General multivariate arctangent function activated neural network approximations

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Abstract

Here we expose multivariate quantitative approximations of Banach space valued continuous multivariate functions on a box or $\mathbb{R}^N$, $N \in \mathbb{N}$, by the multivariate normalized, quasi-interpolation, Kantorovich type and quadrature type neural network operators. We treat also the case of approximation by iterated operators of the last four types. These approximations are derived by establishing multidimensional Jackson type inequalities involving the multivariate modulus of continuity of the engaged function or its high order Fréchet derivatives. Our multivariate operators are defined by using a multidimensional density function induced by the arctangent function. The approximations are pointwise and uniform. The related feed-forward neural network is with one hidden layer.

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1 Introduction

The author in [2] and [3], see chapters 2-5, was the first to establish neural network approximations to continuous functions with rates by very specifically defined neural network operators of Cardaliagnet-Euvrard and "Squashing" types, by employing the modulus of continuity of the engaged function or its high order derivative, and producing very tight Jackson type inequalities. He treats

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there both the univariate and multivariate cases. The defining these operators "bell-shaped" and "squashing" functions are assumed to be of compact support. Also in [3] he gives the Nth order asymptotic expansion for the error of weak approximation of these two operators to a special natural class of smooth functions, see chapters 4-5 there.

For this article the author is motivated by the article [13] of Z. Chen and F. Cao, also by [4], [5], [6], [7], [8], [9], [10], [11], [14], [15].

The author here performs multivariate arctangent function based neural network approximations to continuous functions over boxes or over the whole $\mathbb{R}^N$, $N \in \mathbb{N}$. Also he does iterated approximation. All convergences here are with rates expressed via the multivariate modulus of continuity of the involved function or its high order Fréchet derivative and given by very tight multidimensional Jackson type inequalities.

The author here comes up with the "right" precisely defined multivariate normalized, quasi-interpolation neural network operators related to boxes or $\mathbb{R}^N$, as well as Kantorovich type and quadrature type related operators on $\mathbb{R}^N$. Our boxes are not necessarily symmetric to the origin. In preparation to prove our results we establish important properties of the basic multivariate density function induced by arctangent function and defining our operators.

Feed-forward neural networks (FNNs) with one hidden layer, the only type of networks we deal with in this chapter, are mathematically expressed as

$$N_n(x) = \sum_{j=0}^{n} c_j \sigma(\langle a_j \cdot x \rangle + b_j), \quad x \in \mathbb{R}^s, \quad s \in \mathbb{N},$$

where for $0 \le j \le n$, $b_j \in \mathbb{R}$ are the thresholds, $a_j \in \mathbb{R}^s$ are the connection weights, $c_j \in \mathbb{R}$ are the coefficients, $\langle a_j \cdot x \rangle$ is the inner product of a_j and x, and σ is the activation function of the network. In many fundamental network models, the activation function is the arctangent function. About neural networks read [16], [17], [18].

2 Auxiliary Notions

We consider the

$$\arctan x = \int_0^x \frac{dz}{1+z^2}, \quad x \in \mathbb{R}. \tag{1}$$

We will be using

$$h(x) := \frac{2}{\pi} \arctan\left(\frac{\pi}{2} x\right) = \frac{2}{\pi} \int_0^{\frac{\pi x}{2}} \frac{dz}{1+z^2}, \quad x \in \mathbb{R}, \tag{2}$$

which is a sigmoid type function and it is strictly increasing. We have that

$$h(0) = 0, \ \ h(-x) = -h(x), \ \ h(+\infty) = 1, \ \ h(-\infty) = -1,$$

and

$$h'(x) = \frac{4}{4+\pi^2 x^2} > 0, \text{ all } x \in \mathbb{R}. \tag{3}$$

We consider the activation function

$$(x) := \frac{1}{4}\left(h(x+1) - h(x-1)\right), \ \ x \in \mathbb{R}, \tag{4}$$

and we notice that

$$(-x) = \psi(x), \tag{5}$$

it is an even function.

Since $x+1 > x-1$, then $h(x+1) > h(x-1)$, and $\psi(x) > 0$, all $x \in \mathbb{R}$.

We see that

$$(0) = \frac{1}{\pi}\arctan\frac{\pi}{2} \cong 18.31. \tag{6}$$

Let $x > 0$, we have that

$$'(x) = \frac{1}{4}\left(h'(x+1) - h'(x-1)\right) =$$

$$\frac{-4\pi^2 x}{\left(4+\pi^2(x+1)^2\right)\left(4+\pi^2(x-1)^2\right)} < 0. \tag{7}$$

That is

$$'(x) < 0, \text{ for } x > 0. \tag{8}$$

That is ψ is strictly decreasing on $[0,\infty)$ and clearly is strictly increasing on $(-\infty, 0]$, and $\psi'(0) = 0$.

Observe that

$$\begin{array}{l} \lim_{x\to+\infty} (x) = \frac{1}{4}\left(h(+\infty) - h(+\infty)\right) = 0, \\ \text{and} \\ \lim_{x\to-\infty} (x) = \frac{1}{4}\left(h(-\infty) - h(-\infty)\right) = 0. \end{array} \tag{9}$$

That is the x-axis is the horizontal asymptote on ψ.

All in all, ψ is a bell symmetric function with maximum $\psi(0) \cong 18.31$.

We need

Theorem 1 *([11], p. 286) We have that*

$$\sum_{i=-\infty}^{\infty} (x-i) = 1, \ \ \forall\, x \in \mathbb{R}. \tag{10}$$

Theorem 2 *([11], p. 287) It holds*

$$\int_{-\infty}^{\infty} (x)\, dx = 1. \tag{11}$$

So that $\psi(x)$ is a density function on $\mathbb{R}$.

We mention

Theorem 3 *([11], p. 288) Let $0 < \alpha < 1$, and $n \in \mathbb{N}$ with $n^{1-\alpha} > 2$. It holds*

$$\sum_{\begin{cases} k=-\infty \\ : |nx-k| \geq n^{1-\alpha} \end{cases}}^{\infty} \quad (nx-k) < \frac{2}{\pi^2 \left(n^{1-\alpha}-2\right)}. \tag{12}$$

Denote by $\lfloor \cdot \rfloor$ the integral part of the number and by $\lceil \cdot \rceil$ the ceiling of the number.

We need

Theorem 4 *([11], p. 289) Let $x \in [a,b] \subset \mathbb{R}$ and $n \in \mathbb{N}$ so that $\lceil na \rceil \leq \lfloor nb \rfloor$. It holds*

$$\frac{1}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \quad (nx-k)} < \frac{1}{(1)} \cong 0.0868, \quad \forall\, x \in [a,b]. \tag{13}$$

Note 5 *([11], pp. 290-291)*

i) We have that

$$\lim_{n\to\infty} \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \quad (nx-k) \neq 1, \tag{14}$$

for at least some $x \in [a,b]$.

ii) For large enough $n \in \mathbb{N}$ we always obtain $\lceil na \rceil \leq \lfloor nb \rfloor$. Also $a \leq \frac{k}{n} \leq b$, iff $\lceil na \rceil \leq k \leq \lfloor nb \rfloor$.

In general, by Theorem 1, it holds

$$\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \quad (nx-k) \leq 1. \tag{15}$$

We introduce

$$Z(x_1,...,x_N) := Z(x) := \prod_{i=1}^{N} \quad (x_i), \quad x = (x_1,...,x_N) \in \mathbb{R}^N, \ N \in \mathbb{N}. \tag{16}$$

It has the properties:

(i) $Z(x) > 0, \ \forall\, x \in \mathbb{R}^N$,

(ii)

$$\sum_{k=-\infty}^{\infty} Z(x-k) := \sum_{k_1=-\infty}^{\infty} \sum_{k_2=-\infty}^{\infty} ... \sum_{k_N=-\infty}^{\infty} Z(x_1-k_1,...,x_N-k_N) = 1, \tag{17}$$

where $k := (k_1,...,k_n) \in \mathbb{Z}^N$, $\forall\, x \in \mathbb{R}^N$, hence

(iii)

$$\sum_{k=-\infty}^{\infty} Z\left(nx-k\right)=1, \tag{18}$$

$\forall\ x \in \mathbb{R}^N$; $n \in \mathbb{N}$,

and

(iv)

$$\int_{\mathbb{R}^N} Z\left(x\right)dx=1, \tag{19}$$

that is Z is a multivariate density function.

Here denote $\left\|x\right\|_{\infty}:=\max\left\{\left|x_1\right|,...,\left|x_N\right|\right\}$, $x \in \mathbb{R}^N$, also set $\infty:=\left(\infty,...,\infty\right)$, $-\infty:=\left(-\infty,...,-\infty\right)$ upon the multivariate context, and

$$\begin{aligned} \lceil na \rceil &:= \left(\lceil na_1 \rceil,...,\lceil na_N \rceil\right), \\ \lfloor nb \rfloor &:= \left(\lfloor nb_1 \rfloor,...,\lfloor nb_N \rfloor\right), \end{aligned} \tag{20}$$

where $a:=\left(a_1,...,a_N\right)$, $b:=\left(b_1,...,b_N\right)$.

We obviously see that

$$\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} Z\left(nx-k\right)=\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor}\left(\prod_{i=1}^{N}\quad\left(nx_i-k_i\right)\right)=$$

$$\sum_{k_1=\lceil na_1 \rceil}^{\lfloor nb_1 \rfloor}\cdots\sum_{k_N=\lceil na_N \rceil}^{\lfloor nb_N \rfloor}\left(\prod_{i=1}^{N}\quad\left(nx_i-k_i\right)\right)=\prod_{i=1}^{N}\left(\sum_{k_i=\lceil na_i \rceil}^{\lfloor nb_i \rfloor}\quad\left(nx_i-k_i\right)\right). \tag{21}$$

For $0<\beta<1$ and $n \in \mathbb{N}$, a fixed $x \in \mathbb{R}^N$, we have that

$$\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor}\quad\left(nx-k\right)=$$

$$\sum_{\begin{cases} k=\lceil na \rceil \\ \left\|\frac{k}{n}-x\right\|_{\infty}\leq\frac{1}{n^{\beta}} \end{cases}}^{\lfloor nb \rfloor}\quad\left(nx-k\right)+\sum_{\begin{cases} k=\lceil na \rceil \\ \left\|\frac{k}{n}-x\right\|_{\infty}>\frac{1}{n^{\beta}} \end{cases}}^{\lfloor nb \rfloor}\quad\left(nx-k\right). \tag{22}$$

In the last two sums the counting is over disjoint vector sets of k's, because the condition $\left\|\frac{k}{n}-x\right\|_{\infty}>\frac{1}{n^{\beta}}$ implies that there exists at least one $\left|\frac{k_r}{n}-x_r\right|>\frac{1}{n^{\beta}}$, where $r \in \left\{1,...,N\right\}$.

(v) As in [10], pp. 379-380, we derive that

$$\sum_{\begin{cases} k=\lceil na \rceil \\ \left\|\frac{k}{n}-x\right\|_{\infty}>\frac{1}{n^{\beta}} \end{cases}}^{\lfloor nb \rfloor} Z\left(nx-k\right)\overset{(12)}{<}\frac{2}{\pi^2\left(n^{1-\beta}-2\right)},\ \ 0<\beta<1, \tag{23}$$

with $n \in \mathbb{N} : n^{1-\beta} > 2$, $x \in \prod_{i=1}^{N} [a_i, b_i]$.

(vi) By Theorem 4 we get that

$$0 < \frac{1}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} Z(nx-k)} < \frac{1}{(\psi(1))^N} \cong (0.0868)^N, \tag{24}$$

$\forall\, x \in \left(\prod_{i=1}^{N} [a_i, b_i]\right)$, $n \in \mathbb{N}$.

It is also clear that

(vii)

$$\sum_{\substack{k=-\infty \\ \left\| \frac{k}{n} - x \right\|_\infty > \frac{1}{n^\beta}}}^{\infty} Z(nx-k) < \frac{2}{\pi^2 (n^{1-\beta} - 2)}, \tag{25}$$

$0 < \beta < 1$, $n \in \mathbb{N} : n^{1-\beta} > 2$, $x \in \mathbb{R}^N$.

Furthermore it holds

$$\lim_{n \to \infty} \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} Z(nx-k) \neq 1, \tag{26}$$

for at least some $x \in \left(\prod_{i=1}^{N} [a_i, b_i]\right)$.

Here $\left(X, \|\cdot\|_\gamma\right)$ is a Banach space.

Let $f \in C\left(\prod_{i=1}^{N} [a_i, b_i], X\right)$, $x = (x_1, ..., x_N) \in \prod_{i=1}^{N} [a_i, b_i]$, $n \in \mathbb{N}$ such that $\lceil na_i \rceil \leq \lfloor nb_i \rfloor$, $i = 1, ..., N$.

We introduce and define the following multivariate linear normalized neural network operator ($x := (x_1, ..., x_N) \in \left(\prod_{i=1}^{N} [a_i, b_i]\right)$):

$$A_n(f, x_1, ..., x_N) := A_n(f, x) := \frac{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} f\left(\frac{k}{n}\right) Z(nx-k)}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} Z(nx-k)} =$$

$$\frac{\sum_{k_1=\lceil na_1 \rceil}^{\lfloor nb_1 \rfloor} \sum_{k_2=\lceil na_2 \rceil}^{\lfloor nb_2 \rfloor} \cdots \sum_{k_N=\lceil na_N \rceil}^{\lfloor nb_N \rfloor} f\left(\frac{k_1}{n}, ..., \frac{k_N}{n}\right) \left(\prod_{i=1}^{N} \psi(nx_i - k_i)\right)}{\prod_{i=1}^{N} \left(\sum_{k_i=\lceil na_i \rceil}^{\lfloor nb_i \rfloor} (nx_i - k_i)\right)}. \tag{27}$$

For large enough $n \in \mathbb{N}$ we always obtain $\lceil na_i \rceil \leq \lfloor nb_i \rfloor$, $i = 1, ..., N$. Also $a_i \leq \frac{k_i}{n} \leq b_i$, iff $\lceil na_i \rceil \leq k_i \leq \lfloor nb_i \rfloor$, $i = 1, ..., N$.

When $g \in C\left(\prod_{i=1}^{N} [a_i, b_i]\right)$ we define the companion operator

$$\widetilde{A}_n(g, x) := \frac{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} g\left(\frac{k}{n}\right) Z(nx-k)}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} Z(nx-k)}. \tag{28}$$

Clearly $\widetilde{A}_n$ is a positive linear operator. We have that

$$\widetilde{A}_n(1,x)=1, \ \forall\, x\in\left(\prod_{i=1}^{N}[a_i,b_i]\right).$$

Notice that $A_n(f)\in C\left(\prod_{i=1}^{N}[a_i,b_i],X\right)$ and $\widetilde{A}_n(g)\in C\left(\prod_{i=1}^{N}[a_i,b_i]\right)$.

Furthermore it holds

$$\|A_n(f,x)\|_\gamma\le\frac{\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}\left\|f\left(\frac{k}{n}\right)\right\|_\gamma Z(nx-k)}{\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}Z(nx-k)}=\widetilde{A}_n\left(\|f\|_\gamma,x\right), \tag{29}$$

$\forall\, x\in\prod_{i=1}^{N}[a_i,b_i]$.

Clearly $\|f\|_\gamma\in C\left(\prod_{i=1}^{N}[a_i,b_i]\right)$.

So, we have that

$$\|A_n(f,x)\|_\gamma\le\widetilde{A}_n\left(\|f\|_\gamma,x\right), \tag{30}$$

$\forall\, x\in\prod_{i=1}^{N}[a_i,b_i]$, $\forall\, n\in\mathbb{N}$, $\forall\, f\in C\left(\prod_{i=1}^{N}[a_i,b_i],X\right)$.

Let $c\in X$ and $g\in C\left(\prod_{i=1}^{N}[a_i,b_i]\right)$, then $cg\in C\left(\prod_{i=1}^{N}[a_i,b_i],X\right)$.

Furthermore it holds

$$A_n(cg,x)=c\widetilde{A}_n(g,x), \ \forall\, x\in\prod_{i=1}^{N}[a_i,b_i]. \tag{31}$$

Since $\widetilde{A}_n(1)=1$, we get that

$$A_n(c)=c, \ \forall\, c\in X. \tag{32}$$

We call $\widetilde{A}_n$ the companion operator of A_n.

For convinience we call

$$A_n^*(f,x):=\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}f\left(\frac{k}{n}\right)Z(nx-k)=$$

$$\sum_{k_1=\lceil na_1\rceil}^{\lfloor nb_1\rfloor}\sum_{k_2=\lceil na_2\rceil}^{\lfloor nb_2\rfloor}\cdots\sum_{k_N=\lceil na_N\rceil}^{\lfloor nb_N\rfloor}f\left(\frac{k_1}{n},...,\frac{k_N}{n}\right)\left(\prod_{i=1}^{N}\ (nx_i-k_i)\right), \tag{33}$$

$\forall\, x\in\left(\prod_{i=1}^{N}[a_i,b_i]\right)$.

That is

$$A_n(f,x):=\frac{A_n^*(f,x)}{\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}Z(nx-k)}, \tag{34}$$

$\forall\ x \in \left(\prod_{i=1}^{N}[a_i,b_i]\right)$, $n \in \mathbb{N}$.

Hence

$$A_n(f,x) - f(x) = \frac{A_n^*(f,x) - f(x)\left(\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} Z(nx-k)\right)}{\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} Z(nx-k)}. \tag{35}$$

Consequently we derive

$$\|A_n(f,x) - f(x)\|_\gamma \overset{(24)}{\leq} (0.0868)^N \left\| A_n^*(f,x) - f(x)\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} Z(nx-k)\right\|_\gamma, \tag{36}$$

$\forall\ x \in \left(\prod_{i=1}^{N}[a_i,b_i]\right)$.

We will estimate the right hand side of (36).

For the last and others we need

Definition 6 *([11], p. 274) Let M be a convex and compact subset of $\left(\mathbb{R}^N, \|\cdot\|_p\right)$, $p \in [1,\infty]$, and $\left(X, \|\cdot\|_\gamma\right)$ be a Banach space. Let $f \in C(M,X)$. We define the first modulus of continuity of f as*

$$\omega_1(f,\delta) := \sup_{\substack{x,y\in M:\\ \|x-y\|_p \leq \delta}} \|f(x)-f(y)\|_\gamma,\ \ 0<\delta\leq diam(M). \tag{37}$$

If $\delta > diam(M)$, then

$$\omega_1(f,\delta) = \omega_1(f, diam(M)). \tag{38}$$

Notice $\omega_1(f,\delta)$ is increasing in $\delta > 0$. For $f \in C_B(M,X)$ (continuous and bounded functions) $\omega_1(f,\delta)$ is defined similarly.

Lemma 7 *([11], p. 274) We have $\omega_1(f,\delta) \to 0$ as $\delta \downarrow 0$, iff $f \in C(M,X)$, where M is a convex compact subset of $\left(\mathbb{R}^N, \|\cdot\|_p\right)$, $p \in [1,\infty]$.*

Clearly we have also: $f \in C_U\left(\mathbb{R}^N, X\right)$ (uniformly continuous functions), iff $\omega_1(f,\delta) \to 0$ as $\delta \downarrow 0$, where ω_1 is defined similarly to (37). The space $C_B\left(\mathbb{R}^N, X\right)$ denotes the continuous and bounded functions on $\mathbb{R}^N$.

When $f \in C_B\left(\mathbb{R}^N, X\right)$ we define,

$$B_n(f,x) := B_n(f,x_1,...,x_N) := \sum_{k=-\infty}^{\infty} f\left(\frac{k}{n}\right) Z(nx-k) :=$$

$$\sum_{k_1=-\infty}^{\infty}\sum_{k_2=-\infty}^{\infty}\cdots\sum_{k_N=-\infty}^{\infty} f\left(\frac{k_1}{n},\frac{k_2}{n},...,\frac{k_N}{n}\right)\left(\prod_{i=1}^{N}(nx_i-k_i)\right), \tag{39}$$

$n \in \mathbb{N}$, $\forall\, x \in \mathbb{R}^N$, $N \in \mathbb{N}$, the multivariate quasi-interpolation neural network operator.

Also for $f \in C_B\left(\mathbb{R}^N, X\right)$ we define the multivariate Kantorovich type neural network operator

$$C_n(f,x) := C_n(f, x_1, ..., x_N) := \sum_{k=-\infty}^{\infty} \left(n^N \int_{\frac{k}{n}}^{\frac{k+1}{n}} f(t)\, dt \right) Z(nx-k) =$$

$$\sum_{k_1=-\infty}^{\infty} \sum_{k_2=-\infty}^{\infty} ... \sum_{k_N=-\infty}^{\infty} \left(n^N \int_{\frac{k_1}{n}}^{\frac{k_1+1}{n}} \int_{\frac{k_2}{n}}^{\frac{k_2+1}{n}} ... \int_{\frac{k_N}{n}}^{\frac{k_N+1}{n}} f(t_1, ..., t_N)\, dt_1 ... dt_N \right)$$

$$\cdot \left(\prod_{i=1}^{N} \quad (nx_i - k_i) \right), \tag{40}$$

$n \in \mathbb{N}$, $\forall\, x \in \mathbb{R}^N$.

Again for $f \in C_B\left(\mathbb{R}^N, X\right)$, $N \in \mathbb{N}$, we define the multivariate neural network operator of quadrature type $D_n(f,x)$, $n \in \mathbb{N}$, as follows.

Let $\theta = (\theta_1, ..., \theta_N) \in \mathbb{N}^N$, $r = (r_1, ..., r_N) \in \mathbb{Z}_+^N$, $w_r = w_{r_1, r_2, ... r_N} \geq 0$, such that $\sum\limits_{r=0}^{\theta} w_r = \sum\limits_{r_1=0}^{\theta_1} \sum\limits_{r_2=0}^{\theta_2} ... \sum\limits_{r_N=0}^{\theta_N} w_{r_1, r_2, ... r_N} = 1$; $k \in \mathbb{Z}^N$ and

$$\delta_{nk}(f) := \delta_{n, k_1, k_2, ..., k_N}(f) := \sum_{r=0}^{\theta} w_r f\left(\frac{k}{n} + \frac{r}{n\theta} \right) =$$

$$\sum_{r_1=0}^{\theta_1} \sum_{r_2=0}^{\theta_2} ... \sum_{r_N=0}^{\theta_N} w_{r_1, r_2, ... r_N} f\left(\frac{k_1}{n} + \frac{r_1}{n\theta_1}, \frac{k_2}{n} + \frac{r_2}{n\theta_2}, ..., \frac{k_N}{n} + \frac{r_N}{n\theta_N} \right), \tag{41}$$

where $\frac{r}{\theta} := \left(\frac{r_1}{\theta_1}, \frac{r_2}{\theta_2}, ..., \frac{r_N}{\theta_N} \right)$.

We set

$$D_n(f,x) := D_n(f, x_1, ..., x_N) := \sum_{k=-\infty}^{\infty} \delta_{nk}(f)\, Z(nx-k) = \tag{42}$$

$$\sum_{k_1=-\infty}^{\infty} \sum_{k_2=-\infty}^{\infty} ... \sum_{k_N=-\infty}^{\infty} \delta_{n, k_1, k_2, ..., k_N}(f) \left(\prod_{i=1}^{N} \quad (nx_i - k_i) \right),$$

$\forall\, x \in \mathbb{R}^N$.

In this article we study the approximation properties of A_n, B_n, C_n, D_n neural network operators and as well of their iterates. That is, the quantitative pointwise and uniform convergence of these operators to the unit operator I.

3 Multivariate general Neural Network Approximations

Here we present several vectorial neural network approximations to Banach space valued functions given with rates.

We give

Theorem 8 *Let* $f \in C\left(\prod_{i=1}^{N}[a_i,b_i], X\right)$, $0<\beta<1$, $x \in \left(\prod_{i=1}^{N}[a_i,b_i]\right)$, $N, n \in \mathbb{N}$ *with* $n^{1-\beta}>2$. *Then*

1)

$$\left\| A_n(f,x) - f(x) \right\|_\gamma \leq (0.0868)^N \left[\omega_1\left(f, \frac{1}{n^\beta}\right) + \frac{4\left\| \|f\|_\gamma \right\|_\infty}{\pi^2\left(n^{1-\beta}-2\right)} \right] =: \lambda_1(n), \tag{43}$$

and

2)

$$\left\| \left\| A_n(f) - f \right\|_\gamma \right\|_\infty \leq \lambda_1(n). \tag{44}$$

We notice that $\lim_{n\to\infty} A_n(f) \overset{\|\cdot\|_\gamma}{=} f$, *pointwise and uniformly.*

Above ω_1 *is with respect to* $p=\infty$.

Proof. We observe that

$$\Delta(x) := A_n^*(f,x) - f(x) \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} Z(nx-k) =$$

$$\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} f\left(\frac{k}{n}\right) Z(nx-k) - \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} f(x) Z(nx-k) =$$

$$\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \left(f\left(\frac{k}{n}\right) - f(x) \right) Z(nx-k). \tag{45}$$

Thus

$$\|\Delta(x)\|_\gamma \leq \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \left\| f\left(\frac{k}{n}\right) - f(x) \right\|_\gamma Z(nx-k) =$$

$$\sum_{\substack{\begin{cases} k=\lceil na \rceil \\ \left\|\frac{k}{n}-x\right\|_\infty \leq \frac{1}{n^\beta} \end{cases}}}^{\lfloor nb \rfloor} \left\| f\left(\frac{k}{n}\right) - f(x) \right\|_\gamma Z(nx-k) +$$

$$\sum_{\begin{cases} k=\lceil na\rceil \\ \left\|\frac{k}{n}-x\right\|_{\infty}>\frac{1}{n^{\beta}} \end{cases}}^{\lfloor nb\rfloor} \left\|f\left(\frac{k}{n}\right)-f(x)\right\|_{\gamma} Z(nx-k) \overset{(18)}{\leq}$$

$$\omega_1\left(f,\frac{1}{n^{\beta}}\right)+2\left\|\|f\|_{\gamma}\right\|_{\infty} \sum_{\begin{cases} k=\lceil na\rceil \\ \left\|\frac{k}{n}-x\right\|_{\infty}>\frac{1}{n^{\beta}} \end{cases}}^{\lfloor nb\rfloor} Z(nx-k) \overset{(23)}{\leq}$$

$$\omega_1\left(f,\frac{1}{n^{\beta}}\right)+\frac{4\left\|\|f\|_{\gamma}\right\|_{\infty}}{\pi^2\left(n^{1-\beta}-2\right)}. \tag{46}$$

So that

$$\|\Delta(x)\|_{\gamma} \leq \omega_1\left(f,\frac{1}{n^{\beta}}\right)+\frac{4\left\|\|f\|_{\gamma}\right\|_{\infty}}{\pi^2\left(n^{1-\beta}-2\right)}. \tag{47}$$

Now using (36) we finish the proof. ■

We make

Remark 9 *([11], pp. 263-266) Let* $\left(\mathbb{R}^N,\|\cdot\|_p\right)$, $N\in\mathbb{N}$; *where* $\|\cdot\|_p$ *is the* L_p*-norm,* $1\leq p\leq\infty$. $\mathbb{R}^N$ *is a Banach space, and* $\left(\mathbb{R}^N\right)^j$ *denotes the* j*-fold product space* $\mathbb{R}^N\times\ldots\times\mathbb{R}^N$ *endowed with the max-norm* $\|x\|_{(\mathbb{R}^N)^j}:=\max_{1\leq\lambda\leq j}\|x_{\lambda}\|_p$, *where* $x:=(x_1,\ldots,x_j)\in\left(\mathbb{R}^N\right)^j$.

Let $\left(X,\|\cdot\|_{\gamma}\right)$ *be a general Banach space. Then the space* $L_j:=L_j\left(\left(\mathbb{R}^N\right)^j;X\right)$ *of all* j*-multilinear continuous maps* $g:\left(\mathbb{R}^N\right)^j\rightarrow X$, $j=1,\ldots,m$, *is a Banach space with norm*

$$\|g\|:=\|g\|_{L_j}:=\sup_{\left(\|x\|_{(\mathbb{R}^N)^j}=1\right)}\|g(x)\|_{\gamma}=\sup\frac{\|g(x)\|_{\gamma}}{\|x_1\|_p\ldots\|x_j\|_p}. \tag{48}$$

Let M *be a non-empty convex and compact subset of* $\mathbb{R}^k$ *and* $x_0\in M$ *is fixed.*

Let O *be an open subset of* $\mathbb{R}^N: M\subset O$. *Let* $f:O\rightarrow X$ *be a continuous function, whose Fréchet derivatives (see [19])* $f^{(j)}:O\rightarrow L_j=L_j\left(\left(\mathbb{R}^N\right)^j;X\right)$ *exist and are continuous for* $1\leq j\leq m$, $m\in\mathbb{N}$.

Call $(x-x_0)^j:=(x-x_0,\ldots,x-x_0)\in\left(\mathbb{R}^N\right)^j$, $x\in M$.

We will work with $f|_M$.

Then, by Taylor's formula ([12]), ([19], p. 124), we get

$$f(x)=\sum_{j=0}^{m}\frac{f^{(j)}(x_0)(x-x_0)^j}{j!}+R_m(x,x_0),\quad \text{all } x\in M, \tag{49}$$

where the remainder is the Riemann integral

$$R_m(x, x_0) := \int_0^1 \frac{(1-u)^{m-1}}{(m-1)!} \left(f^{(m)}(x_0 + u(x - x_0)) - f^{(m)}(x_0) \right) (x - x_0)^m \, du, \tag{50}$$

here we set $f^{(0)}(x_0)(x-x_0)^0 = f(x_0)$.

We consider

$$w := \omega_1 \left(f^{(m)}, h \right) := \sup_{\substack{x,y \in M: \\ \|x-y\|_p \leq h}} \left\| f^{(m)}(x) - f^{(m)}(y) \right\|, \tag{51}$$

$h > 0$.

We obtain

$$\left\| \left(f^{(m)}(x_0 + u(x - x_0)) - f^{(m)}(x_0) \right) (x - x_0)^m \right\|_\gamma \leq$$

$$\left\| f^{(m)}(x_0 + u(x - x_0)) - f^{(m)}(x_0) \right\| \cdot \|x - x_0\|_p^m \leq$$

$$w \|x - x_0\|_p^m \left\lceil \frac{u \|x - x_0\|_p}{h} \right\rceil, \tag{52}$$

by Lemma 7.1.1, [1], p. 208, where $\lceil \cdot \rceil$ *is the ceiling.*

Therefore for all $x \in M$ *(see [1], pp. 121-122):*

$$\|R_m(x, x_0)\|_\gamma \leq w \|x - x_0\|_p^m \int_0^1 \left\lceil \frac{u \|x - x_0\|_p}{h} \right\rceil \frac{(1-u)^{m-1}}{(m-1)!} du$$

$$= w \Phi_m \left(\|x - x_0\|_p \right) \tag{53}$$

by a change of variable, where

$$\Phi_m(t) := \int_0^{|t|} \left\lceil \frac{s}{h} \right\rceil \frac{(|t| - s)^{m-1}}{(m-1)!} ds = \frac{1}{m!} \left(\sum_{j=0}^{\infty} (|t| - jh)_+^m \right), \quad \forall \, t \in \mathbb{R}, \tag{54}$$

is a (polynomial) spline function, see [1], p. 210-211.

Also from there we get

$$\Phi_m(t) \leq \left(\frac{|t|^{m+1}}{(m+1)!h} + \frac{|t|^m}{2m!} + \frac{h |t|^{m-1}}{8(m-1)!} \right), \quad \forall \, t \in \mathbb{R}, \tag{55}$$

with equality true only at $t = 0$.

Therefore it holds

$$\|R_m(x, x_0)\|_\gamma \leq w \left(\frac{\|x - x_0\|_p^{m+1}}{(m+1)!h} + \frac{\|x - x_0\|_p^m}{2m!} + \frac{h \|x - x_0\|_p^{m-1}}{8(m-1)!} \right), \quad \forall \, x \in M. \tag{56}$$

We have found that

$$\left\| f(x) - \sum_{j=0}^{m} \frac{f^{(j)}(x_0)(x-x_0)^j}{j!} \right\|_\gamma \leq$$

$$\omega_1\left(f^{(m)}, h\right)\left(\frac{\|x-x_0\|_p^{m+1}}{(m+1)!h} + \frac{\|x-x_0\|_p^{m}}{2m!} + \frac{h\|x-x_0\|_p^{m-1}}{8(m-1)!}\right) < \infty, \tag{57}$$

$\forall\ x, x_0 \in M$.

Here $0 < \omega_1\left(f^{(m)}, h\right) < \infty$, *by* M *being compact and* $f^{(m)}$ *being continuous on* M.

One can rewrite (57) as follows:

$$\left\| f(\cdot) - \sum_{j=0}^{m} \frac{f^{(j)}(x_0)(\cdot-x_0)^j}{j!} \right\|_\gamma \leq$$

$$\omega_1\left(f^{(m)}, h\right)\left(\frac{\|\cdot-x_0\|_p^{m+1}}{(m+1)!h} + \frac{\|\cdot-x_0\|_p^{m}}{2m!} + \frac{h\|\cdot-x_0\|_p^{m-1}}{8(m-1)!}\right),\ \forall\ x_0 \in M, \tag{58}$$

a pointwise functional inequality on M.

Here $(\cdot - x_0)^j$ *maps* M *into* $\left(\mathbb{R}^N\right)^j$ *and it is continuous, also* $f^{(j)}(x_0)$ *maps* $\left(\mathbb{R}^N\right)^j$ *into* X *and it is continuous. Hence their composition* $f^{(j)}(x_0)(\cdot - x_0)^j$ *is continuous from* M *into* X.

Clearly $f(\cdot) - \sum_{j=0}^{m} \frac{f^{(j)}(x_0)(\cdot-x_0)^j}{j!} \in C(M,X)$, *hence* $\left\| f(\cdot) - \sum_{j=0}^{m} \frac{f^{(j)}(x_0)(\cdot-x_0)^j}{j!} \right\|_\gamma \in C(M)$.

Let $\left\{\widetilde{L}_N\right\}_{N\in\mathbb{N}}$ *be a sequence of positive linear operators mapping* $C(M)$ *into* $C(M)$.

Therefore we obtain

$$\left(\widetilde{L}_N\left(\left\| f(\cdot) - \sum_{j=0}^{m} \frac{f^{(j)}(x_0)(\cdot-x_0)^j}{j!} \right\|_\gamma\right)\right)(x_0) \leq$$

$$\omega_1\left(f^{(m)}, h\right)\left[\frac{\left(\widetilde{L}_N\left(\|\cdot-x_0\|_p^{m+1}\right)\right)(x_0)}{(m+1)!h} + \frac{\left(\widetilde{L}_N\left(\|\cdot-x_0\|_p^{m}\right)\right)(x_0)}{2m!}\right.$$

$$\left. + \frac{h\left(\widetilde{L}_N\left(\|\cdot-x_0\|_p^{m-1}\right)\right)(x_0)}{8(m-1)!}\right], \tag{59}$$

$\forall\ N \in \mathbb{N}$, $\forall\ x_0 \in M$.

Clearly (59) is valid when $M=\prod_{i=1}^{N}[a_i,b_i]$ and $\widetilde{L}_n=\widetilde{A}_n$, see (28).

All the above is preparation for the following theorem, where we assume Fréchet differentiability of functions.

This will be a direct application of Theorem 10.2, [11], pp. 268-270. The operators A_n, $\widetilde{A}_n$ fulfill its assumptions, see (27), (28), (30), (31) and (32).

We present the following high order approximation results.

Theorem 10 *Let O open subset of $\left(\mathbb{R}^N,\|\cdot\|_p\right)$, $p\in[1,\infty]$, such that $\prod_{i=1}^{N}[a_i,b_i]\subset O\subseteq\mathbb{R}^N$, and let $\left(X,\|\cdot\|_\gamma\right)$ be a general Banach space. Let $m\in\mathbb{N}$ and $f\in C^m(O,X)$, the space of m-times continuously Fréchet differentiable functions from O into X. We study the approximation of $f|_{\prod_{i=1}^{N}[a_i,b_i]}$. Let $x_0\in\left(\prod_{i=1}^{N}[a_i,b_i]\right)$ and $r>0$. Then*

1)

$$\left\|\left(A_n(f)\right)(x_0)-\sum_{j=0}^{m}\frac{1}{j!}\left(A_n\left(f^{(j)}(x_0)(\cdot-x_0)^j\right)\right)(x_0)\right\|_\gamma\leq$$

$$\frac{\omega_1\left(f^{(m)},r\left(\left(\widetilde{A}_n\left(\|\cdot-x_0\|_p^{m+1}\right)\right)(x_0)\right)^{\frac{1}{m+1}}\right)}{rm!}\left(\left(\widetilde{A}_n\left(\|\cdot-x_0\|_p^{m+1}\right)\right)(x_0)\right)^{\left(\frac{m}{m+1}\right)}$$
$$\left[\frac{1}{(m+1)}+\frac{r}{2}+\frac{mr^2}{8}\right], \tag{60}$$

2) additionally if $f^{(j)}(x_0)=0$, $j=1,...,m$, we have

$$\left\|\left(A_n(f)\right)(x_0)-f(x_0)\right\|_\gamma\leq$$

$$\frac{\omega_1\left(f^{(m)},r\left(\left(\widetilde{A}_n\left(\|\cdot-x_0\|_p^{m+1}\right)\right)(x_0)\right)^{\frac{1}{m+1}}\right)}{rm!}\left(\left(\widetilde{A}_n\left(\|\cdot-x_0\|_p^{m+1}\right)\right)(x_0)\right)^{\left(\frac{m}{m+1}\right)} \tag{61}$$
$$\left[\frac{1}{(m+1)}+\frac{r}{2}+\frac{mr^2}{8}\right],$$

3)

$$\left\|\left(A_n(f)\right)(x_0)-f(x_0)\right\|_\gamma\leq\sum_{j=1}^{m}\frac{1}{j!}\left\|\left(A_n\left(f^{(j)}(x_0)(\cdot-x_0)^j\right)\right)(x_0)\right\|_\gamma+$$

$$\frac{\omega_1\left(f^{(m)},r\left(\left(\widetilde{A}_n\left(\|\cdot-x_0\|_p^{m+1}\right)\right)(x_0)\right)^{\frac{1}{m+1}}\right)}{rm!}\left(\left(\widetilde{A}_n\left(\|\cdot-x_0\|_p^{m+1}\right)\right)(x_0)\right)^{\left(\frac{m}{m+1}\right)} \tag{62}$$

$$\left[\frac{1}{(m+1)}+\frac{r}{2}+\frac{mr^2}{8}\right],$$

and

4)

$$\left\|\left\|A_n(f)-f\right\|_\gamma\right\|_{\infty,\prod_{i=1}^{N}[a_i,b_i]}\leq$$

$$\sum_{j=1}^{m}\frac{1}{j!}\left\|\left\|\left(A_n\left(f^{(j)}(x_0)(\cdot-x_0)^j\right)\right)(x_0)\right\|_\gamma\right\|_{\infty,x_0\in\prod_{i=1}^{N}[a_i,b_i]}+$$

$$\frac{\omega_1\left(f^{(m)},r\left\|\left(\widetilde{A}_n\left(\|\cdot-x_0\|_p^{m+1}\right)\right)(x_0)\right\|_{\infty,x_0\in\prod_{i=1}^{N}[a_i,b_i]}^{\frac{1}{m+1}}\right)}{rm!}$$

$$\left\|\left(\widetilde{A}_n\left(\|\cdot-x_0\|_p^{m+1}\right)\right)(x_0)\right\|_{\infty,x_0\in\prod_{i=1}^{N}[a_i,b_i]}^{\left(\frac{m}{m+1}\right)} \tag{63}$$

$$\left[\frac{1}{(m+1)}+\frac{r}{2}+\frac{mr^2}{8}\right].$$

We need

Lemma 11 *The function* $\left(\widetilde{A}_n\left(\|\cdot-x_0\|_p^m\right)\right)(x_0)$ *is continuous in* $x_0\in\left(\prod_{i=1}^{N}[a_i,b_i]\right)$, $m\in\mathbb{N}$.

Proof. By Lemma 10.3, [11], p. 272. ■

We make

Remark 12 *By Remark 10.4, [11], p. 273, we get that*

$$\left\|\left(\widetilde{A}_n\left(\|\cdot-x_0\|_p^k\right)\right)(x_0)\right\|_{\infty,x_0\in\prod_{i=1}^{N}[a_i,b_i]}\leq$$

$$\left\|\left(\widetilde{A}_n\left(\|\cdot-x_0\|_p^{m+1}\right)\right)(x_0)\right\|_{\infty,x_0\in\prod_{i=1}^{N}[a_i,b_i]}^{\left(\frac{k}{m+1}\right)}, \tag{64}$$

for all $k=1,...,m$.

We give

Corollary 13 *(to Theorem 10, case of* $m=1$*) Then*

1)

$$\left\|(A_n(f))(x_0)-f(x_0)\right\|_\gamma\leq\left\|\left(A_n\left(f^{(1)}(x_0)(\cdot-x_0)\right)\right)(x_0)\right\|_\gamma+$$

$$\frac{1}{2r}\omega_1\left(f^{(1)}, r\left(\left(\widetilde{A}_n\left(\|\cdot - x_0\|_p^2\right)\right)(x_0)\right)^{\frac{1}{2}}\right)\left(\left(\widetilde{A}_n\left(\|\cdot - x_0\|_p^2\right)\right)(x_0)\right)^{\frac{1}{2}} \tag{65}$$

$$\left[1 + r + \frac{r^2}{4}\right],$$

and

2)

$$\left\|\|(A_n(f)) - f\|_\gamma\right\|_{\infty, \prod_{i=1}^N [a_i, b_i]} \leq$$

$$\left\|\left\|\left(A_n\left(f^{(1)}(x_0)(\cdot - x_0)\right)\right)(x_0)\right\|_\gamma\right\|_{\infty, x_0 \in \prod_{i=1}^N [a_i, b_i]} +$$

$$\frac{1}{2r}\omega_1\left(f^{(1)}, r\left\|\left(\widetilde{A}_n\left(\|\cdot - x_0\|_p^2\right)\right)(x_0)\right\|_{\infty, x_0 \in \prod_{i=1}^N [a_i, b_i]}^{\frac{1}{2}}\right)$$

$$\left\|\left(\widetilde{A}_n\left(\|\cdot - x_0\|_p^2\right)\right)(x_0)\right\|_{\infty, x_0 \in \prod_{i=1}^N [a_i, b_i]}^{\frac{1}{2}}\left[1 + r + \frac{r^2}{4}\right], \tag{66}$$

$r > 0$.

We make

Remark 14 *We estimate* $0 < \alpha < 1$, $m, n \in \mathbb{N} : n^{1-\alpha} > 2$,

$$\widetilde{A}_n\left(\|\cdot - x_0\|_\infty^{m+1}\right)(x_0) = \frac{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor}\left\|\frac{k}{n} - x_0\right\|_\infty^{m+1} Z(nx_0 - k)}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} Z(nx_0 - k)} \overset{(24)}{<}$$

$$(0.0868)^N \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor}\left\|\frac{k}{n} - x_0\right\|_\infty^{m+1} Z(nx_0 - k) = \tag{67}$$

$$(0.0868)^N \left\{ \sum_{\substack{k = \lceil na \rceil \\ : \left\|\frac{k}{n} - x_0\right\|_\infty \leq \frac{1}{n^\alpha}}}^{\lfloor nb \rfloor} \left\|\frac{k}{n} - x_0\right\|_\infty^{m+1} Z(nx_0 - k) + \right.$$

$$\left. \sum_{\substack{k = \lceil na \rceil \\ : \left\|\frac{k}{n} - x_0\right\|_\infty > \frac{1}{n^\alpha}}}^{\lfloor nb \rfloor} \left\|\frac{k}{n} - x_0\right\|_\infty^{m+1} Z(nx_0 - k) \right\} \overset{(25)}{\leq}$$

$$(0.0868)^N \left\{ \frac{1}{n^{\alpha(m+1)}} + \frac{2 \|b-a\|_\infty^{m+1}}{\pi^2 (n^{1-\alpha} - 2)} \right\}, \tag{68}$$

(where $b - a = (b_1 - a_1, ..., b_N - a_N)$*).*

We have proved that $(\forall\ x_0 \in \prod_{i=1}^{N} [a_i, b_i])$

$$\widetilde{A}_n \left(\|\cdot - x_0\|_\infty^{m+1} \right)(x_0) < (0.0868)^N \left\{ \frac{1}{n^{\alpha(m+1)}} + \frac{2 \|b-a\|_\infty^{m+1}}{\pi^2 (n^{1-\alpha} - 2)} \right\} =: \varphi_1(n) \tag{69}$$

$(0 < \alpha < 1,\ m, n \in \mathbb{N} : n^{1-\alpha} > 2)$*.*

And, consequently it holds

$$\left\| \widetilde{A}_n \left(\|\cdot - x_0\|_\infty^{m+1} \right)(x_0) \right\|_{\infty, x_0 \in \prod_{i=1}^{N} [a_i, b_i]} <$$

$$(0.0868)^N \left\{ \frac{1}{n^{\alpha(m+1)}} + \frac{2 \|b-a\|_\infty^{m+1}}{\pi^2 (n^{1-\alpha} - 2)} \right\} = \varphi_1(n) \to 0, \quad \text{as } n \to +\infty. \tag{70}$$

So, we have that $\varphi_1(n) \to 0$*, as* $n \to +\infty$*. Thus, when* $p \in [1, \infty]$*, from Theorem 10 we have the convergence to zero in the right hand sides of parts (1), (2).*

Next we estimate $\left\| \left(\widetilde{A}_n \left(f^{(j)}(x_0) (\cdot - x_0)^j \right) \right)(x_0) \right\|_\gamma$.

We have that

$$\left(\widetilde{A}_n \left(f^{(j)}(x_0) (\cdot - x_0)^j \right) \right)(x_0) = \frac{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} f^{(j)}(x_0) \left(\frac{k}{n} - x_0 \right)^j Z(nx_0 - k)}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} Z(nx_0 - k)}. \tag{71}$$

When $p = \infty$*,* $j = 1, ..., m$*, we obtain*

$$\left\| f^{(j)}(x_0) \left(\frac{k}{n} - x_0 \right)^j \right\|_\gamma \leq \left\| f^{(j)}(x_0) \right\| \left\| \frac{k}{n} - x_0 \right\|_\infty^j. \tag{72}$$

We further have that

$$\left\| \left(\widetilde{A}_n \left(f^{(j)}(x_0) (\cdot - x_0)^j \right) \right)(x_0) \right\|_\gamma \overset{(24)}{<}$$

$$(0.0868)^N \left(\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \left\| f^{(j)}(x_0) \left(\frac{k}{n} - x_0 \right)^j \right\|_\gamma Z(nx_0 - k) \right) \leq$$

$$(0.0868)^N \left(\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \left\| f^{(j)}(x_0) \right\| \left\| \frac{k}{n} - x_0 \right\|_\infty^j Z(nx_0 - k) \right) = \tag{73}$$

$$(0.0868)^N \left\| f^{(j)}(x_0) \right\| \left(\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \left\| \frac{k}{n} - x_0 \right\|_\infty^j Z(nx_0 - k) \right) =$$

$$(0.0868)^N \left\| f^{(j)}(x_0) \right\| \left\{ \sum_{\begin{cases} k = \lceil na \rceil \\ : \left\| \frac{k}{n} - x_0 \right\|_\infty \leq \frac{1}{n^\alpha} \end{cases}}^{\lfloor nb \rfloor} \left\| \frac{k}{n} - x_0 \right\|_\infty^j Z(nx_0 - k) \right.$$

$$\left. + \sum_{\begin{cases} k = \lceil na \rceil \\ : \left\| \frac{k}{n} - x_0 \right\|_\infty > \frac{1}{n^\alpha} \end{cases}}^{\lfloor nb \rfloor} \left\| \frac{k}{n} - x_0 \right\|_\infty^j Z(nx_0 - k) \right\} \overset{(25)}{\leq} \tag{74}$$

$$(0.0868)^N \left\| f^{(j)}(x_0) \right\| \left\{ \frac{1}{n^{\alpha j}} + \frac{2 \|b-a\|_\infty^j}{\pi^2 (n^{1-\alpha} - 2)} \right\} \to 0, \text{ as } n \to \infty.$$

That is

$$\left\| \left(\widetilde{A}_n \left(f^{(j)}(x_0) (\cdot - x_0)^j \right) \right) (x_0) \right\|_\gamma \to 0, \text{ as } n \to \infty.$$

Therefore when $p = \infty$, for $j = 1, ..., m$, we have proved:

$$\left\| \left(\widetilde{A}_n \left(f^{(j)}(x_0) (\cdot - x_0)^j \right) \right) (x_0) \right\|_\gamma <$$

$$(0.0868)^N \left\| f^{(j)}(x_0) \right\| \left\{ \frac{1}{n^{\alpha j}} + \frac{2 \|b-a\|_\infty^j}{\pi^2 (n^{1-\alpha} - 2)} \right\} \leq \tag{75}$$

$$(0.0868)^N \left\| f^{(j)} \right\|_\infty \left\{ \frac{1}{n^{\alpha j}} + \frac{2 \|b-a\|_\infty^j}{\pi^2 (n^{1-\alpha} - 2)} \right\} =: \varphi_{2j}(n) < \infty,$$

and converges to zero, as $n \to \infty$.

We conclude:

In Theorem 10, the right hand sides of (62) and (63) converge to zero as $n \to \infty$, for any $p \in [1, \infty]$.

Also in Corollary 13, the right hand sides of (65) and (66) converge to zero as $n \to \infty$, for any $p \in [1, \infty]$.

Conclusion 15 *We have proved that the left hand sides of (60), (61), (62), (63) and (65), (66) converge to zero as $n \to \infty$, for $p \in [1, \infty]$. Consequently $A_n \to I$ (unit operator) pointwise and uniformly, as $n \to \infty$, where $p \in [1, \infty]$. In the presence of initial conditions we achieve a higher speed of convergence, see (61). Higher speed of convergence happens also to the left hand side of (60).*

We give

Corollary 16 *(to Theorem 10) Let O open subset of $\left(\mathbb{R}^N, \|\cdot\|_\infty\right)$, such that $\prod_{i=1}^{N}[a_i, b_i] \subset O \subseteq \mathbb{R}^N$, and let $\left(X, \|\cdot\|_\gamma\right)$ be a general Banach space. Let $m \in \mathbb{N}$ and $f \in C^m(O, X)$, the space of m-times continuously Fréchet differentiable functions from O into X. We study the approximation of $f|_{\prod_{i=1}^{N}[a_i,b_i]}$. Let $x_0 \in \left(\prod_{i=1}^{N}[a_i, b_i]\right)$ and $r > 0$. Here $\varphi_1(n)$ as in (69) and $\varphi_{2j}(n)$ as in (75), where $n \in \mathbb{N} : n^{1-\alpha} > 2$, $0 < \alpha < 1$, $j = 1, ..., m$. Then*

1)

$$\left\| (A_n(f))(x_0) - \sum_{j=0}^{m} \frac{1}{j!} \left(A_n \left(f^{(j)}(x_0)(\cdot - x_0)^j \right) \right)(x_0) \right\|_\gamma \leq$$

$$\frac{\omega_1 \left(f^{(m)}, r(\varphi_1(n))^{\frac{1}{m+1}} \right)}{rm!} (\varphi_1(n))^{\left(\frac{m}{m+1}\right)} \left[\frac{1}{(m+1)} + \frac{r}{2} + \frac{mr^2}{8} \right], \tag{76}$$

2) additionally, if $f^{(j)}(x_0) = 0$, $j = 1, ..., m$, we have

$$\|(A_n(f))(x_0) - f(x_0)\|_\gamma \leq$$

$$\frac{\omega_1 \left(f^{(m)}, r(\varphi_1(n))^{\frac{1}{m+1}} \right)}{rm!} (\varphi_1(n))^{\left(\frac{m}{m+1}\right)} \left[\frac{1}{(m+1)} + \frac{r}{2} + \frac{mr^2}{8} \right], \tag{77}$$

3)

$$\left\| \|A_n(f) - f\|_\gamma \right\|_{\infty, \prod_{i=1}^{N}[a_i,b_i]} \leq \sum_{j=1}^{m} \frac{\varphi_{2j}(n)}{j!} +$$

$$\frac{\omega_1 \left(f^{(m)}, r(\varphi_1(n))^{\frac{1}{m+1}} \right)}{rm!} (\varphi_1(n))^{\left(\frac{m}{m+1}\right)} \tag{78}$$

$$\left[\frac{1}{(m+1)} + \frac{r}{2} + \frac{mr^2}{8} \right] =: \varphi_3(n) \to 0, \text{ as } n \to \infty.$$

We continue with

Theorem 17 *Let $f \in C_B\left(\mathbb{R}^N, X\right)$, $0 < \beta < 1$, $x \in \mathbb{R}^N$, $N, n \in \mathbb{N}$ with $n^{1-\beta} > 2$, ω_1 is for $p = \infty$. Then*

1)

$$\|B_n(f, x) - f(x)\|_\gamma \leq \omega_1\left(f, \frac{1}{n^\beta}\right) + \frac{4 \left\| \|f\|_\gamma \right\|_\infty}{\pi^2 \left(n^{1-\beta} - 2\right)} =: \lambda_2(n), \tag{79}$$

2)

$$\left\| \|B_n(f) - f\|_\gamma \right\|_\infty \leq \lambda_2(n). \tag{80}$$

Given that $f \in \left(C_U\left(\mathbb{R}^N, X\right) \cap C_B\left(\mathbb{R}^N, X\right)\right)$, *we obtain* $\lim_{n\to\infty} B_n(f) = f$, *uniformly.*

Proof. We have that

$$B_n(f,x) - f(x) \overset{(18)}{=} \sum_{k=-\infty}^{\infty} f\left(\frac{k}{n}\right) Z(nx-k) - f(x) \sum_{k=-\infty}^{\infty} Z(nx-k) = \tag{81}$$

$$\sum_{k=-\infty}^{\infty} \left(f\left(\frac{k}{n}\right) - f(x) \right) Z(nx-k).$$

Hence

$$\|B_n(f,x) - f(x)\|_\gamma \leq \sum_{k=-\infty}^{\infty} \left\| f\left(\frac{k}{n}\right) - f(x) \right\|_\gamma Z(nx-k) =$$

$$\sum_{\substack{k=-\infty \\ \left\| \frac{k}{n} - x \right\|_\infty \leq \frac{1}{n^\beta}}}^{\infty} \left\| f\left(\frac{k}{n}\right) - f(x) \right\|_\gamma Z(nx-k) +$$

$$\sum_{\substack{k=-\infty \\ \left\| \frac{k}{n} - x \right\|_\infty > \frac{1}{n^\beta}}}^{\infty} \left\| f\left(\frac{k}{n}\right) - f(x) \right\|_\gamma Z(nx-k) \overset{(18)}{\leq}$$

$$\omega_1\left(f, \frac{1}{n^\beta}\right) + 2 \left\| \|f\|_\gamma \right\|_\infty \sum_{\substack{k=-\infty \\ \left\| \frac{k}{n} - x \right\|_\infty > \frac{1}{n^\beta}}}^{\infty} Z(nx-k) \overset{(25)}{\leq}$$

$$\omega_1\left(f, \frac{1}{n^\beta}\right) + \frac{4 \left\| \|f\|_\gamma \right\|_\infty}{\pi^2 \left(n^{1-\beta} - 2\right)}, \tag{82}$$

proving the claim. ■

We give

Theorem 18 *Let* $f \in C_B\left(\mathbb{R}^N, X\right)$, $0 < \beta < 1$, $x \in \mathbb{R}^N$, $N, n \in \mathbb{N}$ *with* $n^{1-\beta} > 2$, ω_1 *is for* $p = \infty$. *Then*

1)

$$\|C_n(f,x) - f(x)\|_\gamma \leq \omega_1\left(f, \frac{1}{n} + \frac{1}{n^\beta}\right) + \frac{4 \left\| \|f\|_\gamma \right\|_\infty}{\pi^2 \left(n^{1-\beta} - 2\right)} =: \lambda_3(n), \tag{83}$$

2)
$$\left\| \left\| C_n(f) - f \right\|_\gamma \right\|_\infty \leq \lambda_3(n). \tag{84}$$

Given that $f \in \left(C_U\left(\mathbb{R}^N, X\right) \cap C_B\left(\mathbb{R}^N, X\right)\right)$, *we obtain* $\lim_{n\to\infty} C_n(f) = f$, *uniformly.*

Proof. We notice that

$$\int_{\frac{k}{n}}^{\frac{k+1}{n}} f(t)\,dt = \int_{\frac{k_1}{n}}^{\frac{k_1+1}{n}} \int_{\frac{k_2}{n}}^{\frac{k_2+1}{n}} \dots \int_{\frac{k_N}{n}}^{\frac{k_N+1}{n}} f(t_1, t_2, \dots, t_N)\,dt_1 dt_2 \dots dt_N =$$

$$\int_0^{\frac{1}{n}} \int_0^{\frac{1}{n}} \dots \int_0^{\frac{1}{n}} f\left(t_1 + \frac{k_1}{n}, t_2 + \frac{k_2}{n}, \dots, t_N + \frac{k_N}{n}\right) dt_1 \dots dt_N = \int_0^{\frac{1}{n}} f\left(t + \frac{k}{n}\right) dt. \tag{85}$$

Thus it holds (by (40))

$$C_n(f, x) = \sum_{k=-\infty}^{\infty} \left(n^N \int_0^{\frac{1}{n}} f\left(t + \frac{k}{n}\right) dt \right) Z(nx - k). \tag{86}$$

We observe that

$$\|C_n(f,x) - f(x)\|_\gamma =$$

$$\left\| \sum_{k=-\infty}^{\infty} \left(n^N \int_0^{\frac{1}{n}} f\left(t + \frac{k}{n}\right) dt \right) Z(nx-k) - \sum_{k=-\infty}^{\infty} f(x) Z(nx-k) \right\|_\gamma =$$

$$\left\| \sum_{k=-\infty}^{\infty} \left(\left(n^N \int_0^{\frac{1}{n}} f\left(t + \frac{k}{n}\right) dt \right) - f(x) \right) Z(nx-k) \right\|_\gamma =$$

$$\left\| \sum_{k=-\infty}^{\infty} \left(n^N \int_0^{\frac{1}{n}} \left(f\left(t + \frac{k}{n}\right) - f(x) \right) dt \right) Z(nx-k) \right\|_\gamma \leq \tag{87}$$

$$\sum_{k=-\infty}^{\infty} \left(n^N \int_0^{\frac{1}{n}} \left\| f\left(t + \frac{k}{n}\right) - f(x) \right\|_\gamma dt \right) Z(nx-k) =$$

$$\sum_{\substack{k=-\infty \\ \left\|\frac{k}{n} - x\right\|_\infty \leq \frac{1}{n^\beta}}}^{\infty} \left(n^N \int_0^{\frac{1}{n}} \left\| f\left(t + \frac{k}{n}\right) - f(x) \right\|_\gamma dt \right) Z(nx-k) +$$

$$\sum_{\substack{k=-\infty \\ \left\|\frac{k}{n} - x\right\|_\infty > \frac{1}{n^\beta}}}^{\infty} \left(n^N \int_0^{\frac{1}{n}} \left\| f\left(t + \frac{k}{n}\right) - f(x) \right\|_\gamma dt \right) Z(nx-k) \leq$$

$$\sum_{\substack{k=-\infty \\ \left\| \frac{k}{n} - x \right\|_\infty \leq \frac{1}{n^\beta}}}^{\infty} \left(n^N \int_0^{\frac{1}{n}} \omega_1 \left(f, \|t\|_\infty + \left\| \frac{k}{n} - x \right\|_\infty \right) dt \right) Z(nx-k) +$$

$$2 \left\| \|f\|_\gamma \right\|_\infty \left(\sum_{\substack{k=-\infty \\ \left\| \frac{k}{n} - x \right\|_\infty > \frac{1}{n^\beta}}}^{\infty} Z(|nx-k|) \right) \leq$$

$$\omega_1 \left(f, \frac{1}{n} + \frac{1}{n^\beta} \right) + \frac{4 \left\| \|f\|_\gamma \right\|_\infty}{\pi^2 (n^{1-\beta} - 2)}, \tag{88}$$

proving the claim. ∎

We also present

Theorem 19 *Let $f \in C_B(\mathbb{R}^N, X)$, $0 < \beta < 1$, $x \in \mathbb{R}^N$, $N, n \in \mathbb{N}$ with $n^{1-\beta} > 2$, ω_1 is for $p = \infty$. Then*

1)

$$\|D_n(f,x) - f(x)\|_\gamma \leq \omega_1 \left(f, \frac{1}{n} + \frac{1}{n^\beta} \right) + \frac{4 \left\| \|f\|_\gamma \right\|_\infty}{\pi^2 (n^{1-\beta} - 2)} = \lambda_4(n), \tag{89}$$

2)

$$\left\| \|D_n(f) - f\|_\gamma \right\|_\infty \leq \lambda_4(n). \tag{90}$$

Given that $f \in \left(C_U(\mathbb{R}^N, X) \cap C_B(\mathbb{R}^N, X) \right)$, we obtain $\lim_{n \to \infty} D_n(f) = f$, uniformly.

Proof. We have that (by (42))

$$\|D_n(f,x) - f(x)\|_\gamma = \left\| \sum_{k=-\infty}^{\infty} \delta_{nk}(f) Z(nx-k) - \sum_{k=-\infty}^{\infty} f(x) Z(nx-k) \right\|_\gamma = \tag{91}$$

$$\left\| \sum_{k=-\infty}^{\infty} (\delta_{nk}(f) - f(x)) Z(nx-k) \right\|_\gamma =$$

$$\left\| \sum_{k=-\infty}^{\infty} \left(\sum_{r=0}^{\theta} w_r \left(f\left(\frac{k}{n} + \frac{r}{n\theta} \right) - f(x) \right) \right) Z(nx-k) \right\|_\gamma \leq$$

$$\sum_{k=-\infty}^{\infty} \left(\sum_{r=0}^{\theta} w_r \left\| f\left(\frac{k}{n} + \frac{r}{n\theta} \right) - f(x) \right\|_\gamma \right) Z(nx-k) =$$

$$\sum_{\substack{k=-\infty \\ \left\{ \left\| \frac{k}{n}-x \right\|_{\infty} \leq \frac{1}{n^{\beta}} \right.}}^{\infty} \left(\sum_{r=0}^{\theta} w_r \left\| f\left(\frac{k}{n}+\frac{r}{n\theta}\right) - f(x) \right\|_{\gamma} \right) Z(nx-k) +$$

$$\sum_{\substack{k=-\infty \\ \left\{ \left\| \frac{k}{n}-x \right\|_{\infty} > \frac{1}{n^{\beta}} \right.}}^{\infty} \left(\sum_{r=0}^{\theta} w_r \left\| f\left(\frac{k}{n}+\frac{r}{n\theta}\right) - f(x) \right\|_{\gamma} \right) Z(nx-k) \leq$$

$$\sum_{\substack{k=-\infty \\ \left\{ \left\| \frac{k}{n}-x \right\|_{\infty} \leq \frac{1}{n^{\beta}} \right.}}^{\infty} \left(\sum_{r=0}^{\theta} w_r \omega_1 \left(f, \left\| \frac{k}{n}-x \right\|_{\infty} + \left\| \frac{r}{n\theta} \right\|_{\infty} \right) \right) Z(nx-k) +$$

$$2 \left\| \|f\|_{\gamma} \right\|_{\infty} \left(\sum_{\substack{k=-\infty \\ \left\{ \left\| \frac{k}{n}-x \right\|_{\infty} > \frac{1}{n^{\beta}} \right.}}^{\infty} Z(nx-k) \right) \leq$$

$$\omega_1 \left(f, \frac{1}{n} + \frac{1}{n^{\beta}} \right) + \frac{4 \left\| \|f\|_{\gamma} \right\|_{\infty}}{\pi^2 \left(n^{1-\beta} - 2 \right)}, \tag{92}$$

proving the claim. ■

We make

Definition 20 *Let $f \in C_B\left(\mathbb{R}^N, X\right)$, $N \in \mathbb{N}$, where $\left(X, \|\cdot\|_{\gamma}\right)$ is a Banach space. We define the general neural network operator*

$$F_n(f,x) := \sum_{k=-\infty}^{\infty} l_{nk}(f) Z(nx-k) =$$

$$\begin{cases} B_n(f,x), & \text{if } l_{nk}(f) = f\left(\frac{k}{n}\right), \\ C_n(f,x), & \text{if } l_{nk}(f) = n^N \int_{\frac{k}{n}}^{\frac{k+1}{n}} f(t)\,dt, \\ D_n(f,x), & \text{if } l_{nk}(f) = \delta_{nk}(f). \end{cases} \tag{93}$$

Clearly $l_{nk}(f)$ is an X-valued bounded linear functional such that $\|l_{nk}(f)\|_{\gamma} \leq \left\| \|f\|_{\gamma} \right\|_{\infty}$.

Hence $F_n(f)$ is a bounded linear operator with $\left\| \|F_n(f)\|_{\gamma} \right\|_{\infty} \leq \left\| \|f\|_{\gamma} \right\|_{\infty}$.

We need

Theorem 21 *Let $f \in C_B\left(\mathbb{R}^N, X\right)$, $N \geq 1$. Then $F_n(f) \in C_B\left(\mathbb{R}^N, X\right)$.*

Proof. Clearly $F_n(f)$ is a bounded function.

Next we prove the continuity of $F_n(f)$. Notice for $N=1$, $Z=\psi$ by (16).

We will use the generalized Weierstrass M test: If a sequence of positive constants $M_1, M_2, M_3, \ldots$, can be found such that in some interval

(a) $\|u_n(x)\|_\gamma \leq M_n$, $n=1,2,3,\ldots$

(b) $\sum M_n$ converges,

then $\sum u_n(x)$ is uniformly and absolutely convergent in the interval.

Also we will use:

If $\{u_n(x)\}$, $n=1,2,3,\ldots$ are continuous in $[a,b]$ and if $\sum u_n(x)$ converges uniformly to the sum $S(x)$ in $[a,b]$, then $S(x)$ is continuous in $[a,b]$. I.e. a uniformly convergent series of continuous functions is a continuous function. First we prove claim for $N=1$.

We will prove that $\sum_{k=-\infty}^{\infty} l_{nk}(f)\,\psi(nx-k)$ is continuous in $x \in \mathbb{R}$.

There always exists $\lambda \in \mathbb{N}$ such that $nx \in [-\lambda, \lambda]$.

Since $nx \leq \lambda$, then $-nx \geq -\lambda$ and $k-nx \geq k-\lambda \geq 0$, when $k \geq \lambda$. Therefore

$$\sum_{k=\lambda}^{\infty} \psi(nx-k) = \sum_{k=\lambda}^{\infty} \psi(k-nx) \leq \sum_{k=\lambda}^{\infty} \psi(k-\lambda) = \sum_{k'=0}^{\infty} \psi(k') \leq 1. \tag{94}$$

So for $k \geq \lambda$ we get

$$\|l_{nk}(f)\|_\gamma\,\psi(nx-k) \leq \left\|\|f\|_\gamma\right\|_\infty \psi(k-\lambda),$$

and

$$\left\|\|f\|_\gamma\right\|_\infty \sum_{k=\lambda}^{\infty} \psi(k-\lambda) \leq \left\|\|f\|_\gamma\right\|_\infty.$$

Hence by the generalized Weierstrass M test we obtain that $\sum_{k=\lambda}^{\infty} l_{nk}(f)\,\psi(nx-k)$ is uniformly and absolutely convergent on $\left[-\frac{\lambda}{n}, \frac{\lambda}{n}\right]$.

Since $l_{nk}(f)\,\psi(nx-k)$ is continuous in x, then $\sum_{k=\lambda}^{\infty} l_{nk}(f)\,\psi(nx-k)$ is continuous on $\left[-\frac{\lambda}{n}, \frac{\lambda}{n}\right]$.

Because $nx \geq -\lambda$, then $-nx \leq \lambda$, and $k-nx \leq k+\lambda \leq 0$, when $k \leq -\lambda$. Therefore

$$\sum_{k=-\infty}^{-\lambda} \psi(nx-k) = \sum_{k=-\infty}^{-\lambda} \psi(k-nx) \leq \sum_{k=-\infty}^{-\lambda} \psi(k+\lambda) = \sum_{k'=-\infty}^{0} \psi(k') \leq 1.$$

So for $k \leq -\lambda$ we get

$$\|l_{nk}(f)\|_\gamma\,\psi(nx-k) \leq \left\|\|f\|_\gamma\right\|_\infty \psi(k+\lambda), \tag{95}$$

and

$$\left\|\|f\|_\gamma\right\|_\infty \sum_{k=-\infty}^{-\lambda} \psi(k+\lambda) \leq \left\|\|f\|_\gamma\right\|_\infty.$$

Hence by Weierstrass M test we obtain that $\sum_{k=-\infty}^{-\lambda} l_{nk}(f)\psi(nx-k)$ is uniformly and absolutely convergent on $\left[-\frac{\lambda}{n}, \frac{\lambda}{n}\right]$.

Since $l_{nk}(f)\psi(nx-k)$ is continuous in x, then $\sum_{k=-\infty}^{-\lambda} l_{nk}(f)\psi(nx-k)$ is continuous on $\left[-\frac{\lambda}{n}, \frac{\lambda}{n}\right]$.

So we proved that $\sum_{k=\lambda}^{\infty} l_{nk}(f)\psi(nx-k)$ and $\sum_{k=-\infty}^{-\lambda} l_{nk}(f)\psi(nx-k)$ are continuous on $\mathbb{R}$. Since $\sum_{k=-\lambda+1}^{\lambda-1} l_{nk}(f)\psi(nx-k)$ is a finite sum of continuous functions on $\mathbb{R}$, it is also a continuous function on $\mathbb{R}$.

Writing

$$\sum_{k=-\infty}^{\infty} l_{nk}(f)\psi(nx-k) = \sum_{k=-\infty}^{-\lambda} l_{nk}(f)\psi(nx-k) +$$

$$\sum_{k=-\lambda+1}^{\lambda-1} l_{nk}(f)\psi(nx-k) + \sum_{k=\lambda}^{\infty} l_{nk}(f)\psi(nx-k) \tag{96}$$

we have it as a continuous function on $\mathbb{R}$. Therefore $F_n(f)$, when $N=1$, is a continuous function on $\mathbb{R}$.

When $N=2$ we have

$$F_n(f, x_1, x_2) = \sum_{k_1=-\infty}^{\infty} \sum_{k_2=-\infty}^{\infty} l_{nk}(f)\psi(nx_1-k_1)\psi(nx_2-k_2) =$$

$$\sum_{k_1=-\infty}^{\infty} \psi(nx_1-k_1)\left(\sum_{k_2=-\infty}^{\infty} l_{nk}(f)\psi(nx_2-k_2)\right)$$

(there always exist $\lambda_1, \lambda_2 \in \mathbb{N}$ such that $nx_1 \in [-\lambda_1, \lambda_1]$ and $nx_2 \in [-\lambda_2, \lambda_2]$)

$$= \sum_{k_1=-\infty}^{\infty} \psi(nx_1-k_1)\left[\sum_{k_2=-\infty}^{-\lambda_2} l_{nk}(f)\psi(nx_2-k_2) +\right.$$

$$\left.\sum_{k_2=-\lambda_2+1}^{\lambda_2-1} l_{nk}(f)\psi(nx_2-k_2) + \sum_{k_2=\lambda_2}^{\infty} l_{nk}(f)\psi(nx_2-k_2)\right] =$$

$$= \sum_{k_1=-\infty}^{\infty} \sum_{k_2=-\infty}^{-\lambda_2} l_{nk}(f)\psi(nx_1-k_1)\psi(nx_2-k_2) +$$

$$\sum_{k_1=-\infty}^{\infty} \sum_{k_2=-\lambda_2+1}^{\lambda_2-1} l_{nk}(f)\psi(nx_1-k_1)\psi(nx_2-k_2) +$$

$$\sum_{k_1=-\infty}^{\infty} \sum_{k_2=\lambda_2}^{\infty} l_{nk}(f)\psi(nx_1-k_1)\psi(nx_2-k_2) =: (*).$$

(For convenience call

$$F(k_1,k_2,x_1,x_2) := l_{nk}(f)\,\psi(nx_1-k_1)\,\psi(nx_2-k_2)\,.\,)$$

Thus

$$(*) = \sum_{k_1=-\infty}^{-\lambda_1}\sum_{k_2=-\infty}^{-\lambda_2} F(k_1,k_2,x_1,x_2) + \sum_{k_1=-\lambda_1+1}^{\lambda_1-1}\sum_{k_2=-\infty}^{-\lambda_2} F(k_1,k_2,x_1,x_2) +$$
$$\sum_{k_1=\lambda_1}^{\infty}\sum_{k_2=-\infty}^{-\lambda_2} F(k_1,k_2,x_1,x_2) + \sum_{k_1=-\infty}^{-\lambda_1}\sum_{k_2=-\lambda_2+1}^{\lambda_2-1} F(k_1,k_2,x_1,x_2) +$$
$$\sum_{k_1=-\lambda_1+1}^{\lambda_1-1}\sum_{k_2=-\lambda_2+1}^{\lambda_2-1} F(k_1,k_2,x_1,x_2) + \sum_{k_1=\lambda_1}^{\infty}\sum_{k_2=-\lambda_2+1}^{\lambda_2-1} F(k_1,k_2,x_1,x_2) +$$
$$\sum_{k_1=-\infty}^{-\lambda_1}\sum_{k_2=\lambda_2}^{\infty} F(k_1,k_2,x_1,x_2) + \sum_{k_1=-\lambda_1+1}^{\lambda_1-1}\sum_{k_2=\lambda_2}^{\infty} F(k_1,k_2,x_1,x_2) + \tag{97}$$
$$\sum_{k_1=\lambda_1}^{\infty}\sum_{k_2=\lambda_2}^{\infty} F(k_1,k_2,x_1,x_2)\,.$$

Notice that the finite sum of continuous functions $F(k_1,k_2,x_1,x_2)$, $\sum_{k_1=-\lambda_1+1}^{\lambda_1-1}\sum_{k_2=-\lambda_2+1}^{\lambda_2-1} F(k_1,k_2,x_1,x_2)$ is a continuous function.

The rest of the summands of $F_n(f,x_1,x_2)$ are treated all the same way and similarly to the case of $N=1$. The method is demonstrated as follows.

We will prove that $\sum_{k_1=\lambda_1}^{\infty}\sum_{k_2=-\infty}^{-\lambda_2} l_{nk}(f)\,\psi(nx_1-k_1)\,\psi(nx_2-k_2)$ is continuous in $(x_1,x_2)\in\mathbb{R}^2$.

The continuous function

$$\|l_{nk}(f)\|_\gamma\,\psi(nx_1-k_1)\,\psi(nx_2-k_2) \le \left\|\|f\|_\gamma\right\|_\infty \psi(k_1-\lambda_1)\,\psi(k_2+\lambda_2)\,,$$

and

$$\left\|\|f\|_\gamma\right\|_\infty \sum_{k_1=\lambda_1}^{\infty}\sum_{k_2=-\infty}^{-\lambda_2}\psi(k_1-\lambda_1)\,\psi(k_2+\lambda_2) =$$
$$\left\|\|f\|_\gamma\right\|_\infty \left(\sum_{k_1=\lambda_1}^{\infty}\psi(k_1-\lambda_1)\right)\left(\sum_{k_2=-\infty}^{-\lambda_2}\psi(k_2+\lambda_2)\right) \le$$
$$\left\|\|f\|_\gamma\right\|_\infty \left(\sum_{k_1'=0}^{\infty}\psi(k_1')\right)\left(\sum_{k_2'=-\infty}^{0}\psi(k_2')\right) \le \left\|\|f\|_\gamma\right\|_\infty .$$

So by the Weierstrass M test we get that $\sum_{k_1=\lambda_1}^{\infty}\sum_{k_2=-\infty}^{-\lambda_2} l_{nk}(f)\,\psi(nx_1-k_1)\,\psi(nx_2-k_2)$ is uniformly and absolutely convergent. Therefore it is continuous on $\mathbb{R}^2$.

Next we prove continuity on $\mathbb{R}^2$ of
$\sum_{k_1=-\lambda_1+1}^{\lambda_1-1}\sum_{k_2=-\infty}^{-\lambda_2} l_{nk}(f)\,\psi(nx_1-k_1)\,\psi(nx_2-k_2)$.

Notice here that

$$\|l_{nk}(f)\|_\gamma\,\psi(nx_1-k_1)\,\psi(nx_2-k_2)\le\left\|\|f\|_\gamma\right\|_\infty\psi(nx_1-k_1)\,\psi(k_2+\lambda_2)$$

$$\le\left\|\|f\|_\gamma\right\|_\infty\psi(0)\,\psi(k_2+\lambda_2)=18.31\cdot\left\|\|f\|_\gamma\right\|_\infty\psi(k_2+\lambda_2),$$

and

$$18.31\cdot\left\|\|f\|_\gamma\right\|_\infty\left(\sum_{k_1=-\lambda_1+1}^{\lambda_1-1}1\right)\left(\sum_{k_2=-\infty}^{-\lambda_2}\psi(k_2+\lambda_2)\right)=$$

$$18.31\cdot\left\|\|f\|_\gamma\right\|_\infty(2\lambda_1-1)\left(\sum_{k_2'=-\infty}^{0}\psi(k_2')\right)\le 18.31\cdot(2\lambda_1-1)\left\|\|f\|_\gamma\right\|_\infty. \quad (98)$$

So the double series under consideration is uniformly convergent and continuous. Clearly $F_n(f,x_1,x_2)$ is proved to be continuous on $\mathbb{R}^2$.

Similarly reasoning one can prove easily now, but with more tedious work, that $F_n(f,x_1,...,x_N)$ is continuous on $\mathbb{R}^N$, for any $N\ge 1$. We choose to omit this similar extra work. ■

Remark 22 *By (27) it is obvious that* $\left\|\|A_n(f)\|_\gamma\right\|_\infty\le\left\|\|f\|_\gamma\right\|_\infty<\infty$, *and* $A_n(f)\in C\left(\prod_{i=1}^N[a_i,b_i],X\right)$, *given that* $f\in C\left(\prod_{i=1}^N[a_i,b_i],X\right)$.

Call L_n any of the operators A_n,B_n,C_n,D_n.

Clearly then

$$\left\|\|L_n^2(f)\|_\gamma\right\|_\infty=\left\|\|L_n(L_n(f))\|_\gamma\right\|_\infty\le\left\|\|L_n(f)\|_\gamma\right\|_\infty\le\left\|\|f\|_\gamma\right\|_\infty, \quad (99)$$

etc.

Therefore we get

$$\left\|\|L_n^k(f)\|_\gamma\right\|_\infty\le\left\|\|f\|_\gamma\right\|_\infty,\ \ \forall\, k\in\mathbb{N}, \quad (100)$$

the contraction property.

Also we see that

$$\left\|\|L_n^k(f)\|_\gamma\right\|_\infty\le\left\|\|L_n^{k-1}(f)\|_\gamma\right\|_\infty\le...\le\left\|\|L_n(f)\|_\gamma\right\|_\infty\le\left\|\|f\|_\gamma\right\|_\infty. \quad (101)$$

Here L_n^k are bounded linear operators.

Notation 23 *Here $N\in\mathbb{N}$, $0<\beta<1$. Denote by*

$$c_N:=\begin{cases}(0.0868)^N, & \text{if } L_n=A_n,\\ 1, & \text{if } L_n=B_n,C_n,D_n,\end{cases} \quad (102)$$

$$\varphi(n) := \begin{cases} \frac{1}{n^{\beta}}, & \text{if } L_n = A_n,\ B_n, \\ \frac{1}{n} + \frac{1}{n^{\beta}}, & \text{if } L_n = C_n, D_n, \end{cases} \tag{103}$$

$$\Omega := \begin{cases} C\left(\prod_{i=1}^{N} [a_i, b_i], X\right), & \text{if } L_n = A_n, \\ C_B\left(\mathbb{R}^N, X\right), & \text{if } L_n = B_n, C_n, D_n, \end{cases} \tag{104}$$

and

$$Y := \begin{cases} \prod_{i=1}^{N} [a_i, b_i], & \text{if } L_n = A_n, \\ \mathbb{R}^N, & \text{if } L_n = B_n, C_n, D_n. \end{cases} \tag{105}$$

We give the condensed

Theorem 24 *Let $f \in \Omega$, $0 < \beta < 1$, $x \in Y$; n, $N \in \mathbb{N}$ with $n^{1-\beta} > 2$. Then*
(i)

$$\|L_n(f,x) - f(x)\|_{\gamma} \leq c_N \left[\omega_1(f, \varphi(n)) + \frac{4\left\|\|f\|_{\gamma}\right\|_{\infty}}{\pi^2\left(n^{1-\beta} - 2\right)}\right] =: \tau(n), \tag{106}$$

where ω_1 is for $p = \infty$,
and
(ii)

$$\left\|\|L_n(f) - f\|_{\gamma}\right\|_{\infty} \leq \tau(n) \to 0, \text{ as } n \to \infty. \tag{107}$$

For f uniformly continuous and in Ω we obtain

$$\lim_{n\to\infty} L_n(f) = f,$$

pointwise and uniformly.

Proof. By Theorems 8, 17, 18, 19. ■
Next we do iterated neural network approximation (see also [9]).
We make

Remark 25 *Let $r \in \mathbb{N}$ and L_n as above. We observe that*

$$L_n^r f - f = \left(L_n^r f - L_n^{r-1} f\right) + \left(L_n^{r-1} f - L_n^{r-2} f\right) +$$

$$\left(L_n^{r-2} f - L_n^{r-3} f\right) + \ldots + \left(L_n^2 f - L_n f\right) + \left(L_n f - f\right).$$

Then

$$\left\|\|L_n^r f - f\|_{\gamma}\right\|_{\infty} \leq \left\|\|L_n^r f - L_n^{r-1} f\|_{\gamma}\right\|_{\infty} + \left\|\|L_n^{r-1} f - L_n^{r-2} f\|_{\gamma}\right\|_{\infty} +$$

$$\left\|\|L_n^{r-2} f - L_n^{r-3} f\|_{\gamma}\right\|_{\infty} + \ldots + \left\|\|L_n^2 f - L_n f\|_{\gamma}\right\|_{\infty} + \left\|\|L_n f - f\|_{\gamma}\right\|_{\infty} =$$

$$\left\|\left\|L_n^{r-1}\left(L_nf-f\right)\right\|_\gamma\right\|_\infty+\left\|\left\|L_n^{r-2}\left(L_nf-f\right)\right\|_\gamma\right\|_\infty+\left\|\left\|L_n^{r-3}\left(L_nf-f\right)\right\|_\gamma\right\|_\infty$$
$$+...+\left\|\left\|L_n\left(L_nf-f\right)\right\|_\gamma\right\|_\infty+\left\|\left\|L_nf-f\right\|_\gamma\right\|_\infty\leq r\left\|\left\|L_nf-f\right\|_\gamma\right\|_\infty. \quad (108)$$

That is

$$\left\|\left\|L_n^rf-f\right\|_\gamma\right\|_\infty\leq r\left\|\left\|L_nf-f\right\|_\gamma\right\|_\infty. \quad (109)$$

We give

Theorem 26 *All here as in Theorem 24 and $r\in\mathbb{N}$, $\tau(n)$ as in (106). Then*

$$\left\|\left\|L_n^rf-f\right\|_\gamma\right\|_\infty\leq r\tau(n). \quad (110)$$

So that the speed of convergence to the unit operator of L_n^r is not worse than of L_n.

Proof. By (109) and (107). ■

We make

Remark 27 *Let $m_1,...,m_r\in\mathbb{N}:m_1\leq m_2\leq...\leq m_r$, $0<\beta<1$, $f\in\Omega$. Then $\varphi(m_1)\geq\varphi(m_2)\geq...\geq\varphi(m_r)$, φ as in (103).*

Therefore

$$\omega_1(f,\varphi(m_1))\geq\omega_1(f,\varphi(m_2))\geq...\geq\omega_1(f,\varphi(m_r)). \quad (111)$$

Assume further that $m_i^{1-\beta}>2$, $i=1,...,r$. Then

$$\frac{2}{\pi^2\left(m_1^{1-\beta}-2\right)}\geq\frac{2}{\pi^2\left(m_2^{1-\beta}-2\right)}\geq...\geq\frac{2}{\pi^2\left(m_r^{1-\beta}-2\right)}. \quad (112)$$

Let L_{m_i} as above, $i=1,...,r$, all of the same kind.

We write

$$\begin{gathered}L_{m_r}\left(L_{m_{r-1}}\left(...L_{m_2}\left(L_{m_1}f\right)\right)\right)-f=\\ L_{m_r}\left(L_{m_{r-1}}\left(...L_{m_2}\left(L_{m_1}f\right)\right)\right)-L_{m_r}\left(L_{m_{r-1}}\left(...L_{m_2}f\right)\right)+\\ L_{m_r}\left(L_{m_{r-1}}\left(...L_{m_2}f\right)\right)-L_{m_r}\left(L_{m_{r-1}}\left(...L_{m_3}f\right)\right)+\\ L_{m_r}\left(L_{m_{r-1}}\left(...L_{m_3}f\right)\right)-L_{m_r}\left(L_{m_{r-1}}\left(...L_{m_4}f\right)\right)+...+\\ L_{m_r}\left(L_{m_{r-1}}f\right)-L_{m_r}f+L_{m_r}f-f=\\ L_{m_r}\left(L_{m_{r-1}}\left(...L_{m_2}\right)\right)\left(L_{m_1}f-f\right)+L_{m_r}\left(L_{m_{r-1}}\left(...L_{m_3}\right)\right)\left(L_{m_2}f-f\right)+\\ L_{m_r}\left(L_{m_{r-1}}\left(...L_{m_4}\right)\right)\left(L_{m_3}f-f\right)+...+L_{m_r}\left(L_{m_{r-1}}f-f\right)+L_{m_r}f-f.\end{gathered} \quad (113)$$

Hence by the triangle inequality property of $\left\|\left\|\cdot\right\|_\gamma\right\|_\infty$ we get

$$\left\|\left\|L_{m_r}\left(L_{m_{r-1}}\left(...L_{m_2}\left(L_{m_1}f\right)\right)\right)-f\right\|_\gamma\right\|_\infty\leq$$

$$\left\|\left\|L_{m_r}\left(L_{m_{r-1}}\left(...L_{m_2}\right)\right)\left(L_{m_1}f-f\right)\right\|_\gamma\right\|_\infty+$$

$$\left\|\left\|L_{m_r}\left(L_{m_{r-1}}\left(...L_{m_3}\right)\right)\left(L_{m_2}f-f\right)\right\|_\gamma\right\|_\infty+$$

$$\left\|\left\|L_{m_r}\left(L_{m_{r-1}}\left(...L_{m_4}\right)\right)\left(L_{m_3}f-f\right)\right\|_\gamma\right\|_\infty+...+$$

$$\left\|\left\|L_{m_r}\left(L_{m_{r-1}}f-f\right)\right\|_\gamma\right\|_\infty+\left\|\left\|L_{m_r}f-f\right\|_\gamma\right\|_\infty$$

(repeatedly applying (99))

$$\leq\left\|\left\|L_{m_1}f-f\right\|_\gamma\right\|_\infty+\left\|\left\|L_{m_2}f-f\right\|_\gamma\right\|_\infty+\left\|\left\|L_{m_3}f-f\right\|_\gamma\right\|_\infty+...+$$

$$\left\|\left\|L_{m_{r-1}}f-f\right\|_\gamma\right\|_\infty+\left\|\left\|L_{m_r}f-f\right\|_\gamma\right\|_\infty=\sum_{i=1}^r\left\|\left\|L_{m_i}f-f\right\|_\gamma\right\|_\infty. \tag{114}$$

That is, we proved

$$\left\|\left\|L_{m_r}\left(L_{m_{r-1}}\left(...L_{m_2}\left(L_{m_1}f\right)\right)\right)-f\right\|_\gamma\right\|_\infty\leq\sum_{i=1}^r\left\|\left\|L_{m_i}f-f\right\|_\gamma\right\|_\infty. \tag{115}$$

We give

Theorem 28 *Let $f\in\Omega$; N, $m_1,m_2,...,m_r\in\mathbb{N}:m_1\leq m_2\leq...\leq m_r$, $0<\beta<1$; $m_i^{1-\beta}>2$, $i=1,...,r$, $x\in Y$, and let $\left(L_{m_1},...,L_{m_r}\right)$ as $\left(A_{m_1},...,A_{m_r}\right)$ or $\left(B_{m_1},...,B_{m_r}\right)$ or $\left(C_{m_1},...,C_{m_r}\right)$ or $\left(D_{m_1},...,D_{m_r}\right)$, $p=\infty$. Then*

$$\left\|L_{m_r}\left(L_{m_{r-1}}\left(...L_{m_2}\left(L_{m_1}f\right)\right)\right)(x)-f(x)\right\|_\gamma\leq$$

$$\left\|\left\|L_{m_r}\left(L_{m_{r-1}}\left(...L_{m_2}\left(L_{m_1}f\right)\right)\right)-f\right\|_\gamma\right\|_\infty\leq$$

$$\sum_{i=1}^r\left\|\left\|L_{m_i}f-f\right\|_\gamma\right\|_\infty\leq$$

$$c_N\sum_{i=1}^r\left[\omega_1\left(f,\varphi\left(m_i\right)\right)+\frac{4\left\|\left\|f\right\|_\gamma\right\|_\infty}{\pi^2\left(m_i^{1-\beta}-2\right)}\right]\leq$$

$$rc_N\left[\omega_1\left(f,\varphi\left(m_1\right)\right)+\frac{4\left\|\left\|f\right\|_\gamma\right\|_\infty}{\pi^2\left(m_1^{1-\beta}-2\right)}\right]. \tag{116}$$

Clearly, we notice that the speed of convergence to the unit operator of the multiply iterated operator is not worse than the speed of L_{m_1}.

Proof. Using (115), (111), (112) and (106), (107). ■

We continue with

Theorem 29 *Let all as in Corollary 16, and $r \in \mathbb{N}$. Here $\varphi_3(n)$ is as in (78). Then*

$$\left\| \left\| A_n^r f - f \right\|_\gamma \right\|_\infty \leq r \left\| \left\| A_n f - f \right\|_\gamma \right\|_\infty \leq r \varphi_3(n). \tag{117}$$

Proof. By (109) and (78). ■

Application 30 *A typical application of all of our results is when $\left(X, \|\cdot\|_\gamma\right) = (\mathbb{C}, |\cdot|)$, where $\mathbb{C}$ are the complex numbers.*

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