SOME BASIC RESULTS FOR THE NORMALIZED ENTROPIC DETERMINANT OF POSITIVE OPERATORS IN HILBERT SPACES

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Abstract. For positive invertible operators A on a Hilbert space H and a fixed unit vector $x \in H$, define the normalized entropic determinant by $\eta_x(A) := \exp[-\langle A \ln Ax, x\rangle]$. In this paper we show among others that, if A, $B > 0$, then for all $x \in H$ with $\|x\| = 1$ and $t \in [0,1]$,

$$\eta_x((1-t)A + tB) \geq (\eta_x(A))^{1-t} (\eta_x(B))^t .$$

Also we have the bounds

$$\left(\frac{\langle A^2 x, x\rangle}{\langle Ax, x\rangle}\right)^{-\langle Ax,x\rangle} \leq \eta_x(A) \leq \langle Ax, x\rangle^{-\langle Ax,x\rangle},$$

where $A > 0$ and $x \in H$ with$\|x\| = 1$.

1. Introduction

Let $B(H)$ be the space of all bounded linear operators on a Hilbert space H, and I stands for the identity operator on H. An operator A in $B(H)$ is said to be positive (in symbol: $A \geq 0$) if $\langle Ax, x\rangle \geq 0$ for all $x \in H$. In particular, $A > 0$ means that A is positive and invertible. For a pair A, B of selfadjoint operators the order relation $A \geq B$ means as usual that $A - B$ is positive.

In 1998, Fujii et al. [4], [5], introduced the *normalized determinant* $\Delta_x(A)$ for positive invertible operators A on a Hilbert space H and a fixed unit vector $x \in H$, namely $\|x\| = 1$, defined by $\Delta_x(A) := \exp\langle \ln Ax, x\rangle$ and discussed it as a continuous geometric mean and observed some inequalities around the determinant from this point of view.

Some of the fundamental properties of normalized determinant are as follows, [4].

For each unit vector $x \in H$, see also [7], we have:

(i) *continuity*: the map $A \to \Delta_x(A)$ is norm continuous;
(ii) *bounds*: $\langle A^{-1}x, x\rangle^{-1} \leq \Delta_x(A) \leq \langle Ax, x\rangle$;
(iii) *continuous mean*: $\langle A^p x, x\rangle^{1/p} \downarrow \Delta_x(A)$ for $p \downarrow 0$ and $\langle A^p x, x\rangle^{1/p} \uparrow \Delta_x(A)$ for $p \uparrow 0$;
(iv) *power equality:* $\Delta_x(A^t) = \Delta_x(A)^t$ for all $t > 0$;
(v) *homogeneity*: $\Delta_x(tA) = t\Delta_x(A)$ and $\Delta_x(tI) = t$ for all $t > 0$;
(vi) *monotonicity*: $0 < A \leq B$ implies $\Delta_x(A) \leq \Delta_x(B)$;
(vii) *multiplicativity*: $\Delta_x(AB) = \Delta_x(A)\Delta_x(B)$ for commuting A and B;

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(viii) *Ky Fan type inequality*: $\Delta_x((1-\alpha)A+\alpha B) \geq \Delta_x(A)^{1-\alpha}\Delta_x(B)^{\alpha}$ for $0 < \alpha < 1$.

We define the logarithmic mean of two positive numbers a, b by

$$L(a,b) := \begin{cases} \frac{b-a}{\ln b - \ln a} \text{ if } b \neq a, \\ \\ a \text{ if } b = a. \end{cases} \tag{1.1}$$

In [4] the authors obtained the following additive reverse inequality for the operator A which satisfy the condition $0 < mI \leq A \leq MI$, where m, M are positive numbers,

$$0 \leq \langle Ax, x\rangle - \Delta_x(A) \leq L(m,M)\left[\ln L(m,M) + \frac{M\ln m - m\ln M}{M-m} - 1\right] \tag{1.2}$$

for all $x \in H$, $\|x\| = 1$.

We recall that *Specht's ratio* is defined by [11]

$$S(h) := \begin{cases} \frac{h^{\frac{1}{h-1}}}{e\ln\left(h^{\frac{1}{h-1}}\right)} \text{ if } h \in (0,1)\cup(1,\infty), \\ \\ 1 \text{ if } h = 1. \end{cases} \tag{1.3}$$

It is well known that $\lim_{h\to 1} S(h) = 1$, $S(h) = S\left(\frac{1}{h}\right) > 1$ for $h > 0$, $h \neq 1$. The function is decreasing on $(0,1)$ and increasing on $(1,\infty)$.

In [5], the authors obtained the following multiplicative reverse inequality as well

$$1 \leq \frac{\langle Ax, x\rangle}{\Delta_x(A)} \leq S\left(\frac{M}{m}\right) \tag{1.4}$$

for $0 < mI \leq A \leq MI$ and $x \in H$, $\|x\| = 1$.

For the entropy function $\eta(t) = -t\ln t$, $t > 0$, the operator entropy has the following expression:

$$\eta(A) = -A\ln A$$

for positive A.

For $x \in H$, $\|x\| = 1$, we define the *normalized entropic determinant* $\eta_x(A)$ by

$$\eta_x(A) := \exp(-\langle A\ln Ax, x\rangle) = \exp\langle \eta(A)x, x\rangle. \tag{1.5}$$

Let $x \in H$, $\|x\| = 1$. Observe that the map $A \to \eta_x(A)$ is *norm continuous* and since

$$\begin{aligned} &\exp(-\langle tA\ln(tA)x, x\rangle) \\ &= \exp(-\langle tA(\ln t + \ln A)x, x\rangle) = \exp(-\langle (tA\ln t + tA\ln A)x, x\rangle) \\ &= \exp(-\langle Ax, x\rangle t\ln t)\exp(-t\langle A\ln Ax, x\rangle) \\ &= \exp\ln\left(t^{-\langle Ax,x\rangle t}\right)[\exp(-\langle A\ln Ax, x\rangle)]^{-t}, \end{aligned}$$

hence

$$\eta_x(tA) = t^{-t\langle Ax,x\rangle}[\eta_x(A)]^{-t} \tag{1.6}$$

for $t > 0$ and $A > 0$.

Observe also that

$$\eta_x(I) = 1 \text{ and } \eta_x(tI) = t^{-t} \tag{1.7}$$

for $t > 0$.

Motivated by the above results, in this paper we show among others that, if A, $B>0$, then for all $x\in H$, $\|x\|=1$ and $t\in[0,1]$,

$$\eta_x((1-t)A+tB)\geq(\eta_x(A))^{1-t}(\eta_x(B))^t.$$

Also we have the bounds

$$\left(\frac{\langle A^2x,x\rangle}{\langle Ax,x\rangle}\right)^{-\langle Ax,x\rangle}\leq\eta_x(A)\leq\langle Ax,x\rangle^{-\langle Ax,x\rangle},$$

where $A>0$ and $x\in H$, $\|x\|=1$.

2. Main Results

We have the following upper and lower bounds for *normalized entropic determinant*:

Proposition 1. *If $A>0$, then for all $x\in H$, $\|x\|=1$,*

$$\left(\frac{\langle A^2x,x\rangle}{\langle Ax,x\rangle}\right)^{-\langle Ax,x\rangle}\leq\eta_x(A)\leq\langle Ax,x\rangle^{-\langle Ax,x\rangle}. \tag{2.1}$$

Proof. The entropy function $\eta(t)=-t\ln t$, $t>0$ is operator concave. By utilizing Jensen's inequality for concave function g on $(0,\infty)$, we have

$$\langle g(B)x,x\rangle\leq g(\langle Bx,x\rangle),\ x\in H,\|x\|=1,$$

which gives that

$$\begin{aligned}\eta_x(A)&=\exp\langle\eta(A)x,x\rangle\leq\exp[\eta(\langle Ax,x\rangle)]=\exp\ln\langle Ax,x\rangle^{-\langle Ax,x\rangle}\\&=\langle Ax,x\rangle^{-\langle Ax,x\rangle}.\end{aligned}$$

Also for $x\in H$, $\|x\|=1$

$$\begin{aligned}\eta_x(A)&:=\exp(-\langle A\ln Ax,x\rangle)=\exp\left(-\left\langle(\ln A)A^{1/2}x,A^{1/2}x\right\rangle\right)\\&=\exp\left(\left\|A^{1/2}x\right\|^2\left\langle-(\ln A)\frac{A^{1/2}x}{\|A^{1/2}x\|},\frac{A^{1/2}x}{\|A^{1/2}x\|}\right\rangle\right)\end{aligned}$$

Since the function $-\ln t$ is convex on $(0,\infty)$, then by Jensen's inequality for the convex function $h=-\ln$,

$$\langle h(B)y,y\rangle\geq h(\langle By,y\rangle),\ y\in H,\|y\|=1,$$

by taking $y=\frac{A^{1/2}x}{\|A^{1/2}x\|}$, $x\in H$, $\|x\|=1$, we derive

$$\left\langle-(\ln A)\frac{A^{1/2}x}{\|A^{1/2}x\|},\frac{A^{1/2}x}{\|A^{1/2}x\|}\right\rangle\geq-\ln\left\langle A\frac{A^{1/2}x}{\|A^{1/2}x\|},\frac{A^{1/2}x}{\|A^{1/2}x\|}\right\rangle,$$

which gives that

$$\begin{aligned}&\left\|A^{1/2}x\right\|^2\left\langle-(\ln A)\frac{A^{1/2}x}{\|A^{1/2}x\|},\frac{A^{1/2}x}{\|A^{1/2}x\|}\right\rangle\\&\geq\ln\left\langle A\frac{A^{1/2}x}{\|A^{1/2}x\|},\frac{A^{1/2}x}{\|A^{1/2}x\|}\right\rangle^{-\|A^{1/2}x\|^2}\end{aligned}$$

and by taking the exponential, we get

$$\begin{aligned}\eta_x(A) &\geq \left\langle A\frac{A^{1/2}x}{\left\|A^{1/2}x\right\|},\frac{A^{1/2}x}{\left\|A^{1/2}x\right\|}\right\rangle^{-\left\|A^{1/2}x\right\|^2}\\ &= \left[\frac{1}{\langle Ax,x\rangle}\left\langle A^2x,x\right\rangle\right]^{-\langle Ax,x\rangle} = \left(\frac{\langle Ax,x\rangle}{\langle A^2x,x\rangle}\right)^{\langle Ax,x\rangle},\end{aligned}$$

which proves the first part of (2.1). □

Proposition 2. *If A, $B>0$, then for all $x\in H, \|x\|=1$ and $t\in[0,1]$, then we have the Ky Fan type inequality*

$$\eta_x((1-t)A+tB)\geq(\eta_x(A))^{1-t}(\eta_x(B))^t. \tag{2.2}$$

Proof. Since entropy function $\eta(\cdot)$ is operator concave, then

$$\eta((1-t)A+tB)\geq(1-t)\eta(A)+t\eta(B)$$

for all $t\in[0,1]$.

If we take the inner product over $x\in H, \|x\|=1$, then we get

$$\langle\eta((1-t)A+tB)x,x\rangle\geq(1-t)\langle\eta(A)x,x\rangle+t\langle\eta(B)x,x\rangle.$$

If we take the exponential, then we derive that

$$\begin{aligned}\eta_x((1-t)A+tB) &= \exp\langle\eta((1-t)A+tB)x,x\rangle\\ &\geq \exp\left[(1-t)\langle\eta(A)x,x\rangle+t\langle\eta(B)x,x\rangle\right]\\ &= (\exp\langle\eta(A)x,x\rangle)^{1-t}(\exp\langle\eta(B)x,x\rangle)^t\\ &= (\eta_x(A))^{1-t}(\eta_x(B))^t,\end{aligned}$$

which proves the desired inequality (2.2). □

We define the *logarithmic mean* of two positive numbers a, b by

$$L(a,b):=\begin{cases}\frac{b-a}{\ln b-\ln a} \text{ if } b\neq a,\\ \\ a \text{ if } b=a.\end{cases}$$

The following Hermite-Hadamard type integral inequalities hold:

Corollary 1. *With the assumptions of Proposition 2,*

$$\int_0^1\eta_x((1-t)A+tB)dt\geq L(\eta_x(A),\eta_x(B)). \tag{2.3}$$

and

$$\eta_x\left(\frac{A+B}{2}\right)\geq\int_0^1[\eta_x((1-t)A+tB)]^{1/2}[\eta_x(tA+(1-t)B)]^{1/2}dt. \tag{2.4}$$

Proof. If we take the integral over $t\in[0,1]$ in (2.2), then we get

$$\begin{aligned}\int_0^1\eta_x((1-t)A+tB)dt &\geq \int_0^1[\eta_x(A)]^{1-t}[\eta_x(B)]^t dt\\ &= L(\eta_x(A),\eta_x(B))\end{aligned}$$

for all A, $B>0$, which proves (2.3).

We get from (2.2) for $t=1/2$ that

$$\eta_x\left(\frac{A+B}{2}\right)\geq[\eta_x(A)]^{1/2}[\eta_x(B)]^{1/2}.$$

If we replace A by $(1-t)A+tB$ and B by $tA+(1-t)B$ we obtain

$$\eta_x\left(\frac{A+B}{2}\right)\geq[\eta_x((1-t)A+tB)]^{1/2}[\eta_x(tA+(1-t)B)]^{1/2}.$$

By taking the integral, we derive the desired result (2.4). □

Theorem 1. *If $A>0$, then for all $x\in H$, $\|x\|=1$ and $a>0$, we have the following inequalities*

$$\eta_x(A)\leq a^{-\langle Ax,x\rangle}\exp[-\langle Ax,x\rangle+a] \tag{2.5}$$

and

$$\Delta_x(A)\leq a\exp\left[\frac{1}{a}\langle Ax,x\rangle-1\right]. \tag{2.6}$$

Proof. It is well know that, if f is differentiable convex on an interval I, then for all $u,\,v\in I$ we have

$$f'(v)(u-v)\leq f(u)-f(v)\leq f'(u)(u-v). \tag{2.7}$$

Consider the convex function $f(t)=t\ln t$, $t>0$. Since $f'(t)=\ln t+1$, $t>0$, hence by (2.7) we get

$$(\ln v+1)(u-v)\leq u\ln u-v\ln v\leq(\ln u+1)(u-v) \tag{2.8}$$

namely

$$(\ln v+1)(u-v)-u\ln u\leq-v\ln v\leq-u\ln u+(\ln u+1)(u-v)$$

giving that

$$(u-v)\ln v-u\ln u+u-v\leq-v\ln v\leq u-v-v\ln u$$

for $u,\,v>0$.

If we take $u=a$ and use the functional calculus for $v=A>0$, then we get

$$(a-A)\ln A-a\ln a+a-A\leq-A\ln A\leq a-A-A\ln a,$$

namely

$$a\ln A-A\ln A-A-a\ln\left(\frac{a}{e}\right)\leq-A\ln A\leq-\ln(ea)A+a.$$

If we take the inner product over $x\in H$, $\|x\|=1$, then we get

$$\begin{aligned}a\langle\ln Ax,x\rangle-\langle A\ln Ax,x\rangle-\langle Ax,x\rangle-\ln\left(\frac{a}{e}\right)^a&\leq-\langle A\ln Ax,x\rangle\\&\leq\ln(ea)^{-\langle Ax,x\rangle}+a.\end{aligned}$$

If we take the exponential, then we get

$$\begin{aligned}\frac{\exp[a\langle\ln Ax,x\rangle]\exp[-\langle A\ln Ax,x\rangle]}{\left(\frac{a}{e}\right)^a\exp[\langle Ax,x\rangle]}&\leq\exp[-\langle A\ln Ax,x\rangle]\\&\leq(ea)^{-\langle Ax,x\rangle}\exp a\\&=a^{-\langle Ax,x\rangle}\exp[-\langle Ax,x\rangle+a].\end{aligned}$$

From the second inequality, we get (2.5).

From the first inequality, we get

$$\frac{\exp\left[a\left\langle \ln Ax,x\right\rangle\right]}{\left(\frac{a}{e}\right)^{a}\exp\left[\left\langle Ax,x\right\rangle\right]}\leq 1,$$

namely

$$\left[\Delta_{x}(A)\right]^{a}\leq a^{a}\exp\left[\left\langle Ax,x\right\rangle -a\right]$$

and by taking the power $\frac{1}{a}$ we obtain (2.6). $\square$

Remark 1. *For given* $A>0,\ x\in H, \|x\|=1$ *and* $a>0,$ *consider the function*

$$f\left(t\right)=t^{-\left\langle Ax,x\right\rangle}\exp\left[-\left\langle Ax,x\right\rangle +t\right],\ t>0.$$

We have

$$\begin{aligned}
&f'\left(t\right)\\
&=-\left\langle Ax,x\right\rangle t^{-\left\langle Ax,x\right\rangle -1}\exp\left[-\left\langle Ax,x\right\rangle +t\right]+t^{-\left\langle Ax,x\right\rangle}\exp\left[-\left\langle Ax,x\right\rangle +t\right]\\
&=\exp\left[-\left\langle Ax,x\right\rangle +t\right]t^{-\left\langle Ax,x\right\rangle -1}\left(t-\left\langle Ax,x\right\rangle\right).
\end{aligned}$$

We observe that the function f *is decreasing on* $\left(0,\left\langle Ax,x\right\rangle\right)$ *and increasing on* $\left(\left\langle Ax,x\right\rangle ,\infty\right)$ *showing that*

$$\inf_{t\in(0,\infty)}f\left(t\right)=f\left(\left\langle Ax,x\right\rangle\right)=\left\langle Ax,x\right\rangle^{-\left\langle Ax,x\right\rangle}.$$

Therefore the best inequality we can get from (2.5) is for $a=\left\langle Ax,x\right\rangle ,$ *namely the second inequality in (2.1).*

Consider the function

$$g\left(t\right)=t\exp\left[\frac{1}{t}\left\langle Ax,x\right\rangle -1\right],\ t>0,$$

then

$$\begin{aligned}
g'\left(t\right)&=\exp\left[t^{-1}\left\langle Ax,x\right\rangle -1\right]+t\exp\left[t^{-1}\left\langle Ax,x\right\rangle -1\right]\left(-\frac{\left\langle Ax,x\right\rangle}{t^{2}}\right)\\
&=\exp\left[t^{-1}\left\langle Ax,x\right\rangle -1\right]\left(1-\frac{\left\langle Ax,x\right\rangle}{t}\right).
\end{aligned}$$

We have that $g'\left(t_{0}\right)=0$ *for* $t_{0}=\left\langle Ax,x\right\rangle$ *which shows that* f *is strictly decreasing on* $\left(0,\left\langle Ax,x\right\rangle\right)$ *and strictly increasing on* $\left(\left\langle Ax,x\right\rangle ,\infty\right).$ *Therefore*

$$\inf_{t\in(0,\infty)}g\left(t\right)=g\left(\left\langle Ax,x\right\rangle\right)=\left\langle Ax,x\right\rangle ,$$

and we obtain the best inequality from (2.6) that is the second inequality in (ii) from the introduction.

The following result also holds, see [2]:

Lemma 1. *Let* I *be an interval and* $f:I\rightarrow\mathbb{R}$ *be a convex and differentiable function on* $\mathring{I}$ *(the interior of* I*) whose derivative* f' *is continuous on* $\mathring{I}$*. If* A *and* B *are selfadjoint operators on the Hilbert space* H *with* $\operatorname{Sp}\left(A\right),\operatorname{Sp}\left(B\right)\subseteq\left[m,M\right]\subset\mathring{I}$*, then*

$$\begin{aligned}
&\left\langle f'\left(A\right)x,x\right\rangle\left\langle By,y\right\rangle -\left\langle f'\left(A\right)Ax,x\right\rangle\\
&\leq\left\langle f\left(B\right)y,y\right\rangle -\left\langle f\left(A\right)x,x\right\rangle\leq\left\langle f'\left(B\right)By,y\right\rangle -\left\langle Ax,x\right\rangle\left\langle f'\left(B\right)y,y\right\rangle
\end{aligned}\tag{2.9}$$

for any $x,y\in H$ *with* $\|x\|=\|y\|=1.$

In particular, we have

$$\begin{aligned}(2.10)\qquad & \langle f'(A)x,x\rangle\langle Ay,y\rangle-\langle f'(A)Ax,x\rangle\\ & \le\langle f(A)y,y\rangle-\langle f(A)x,x\rangle\le\langle f'(A)Ay,y\rangle-\langle Ax,x\rangle\langle f'(A)y,y\rangle\end{aligned}$$

for any $x,y\in H$ *with* $\|x\|=\|y\|=1$ *and*

$$\begin{aligned}(2.11)\qquad & \langle f'(A)x,x\rangle\langle Bx,x\rangle-\langle f'(A)Ax,x\rangle\\ & \le\langle f(B)x,x\rangle-\langle f(A)x,x\rangle\le\langle f'(B)Bx,x\rangle-\langle Ax,x\rangle\langle f'(B)x,x\rangle\end{aligned}$$

for any $x\in H$ *with* $\|x\|=1$.

We have the following result concerning two operators as well:

Theorem 2. *Assume that* $A,\ B>0$, *then*

$$\eta_x(B)\le\frac{\exp\langle Ax,x\rangle}{\exp\langle By,y\rangle[\Delta_x(A)]^{\langle By,y\rangle}}\tag{2.12}$$

for $x,\ y\in H$ *with* $\|x\|=\|y\|=1$.

In particular,

$$\eta_x(B)\le\frac{\exp\langle Ax,x\rangle}{\exp\langle Bx,x\rangle[\Delta_x(A)]^{\langle Bx,x\rangle}},\tag{2.13}$$

$$\eta_x(A)\le\frac{\exp\langle Ax,x\rangle}{\exp\langle Ay,y\rangle[\Delta_x(A)]^{\langle Ay,y\rangle}}\tag{2.14}$$

and

$$\eta_x(A)\le[\Delta_x(A)]^{-\langle Ax,x\rangle}.\tag{2.15}$$

Proof. If we write the inequality (2.9) for the convex function $f(t)=t\ln t$, $t>0$, then we get for $x,\ y\in H$ with $\|x\|=\|y\|=1$ that

$$\begin{aligned}&\langle(\ln A+1)x,x\rangle\langle By,y\rangle-\langle(\ln A+1)Ax,x\rangle\\ &\le\langle B\ln By,y\rangle-\langle A\ln Ax,x\rangle\\ &\le\langle(\ln B+1)By,y\rangle-\langle Ax,x\rangle\langle(\ln B+1)y,y\rangle,\end{aligned}$$

namely

$$\begin{aligned}(2.16)\qquad &\langle\ln Ax,x\rangle\langle By,y\rangle+\langle By,y\rangle-\langle A\ln Ax,x\rangle-\langle Ax,x\rangle\\ &\le\langle B\ln By,y\rangle-\langle A\ln Ax,x\rangle\\ &\le\langle B\ln By,y\rangle+\langle By,y\rangle-\langle Ax,x\rangle\langle\ln By,y\rangle-\langle Ax,x\rangle.\end{aligned}$$

From the first inequality in (2.16) we have

$$\langle\ln Ax,x\rangle\langle By,y\rangle+\langle By,y\rangle-\langle Ax,x\rangle\le\langle B\ln By,y\rangle,\tag{2.17}$$

while from the second inequality in (2.16) we get

$$-\langle A\ln Ax,x\rangle\le\langle By,y\rangle-\langle Ax,x\rangle\langle\ln By,y\rangle-\langle Ax,x\rangle\tag{2.18}$$

for $x,\ y\in H$ with $\|x\|=\|y\|=1$.

From (2.17) we obtain

$$\langle By,y\rangle-\langle Ax,x\rangle\le\langle B\ln By,y\rangle-\langle\ln Ax,x\rangle\langle By,y\rangle\tag{2.19}$$

for $x,\ y\in H$ with $\|x\|=\|y\|=1$.

If we take the exponential in (2.19), then we get

$$\begin{aligned}\frac{\exp\langle By,y\rangle}{\exp\langle Ax,x\rangle} &\le \frac{\exp\langle B\ln By,y\rangle}{\left[\exp\langle \ln Ax,x\rangle\right]^{\langle By,y\rangle}} = \frac{\left[\exp\langle -B\ln By,y\rangle\right]^{-1}}{\left[\exp\langle \ln Ax,x\rangle\right]^{\langle By,y\rangle}} \\ &= \frac{\left[\eta_x(B)\right]^{-1}}{\left[\Delta_x(A)\right]^{\langle By,y\rangle}}\end{aligned}$$

giving that

$$\frac{\exp\langle By,y\rangle}{\exp\langle Ax,x\rangle} \le \frac{1}{\eta_x(B)\left[\Delta_x(A)\right]^{\langle By,y\rangle}}$$

for x, $y \in H$ with $\|x\| = \|y\| = 1$.

From (2.19) we obtain similar results with A instead of B. □

3. Related Results

In [1] we obtained the following reverse of Jensen's inequality

Lemma 2. *Let I be an interval and $f : I \to \mathbb{R}$ be a convex and differentiable function on $\mathring{I}$ (the interior of I) whose derivative f' is continuous on $\mathring{I}$. If A is a selfadjoint operator on the Hilbert space H with $\operatorname{Sp}(A) \subseteq [m, M] \subset \mathring{I}$, then*

$$\begin{aligned}(0 \le)\ &\langle f(A)x,x\rangle - f(\langle Ax,x\rangle) \\ &\le \begin{cases} \frac{1}{2}(M-m)\left[\|f'(A)x\|^2 - \langle f'(A)x,x\rangle^2\right]^{1/2} \\ \\ \frac{1}{2}(f'(M)-f'(m))\left[\|Ax\|^2 - \langle Ax,x\rangle^2\right]^{1/2}\end{cases} \\ &\le \frac{1}{4}(M-m)(f'(M)-f'(m)),\end{aligned} \tag{3.1}$$

for any $x \in H$ with $\|x\| = 1$.

We also have the inequality

$$\begin{aligned}(0 \le)\ &\langle f(A)x,x\rangle - f(\langle Ax,x\rangle) \\ &\le \frac{1}{4}(M-m)(f'(M)-f'(m)) \\ &- \begin{cases}\left[\langle Mx-Ax, Ax-mx\rangle\langle f'(M)x - f'(A)x, f'(A)x - f'(m)x\rangle\right]^{\frac{1}{2}}, \\ \\ \left|\langle Ax,x\rangle - \frac{M+m}{2}\right|\left|\langle f'(A)x,x\rangle - \frac{f'(M)+f'(m)}{2}\right|\end{cases} \\ &\le \frac{1}{4}(M-m)(f'(M)-f'(m)),\end{aligned} \tag{3.2}$$

for any $x \in H$ with $\|x\| = 1$.

Using these inequalities we can state:

Theorem 3. *Assume that the operator A satisfies the condition $0 < m \le A \le M$ for some constants m and M. Then for any $x \in H$ with $\|x\| = 1$,*

$$1 \le \frac{\langle Ax, x\rangle^{-\langle Ax,x\rangle}}{\eta_x(A)} \le \begin{cases} \exp\left\{\frac{1}{2}(M-m)\left[\|\ln(eA)x\|^2 - \langle \ln(eA)x, x\rangle^2\right]^{1/2}\right\} \\ \\ \left(\frac{M}{m}\right)^{\frac{1}{2}\left[\|Ax\|^2 - \langle Ax,x\rangle^2\right]^{1/2}} \end{cases} \le \left(\frac{M}{m}\right)^{\frac{1}{4}(M-m)} \tag{3.3}$$

and

$$1 \le \frac{\langle Ax, x\rangle^{-\langle Ax,x\rangle}}{\eta_x(A)} \le \frac{\left(\frac{M}{m}\right)^{\frac{1}{4}(M-m)}}{L(A,x,m,M)} \le \left(\frac{M}{m}\right)^{\frac{1}{4}(M-m)}, \tag{3.4}$$

where

$$L(A,x,m,M) := \begin{cases} \exp\left[\langle Mx - Ax, Ax - mx\rangle \langle \ln(M)x - \ln(A)x, \ln(A)x - \ln(m)x\rangle\right]^{\frac{1}{2}}, \\ \\ \exp\left[\left|\langle Ax,x\rangle - \frac{M+m}{2}\right| \left|\langle \ln(A)x,x\rangle - \ln\sqrt{mM}\right|\right]. \end{cases}$$

Proof. Now consider the convex function $f : (0,\infty) \to \mathbb{R}$, $f(t) = t\ln t$, $t > 0$. On utilizing the inequality (3.1), then for any positive definite operator A on the Hilbert space H, we have the inequality

$$0 \le \langle A\ln(A)x, x\rangle - \langle Ax, x\rangle \ln(\langle Ax, x\rangle) \le \begin{cases} \frac{1}{2}(M-m)\left[\|\ln(eA)x\|^2 - \langle \ln(eA)x, x\rangle^2\right]^{1/2} \\ \\ \ln\sqrt{\frac{M}{m}}\left[\|Ax\|^2 - \langle Ax,x\rangle^2\right]^{1/2} \end{cases} \le \frac{1}{2}(M-m)\ln\sqrt{\frac{M}{m}} \tag{3.5}$$

for any $x \in H$ with $\|x\| = 1$.

If we take the exponential in (3.5), then we get

$$1 \le \exp\left[\langle A\ln(A)x, x\rangle - \langle Ax, x\rangle \ln(\langle Ax, x\rangle)\right] \le \begin{cases} \exp\left\{\frac{1}{2}(M-m)\left[\|\ln(eA)x\|^2 - \langle \ln(eA)x, x\rangle^2\right]^{1/2}\right\} \\ \\ \exp\left\{\ln\sqrt{\frac{M}{m}}\left[\|Ax\|^2 - \langle Ax,x\rangle^2\right]^{1/2}\right\} \end{cases} \le \exp\left\{\frac{1}{2}(M-m)\ln\sqrt{\frac{M}{m}}\right\}. \tag{3.6}$$

Observe that

$$\begin{aligned}
&\exp\left[\langle A\ln(A)x,x\rangle-\langle Ax,x\rangle\ln(\langle Ax,x\rangle)\right]\\
&=\frac{\exp\left[-\langle Ax,x\rangle\ln(\langle Ax,x\rangle)\right]}{\exp\left[-\langle A\ln(A)x,x\rangle\right]}\\
&=\frac{\exp\left[\ln\left(\langle Ax,x\rangle^{-\langle Ax,x\rangle}\right)\right]}{\exp\left[-\langle A\ln(A)x,x\rangle\right]}=\frac{\langle Ax,x\rangle^{-\langle Ax,x\rangle}}{\eta_x(A)},
\end{aligned}$$

$$\begin{aligned}
&\exp\left\{\ln\sqrt{\frac{M}{m}}\left[\|Ax\|^2-\langle Ax,x\rangle^2\right]^{1/2}\right\}\\
&=\exp\left\{\ln\left[\left(\frac{M}{m}\right)^{\frac{1}{2}\left[\|Ax\|^2-\langle Ax,x\rangle^2\right]^{1/2}}\right]\right\}=\left(\frac{M}{m}\right)^{\frac{1}{2}\left[\|Ax\|^2-\langle Ax,x\rangle^2\right]^{1/2}}
\end{aligned}$$

and

$$\exp\left\{\frac{1}{2}(M-m)\ln\sqrt{\frac{M}{m}}\right\}=\exp\left[\ln\left(\frac{M}{m}\right)^{\frac{1}{4}(M-m)}\right]=\left(\frac{M}{m}\right)^{\frac{1}{4}(M-m)}.$$

By making use of (3.6) we derive (3.3).

If we apply now the inequality (3.2), then we have the following result as well

$$\begin{aligned}
(3.7)\quad 0&\le\langle A\ln(A)x,x\rangle-\langle Ax,x\rangle\ln(\langle Ax,x\rangle)\\
&\le\frac{1}{2}(M-m)\ln\sqrt{\frac{M}{m}}\\
&-\begin{cases}\left[\langle Mx-Ax,Ax-mx\rangle\langle\ln(M)x-\ln(A)x,\ln(A)x-\ln(m)x\rangle\right]^{\frac{1}{2}},\\[2mm] \left|\langle Ax,x\rangle-\frac{M+m}{2}\right|\left|\langle\ln(A)x,x\rangle-\ln\sqrt{mM}\right|\end{cases}\\
&\le\frac{1}{2}(M-m)\ln\sqrt{\frac{M}{m}}
\end{aligned}$$

for any $x\in H$ with $\|x\|=1$.

If we take the exponential in (3.7), then we get

$$1\le\frac{\langle Ax,x\rangle^{-\langle Ax,x\rangle}}{\eta_x(A)}\le\frac{\left(\frac{M}{m}\right)^{\frac{1}{4}(M-m)}}{L(A,x,m,M)}\le\left(\frac{M}{m}\right)^{\frac{1}{4}(M-m)}$$

and the inequality (3.4) is thus proved. □

Theorem 4. *Assume that the operator A satisfies the condition $0 < m \leq A \leq M$ for some constants m and M. Then for any $x \in H$ with $\|x\| = 1$,*

$$1 \leq \frac{\eta_x(A)}{\left(\frac{\langle A^2x,x\rangle}{\langle Ax,x\rangle}\right)^{-\langle Ax,x\rangle}} \tag{3.8}$$
$$\leq \begin{cases} \exp\left(\frac{1}{2}(M-m)\left[\langle Ax,x\rangle\left\langle A^{-1}x,x\right\rangle - \langle x,x\rangle^2\right]^{1/2}\right) \\ \exp\left(\frac{1}{2mM}(M-m)\left[\langle Ax,x\rangle\left\langle A^3x,x\right\rangle - \left\langle A^2x,x\right\rangle^2\right]^{1/2}\right) \end{cases}$$
$$\leq \exp\left(\frac{1}{4mM}\langle Ax,x\rangle(M-m)^2\right) \leq \exp\left(\frac{1}{4m}(M-m)^2\right).$$

Proof. If we write the inequality (3.1) for the convex function $-\ln$, then we get

$$0 \leq \ln(\langle Ay,y\rangle) - \langle \ln Ay, y\rangle \tag{3.9}$$
$$\leq \begin{cases} \frac{1}{2}(M-m)\left[\left\|A^{-1}y\right\|^2 - \left\langle A^{-1}y,y\right\rangle^2\right]^{1/2} \\ \frac{1}{2mM}(M-m)\left[\|Ay\|^2 - \langle Ay,y\rangle^2\right]^{1/2} \end{cases}$$
$$\leq \frac{1}{4mM}(M-m)^2,$$

for any $y \in H$ with $\|y\| = 1$.

By taking $y = \frac{A^{1/2}x}{\|A^{1/2}x\|}$, $x \in H, \|x\| = 1$ in (3.9), we obtain

$$0 \leq \ln\left(\left\langle A\frac{A^{1/2}x}{\|A^{1/2}x\|}, \frac{A^{1/2}x}{\|A^{1/2}x\|}\right\rangle\right) - \left\langle \ln A\frac{A^{1/2}x}{\|A^{1/2}x\|}, \frac{A^{1/2}x}{\|A^{1/2}x\|}\right\rangle$$
$$\leq \begin{cases} \frac{1}{2}(M-m)\left[\left\|A^{-1}\frac{A^{1/2}x}{\|A^{1/2}x\|}\right\|^2 - \left\langle A^{-1}\frac{A^{1/2}x}{\|A^{1/2}x\|}, \frac{A^{1/2}x}{\|A^{1/2}x\|}\right\rangle^2\right]^{1/2} \\ \frac{1}{2mM}(M-m)\left[\left\|A\frac{A^{1/2}x}{\|A^{1/2}x\|}\right\|^2 - \left\langle A\frac{A^{1/2}x}{\|A^{1/2}x\|}, \frac{A^{1/2}x}{\|A^{1/2}x\|}\right\rangle^2\right]^{1/2} \end{cases}$$
$$\leq \frac{1}{4mM}(M-m)^2,$$

namely

$$0 \leq \ln\left(\frac{1}{\|A^{1/2}x\|^2}\left\langle A^2x,x\right\rangle\right) - \frac{1}{\|A^{1/2}x\|^2}\langle A\ln Ax, x\rangle$$

$$\leq \begin{cases} \frac{1}{2}(M-m)\left[\frac{1}{\|A^{1/2}x\|^2}\left\|A^{-1/2}x\right\|^2 - \frac{1}{\|A^{1/2}x\|^4}\langle x,x\rangle^2\right]^{1/2} \\ \\ \frac{1}{2mM}(M-m)\left[\frac{1}{\|A^{1/2}x\|^2}\left\|A^{3/2}x\right\|^2 - \frac{1}{\|A^{1/2}x\|^4}\left\langle A^2x,x\right\rangle^2\right]^{1/2} \end{cases}$$
$$\leq \frac{1}{4mM}(M-m)^2.$$

By multiplying with $\left\|A^{1/2}x\right\|^2 > 0$, we get

$$(0\leq)\left\|A^{1/2}x\right\|^2 \ln\left(\frac{1}{\|A^{1/2}x\|^2}\left\langle A^2x,x\right\rangle\right) - \langle A\ln Ax, x\rangle$$
$$\leq \begin{cases} \frac{1}{2}(M-m)\left\|A^{1/2}x\right\|^2\left[\frac{1}{\|A^{1/2}x\|^2}\left\|A^{-1/2}x\right\|^2 - \frac{1}{\|A^{1/2}x\|^4}\langle x,x\rangle^2\right]^{1/2} \\ \\ \frac{1}{2mM}(M-m)\left\|A^{1/2}x\right\|^2\left[\frac{1}{\|A^{1/2}x\|^2}\left\|A^{3/2}x\right\|^2 - \frac{1}{\|A^{1/2}x\|^4}\left\langle A^2x,x\right\rangle^2\right]^{1/2} \end{cases}$$
$$\leq \frac{1}{4mM}\left\|A^{1/2}x\right\|^2(M-m)^2,$$

namely

$$0 \leq \langle Ax,x\rangle \ln\left(\frac{\langle A^2x,x\rangle}{\langle Ax,x\rangle}\right) - \langle A\ln Ax,x\rangle \tag{3.10}$$
$$\leq \begin{cases} \frac{1}{2}(M-m)\left[\langle Ax,x\rangle\left\langle A^{-1}x,x\right\rangle - \langle x,x\rangle^2\right]^{1/2} \\ \\ \frac{1}{2mM}(M-m)\left[\langle Ax,x\rangle\left\langle A^3x,x\right\rangle - \left\langle A^2x,x\right\rangle^2\right]^{1/2} \end{cases}$$
$$\leq \frac{1}{4mM}\langle Ax,x\rangle(M-m)^2,$$

for any $y \in H$ with $\|y\| = 1$.

By taking the exponential in (3.10), we obtain

$$1 \leq \frac{\exp\left[-\langle A\ln Ax,x\rangle\right]}{\exp\left[\ln\left(\frac{\langle A^2x,x\rangle}{\langle Ax,x\rangle}\right)^{-\langle Ax,x\rangle}\right]}$$
$$\leq \begin{cases} \exp\left(\frac{1}{2}(M-m)\left[\langle Ax,x\rangle\left\langle A^{-1}x,x\right\rangle - \langle x,x\rangle^2\right]^{1/2}\right) \\ \\ \exp\left(\frac{1}{2mM}(M-m)\left[\langle Ax,x\rangle\left\langle A^3x,x\right\rangle - \left\langle A^2x,x\right\rangle^2\right]^{1/2}\right) \end{cases}$$
$$\leq \exp\left(\frac{1}{4mM}\langle Ax,x\rangle(M-m)^2\right) \leq \exp\left(\frac{1}{4m}(M-m)^2\right),$$

which is equivalent to (3.8). □

We also have:

Theorem 5. *Assume that the operator A satisfies the condition $0 < m \leq A \leq M$ for some constants m and M. Then for any $x \in H$ with $\|x\| = 1$,*

$$\left(\frac{1}{4mM}(M+m)^2\right)^{-M} \leq \left(\frac{1}{4mM}(M+m)^2\right)^{-\langle Ax,x\rangle} \leq \left(\frac{\langle A^2x,x\rangle}{\langle Ax,x\rangle^2}\right)^{-\langle Ax,x\rangle} \leq \frac{\eta_x(A)}{\langle Ax,x\rangle^{-\langle Ax,x\rangle}} \leq 1. \tag{3.11}$$

Proof. From (2.1) we have

$$\left(\frac{\langle A^2x,x\rangle}{\langle Ax,x\rangle^2}\right)^{-\langle Ax,x\rangle} \langle Ax,x\rangle^{-\langle Ax,x\rangle} \leq \eta_x(A) \leq \langle Ax,x\rangle^{-\langle Ax,x\rangle},$$

which gives that

$$\left(\frac{\langle A^2x,x\rangle}{\langle Ax,x\rangle^2}\right)^{-\langle Ax,x\rangle} \leq \frac{\eta_x(A)}{\langle Ax,x\rangle^{-\langle Ax,x\rangle}} \leq 1, \tag{3.12}$$

for any $x \in H$ with $\|x\| = 1$.

We use Kantorovich inequality

$$\frac{\langle A^2x,x\rangle}{\langle Ax,x\rangle^2} \leq \frac{1}{4mM}(M+m)^2$$

that holds for any $x \in H$ with $\|x\| = 1$, which gives that

$$\left(\frac{1}{4mM}(M+m)^2\right)^{-\langle Ax,x\rangle} \leq \left(\frac{\langle A^2x,x\rangle}{\langle Ax,x\rangle^2}\right)^{-\langle Ax,x\rangle}.$$

Also,

$$\left(\frac{1}{4mM}(M+m)^2\right)^{-M} \leq \left(\frac{1}{4mM}(M+m)^2\right)^{-\langle Ax,x\rangle}$$

and by (3.12) we derive (3.11). □

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