambda\\&\leq\frac{K^{2}}{a}\int_{0}^{\infty}(A+B+a+\lambda)^{-2}d\lambda.\end{aligned}\tag{2.16}$$

Now, if we take the derivative over $t$ in (2.2), then we get

$$\begin{aligned}t^{-1}&=\int_{0}^{\infty}(\lambda+1)^{-1}\left(\frac{t-1}{\lambda+t}\right)'d\lambda\\&=\int_{0}^{\infty}(\lambda+1)^{-1}\frac{\lambda+1}{(\lambda+t)^{2}}d\lambda=\int_{0}^{\infty}(\lambda+t)^{-2}d\lambda.\end{aligned}$$

This gives that

$$\int_{0}^{\infty}(A+B+a+\lambda)^{-2}d\lambda=(A+B+a)^{-1}$$

and by (2.16) we obtain by taking the trace that

$$\int_{0}^{\infty}(a+\lambda)^{-1}\operatorname{tr}[PR(\lambda,a,A,B)]\,d\lambda\leq\frac{K^{2}}{a}\operatorname{tr}\left[P(A+B+a)^{-1}\right].$$

Utilizing the second inequality in (2.9) we derive the desired result (2.14). □

## 3. Related Results

We also have the following integral inequalities:

**Theorem 7.** *Let $P\geq 0$ with $P\in\mathcal{B}_{1}(H)$ and $\operatorname{tr}(P)=1$, and $A,\ B\geq 0$ with $AB+BA\geq 0$, then*

$$\begin{aligned}\Delta_{P}(A+B+1)&\leq\int_{0}^{1}\Delta_{P}((1-t)A+tB+1)\,\Delta_{P}((1-t)B+tA+1)\\&\leq\int_{0}^{1}\Delta_{P}^{2}((1-t)A+tB+1)\,dt\end{aligned}\tag{3.1}$$

*and, if $A+B\leq K$, where $K$ is a constant, then also*

$$\begin{aligned}&\int_{0}^{1}\Delta_{P}((1-t)A+tB+1)\,\Delta_{P}((1-t)B+tA+1)\\&\leq\Delta_{P}(A+B+1)\exp\left(K^{2}\operatorname{tr}\left[P(A+B+1)^{-1}\right]\right).\end{aligned}\tag{3.2}$$

*Proof.* We have

$$\begin{aligned}&((1-t)A+tB)((1-t)B+tA)\\&=(1-t)^{2}AB+t(1-t)B^{2}+t(1-t)A^{2}+t^{2}BA\end{aligned}$$

and

$$\begin{aligned}&((1-t)B+tA)((1-t)A+tB)\\&=(1-t)^2BA+t(1-t)A^2+(1-t)tB^2+t^2AB.\end{aligned}$$

Therefore, since $AB+BA\geq 0$, then

$$\begin{aligned}&((1-t)A+tB)((1-t)B+tA)\\&+((1-t)B+tA)((1-t)A+tB)\\&=(1-t)^2AB+t(1-t)B^2+t(1-t)A^2+t^2BA\\&+(1-t)^2BA+t(1-t)A^2+(1-t)tB^2+t^2AB\\&=2t(1-t)A^2+2t(1-t)B^2+\left[(1-t)^2+t^2\right](AB+BA)\\&\geq 0\end{aligned}$$

for all $t\in[0,1]$.

From (2.13) we get

$$\begin{aligned}&\Delta_P((1-t)A+tB+(1-t)B+tA+1)\\&\leq\Delta_P((1-t)A+tB+1)\,\Delta_P((1-t)B+tA+1),\end{aligned}$$

namely

$$\Delta_P(A+B+1)\leq\Delta_P((1-t)A+tB+1)\,\Delta_P((1-t)B+tA+1),$$

for all $t\in[0,1]$.

If we integrate over $t\in[0,1]$, then we get

$$\begin{aligned}&\Delta_P(A+B+1)\\&\leq\int_0^1\Delta_P((1-t)A+tB+1)\,\Delta_P((1-t)B+tA+1)\,dt\\&\leq\left(\int_0^1\Delta_P^2((1-t)A+tB+1)\,dt\right)^{1/2}\left(\int_0^1\Delta_P^2((1-t)B+tA+1)\,dt\right)^{1/2}\\&=\int_0^1\Delta_P^2((1-t)A+tB+1)\,dt,\end{aligned}$$

which proves (3.1).

From (2.15) we get

$$\begin{aligned}&\Delta_P((1-t)A+tB+1)\,\Delta_P((1-t)B+tA+1)\\&\leq\Delta_P(A+B+1)\exp\left(K^2\operatorname{tr}\left[P(A+B+1)^{-1}\right]\right)\end{aligned}$$

for all $t\in[0,1]$.

If we take the integral over $t\in[0,1]$, then we get (3.2). □

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