UPPER AND LOWER BOUNDS FOR THE NORMALIZED ENTROPIC DETERMINANT OF POSITIVE OPERATORS IN HILBERT SPACES

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Abstract. For positive invertible operators A on a Hilbert space H and a fixed unit vector $x \in H$, define the normalized entropic determinant by $\eta_x(A) := \exp[-\langle A \ln Ax, x\rangle]$. In this paper we show among others that, if A satisfies the condition $0 < m \leq A \leq M$, then

$$\left[S\left(\frac{M}{m}\right)\right]^{-M} \leq \left[S\left(\frac{M}{m}\right)\right]^{-\langle Ax,x\rangle} \leq \frac{\left(\frac{\langle A^2x,x\rangle}{\langle Ax,x\rangle^2}\right)^{-\langle Ax,x\rangle} \langle Ax,x\rangle^{\langle Ax,x\rangle}}{\eta_x(A)} \leq 1$$

and

$$\left[S\left(\frac{M}{m}\right)\right]^{-M} \leq \left[S\left(\frac{M}{m}\right)\right]^{-\langle Ax,x\rangle} \leq \frac{\eta_x(A)}{\langle Ax,x\rangle^{-\langle Ax,x\rangle}} \leq 1$$

for $x \in H$, $\|x\| = 1$.

1. Introduction

Let $B(H)$ be the space of all bounded linear operators on a Hilbert space H, and I stands for the identity operator on H. An operator A in $B(H)$ is said to be positive (in symbol: $A \geq 0$) if $\langle Ax, x\rangle \geq 0$ for all $x \in H$. In particular, $A > 0$ means that A is positive and invertible. For a pair A, B of selfadjoint operators the order relation $A \geq B$ means as usual that $A - B$ is positive.

In 1998, Fujii et al. [2], [3], introduced the *normalized determinant* $\Delta_x(A)$ for positive invertible operators A on a Hilbert space H and a fixed unit vector $x \in H$, namely $\|x\| = 1$, defined by $\Delta_x(A) := \exp\langle \ln Ax, x\rangle$ and discussed it as a continuous geometric mean and observed some inequalities around the determinant from this point of view.

Some of the fundamental properties of normalized determinant are as follows, [2].

For each unit vector $x \in H$, see also [5], we have:

(i) *continuity*: the map $A \to \Delta_x(A)$ is norm continuous;
(ii) *bounds*: $\langle A^{-1}x, x\rangle^{-1} \leq \Delta_x(A) \leq \langle Ax, x\rangle$;
(iii) *continuous mean*: $\langle A^p x, x\rangle^{1/p} \downarrow \Delta_x(A)$ for $p \downarrow 0$ and $\langle A^p x, x\rangle^{1/p} \uparrow \Delta_x(A)$ for $p \uparrow 0$;
(iv) *power equality:* $\Delta_x(A^t) = \Delta_x(A)^t$ for all $t > 0$;
(v) *homogeneity*: $\Delta_x(tA) = t\Delta_x(A)$ and $\Delta_x(tI) = t$ for all $t > 0$;

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(vi) *monotonicity*: $0 < A \leq B$ implies $\Delta_x(A) \leq \Delta_x(B)$;
(vii) *multiplicativity*: $\Delta_x(AB) = \Delta_x(A)\Delta_x(B)$ for commuting A and B;
(viii) *Ky Fan type inequality*: $\Delta_x((1-\alpha)A + \alpha B) \geq \Delta_x(A)^{1-\alpha}\Delta_x(B)^{\alpha}$ for $0 < \alpha < 1$.

We define the logarithmic mean of two positive numbers a, b by

$$L(a,b) := \begin{cases} \frac{b-a}{\ln b - \ln a} \text{ if } b \neq a, \\ \\ a \text{ if } b = a. \end{cases}$$

In [2] the authors obtained the following additive reverse inequality for the operator A which satisfy the condition $0 < mI \leq A \leq MI$, where m, M are positive numbers,

$$0 \leq \langle Ax, x\rangle - \Delta_x(A) \leq L(m,M)\left[\ln L(m,M) + \frac{M\ln m - m\ln M}{M-m} - 1\right] \tag{1.1}$$

for all $x \in H$, $\|x\| = 1$.

The famous *Young inequality* for scalars says that if $a, b > 0$ and $\nu \in [0,1]$, then

$$a^{1-\nu}b^{\nu} \leq (1-\nu)a + \nu b \tag{1.2}$$

with equality if and only if $a = b$. The inequality (1.2) is also called *ν-weighted arithmetic-geometric mean inequality.*

We recall that *Specht's ratio* is defined by [7]

$$S(h) := \begin{cases} \dfrac{h^{\frac{1}{h-1}}}{e\ln\left(h^{\frac{1}{h-1}}\right)} \text{ if } h \in (0,1)\cup(1,\infty) \\ \\ 1 \text{ if } h = 1. \end{cases} \tag{1.3}$$

It is well known that $\lim_{h\to 1} S(h) = 1$, $S(h) = S\left(\frac{1}{h}\right) > 1$ for $h > 0$, $h \neq 1$. The function is decreasing on $(0,1)$ and increasing on $(1,\infty)$.

In [3], the authors obtained the following multiplicative reverse inequality as well

$$1 \leq \frac{\langle Ax, x\rangle}{\Delta_x(A)} \leq S\left(\frac{M}{m}\right) \tag{1.4}$$

for $0 < mI \leq A \leq MI$ and $x \in H$, $\|x\| = 1$.

Since $0 < M^{-1}I \leq A^{-1} \leq m^{-1}I$, then by (1.4) for A^{-1} we get

$$1 \leq \frac{\langle A^{-1}x, x\rangle}{\Delta_x(A^{-1})} \leq S\left(\frac{m^{-1}}{M^{-1}}\right) = S\left(\left(\frac{m}{M}\right)^{-1}\right) = S\left(\frac{M}{m}\right),$$

which is equivalent to

$$1 \leq \frac{\Delta_x(A)}{\langle A^{-1}x, x\rangle^{-1}} \leq S\left(\frac{M}{m}\right) \tag{1.5}$$

for $x \in H$, $\|x\| = 1$.

For the entropy function $\eta(t) = -t\ln t$, $t > 0$, the operator entropy has the following expression:

$$\eta(A) = -A\ln A$$

for positive A.

For $x \in H$, $\|x\| = 1$, we define the *normalized entropic determinant* $\eta_x(A)$ by

$$\eta_x(A) := \exp(-\langle A\ln Ax, x\rangle) = \exp\langle \eta(A)x, x\rangle. \tag{1.6}$$

Let $x \in H$, $\|x\| = 1$. Observe that the map $A \to \eta_x(A)$ is *norm continuous* and since

$$\begin{aligned}
&\exp\left(-\langle tA \ln (tA)\, x, x\rangle\right) \\
&= \exp\left(-\langle tA (\ln t + \ln A)\, x, x\rangle\right) = \exp\left(-\langle (tA \ln t + tA \ln A)\, x, x\rangle\right) \\
&= \exp\left(-\langle Ax, x\rangle\, t \ln t\right) \exp\left(-t \langle A \ln Ax, x\rangle\right) \\
&= \exp \ln \left(t^{-\langle Ax,x\rangle t}\right) \left[\exp\left(-\langle A \ln Ax, x\rangle\right)\right]^{-t},
\end{aligned}$$

hence

$$\eta_x(tA) = t^{-t\langle Ax,x\rangle} \left[\eta_x(A)\right]^{-t} \tag{1.7}$$

for $t > 0$ and $A > 0$.

Observe also that

$$\eta_x(I) = 1 \text{ and } \eta_x(tI) = t^{-t} \tag{1.8}$$

for $t > 0$.

Motivated by the above results, in this paper we show among others that, if A satisfies the condition $0 < m \le A \le M$, then

$$\begin{aligned}
\left[S\left(\frac{M}{m}\right)\right]^{-M} &\le \left[S\left(\frac{M}{m}\right)\right]^{-\langle Ax,x\rangle} \\
&\le \frac{\left(\frac{\langle A^2x,x\rangle}{\langle Ax,x\rangle^2}\right)^{-\langle Ax,x\rangle} \langle Ax, x\rangle^{\langle Ax,x\rangle}}{\eta_x(A)} \le 1
\end{aligned}$$

and

$$\left[S\left(\frac{M}{m}\right)\right]^{-M} \le \left[S\left(\frac{M}{m}\right)\right]^{-\langle Ax,x\rangle} \le \frac{\eta_x(A)}{\langle Ax, x\rangle^{-\langle Ax,x\rangle}} \le 1$$

for $x \in H$, $\|x\| = 1$.

2. Simple Bounds

We provide in this section other fundamental results:

Theorem 1. *Assume that $A > 0$ and $x \in H$, $\|x\| = 1$, then we have the double inequality*

$$1 \le \frac{\langle Ax, x\rangle^{-\langle Ax,x\rangle}}{\eta_x(A)} \le \left[\frac{\langle A^2x, x\rangle}{\langle Ax, x\rangle^2}\right]^{\langle Ax,x\rangle}. \tag{2.1}$$

Proof. Observe we have, for $x \in H$, $\|x\| = 1$, that

$$\begin{aligned}
\Delta_{\frac{A^{1/2}x}{\|A^{1/2}x\|}}(A) &= \exp\left\langle \ln A \frac{A^{1/2}x}{\|A^{1/2}x\|}, \frac{A^{1/2}x}{\|A^{1/2}x\|}\right\rangle \\
&= \exp\left[\frac{1}{\|A^{1/2}x\|^2}\left\langle \ln A A^{1/2}x, A^{1/2}x\right\rangle\right] \\
&= \exp\left[\frac{1}{\langle Ax, x\rangle}\langle A \ln Ax, x\rangle\right] = \exp\left[-\frac{1}{\langle Ax, x\rangle}\langle -A \ln Ax, x\rangle\right] \\
&= \left(\exp\left[\langle -A \ln Ax, x\rangle\right]\right)^{-\frac{1}{\langle Ax,x\rangle}} = \left[\eta_x(A)\right]^{-\frac{1}{\langle Ax,x\rangle}}
\end{aligned}$$

and by taking the power $-\langle Ax,x\rangle$, we derive the equality of interest

$$\eta_x(A)=\left(\Delta_{\frac{A^{1/2}x}{\|A^{1/2}x\|}}(A)\right)^{-\langle Ax,x\rangle} \tag{2.2}$$

for all $x\in H$, $\|x\|=1$.

If we use the inequalities (ii) from the introduction, then we get

$$\left\langle A^{-1}\frac{A^{1/2}x}{\|A^{1/2}x\|},\frac{A^{1/2}x}{\|A^{1/2}x\|}\right\rangle^{-1}\le\Delta_{\frac{A^{1/2}x}{\|A^{1/2}x\|}}(A)\le\left\langle A\frac{A^{1/2}x}{\|A^{1/2}x\|},\frac{A^{1/2}x}{\|A^{1/2}x\|}\right\rangle$$

and by taking the power $-\langle Ax,x\rangle<0$ we derive

$$\begin{aligned}\left\langle A\frac{A^{1/2}x}{\|A^{1/2}x\|},\frac{A^{1/2}x}{\|A^{1/2}x\|}\right\rangle^{-\langle Ax,x\rangle}&\le\left(\Delta_{\frac{A^{1/2}x}{\|A^{1/2}x\|}}(A)\right)^{-\langle Ax,x\rangle}=\eta_x(A)\\&\le\left\langle A^{-1}\frac{A^{1/2}x}{\|A^{1/2}x\|},\frac{A^{1/2}x}{\|A^{1/2}x\|}\right\rangle^{\langle Ax,x\rangle}\end{aligned} \tag{2.3}$$

for all $x\in H$, $\|x\|=1$.

Now, observe that

$$\begin{aligned}\left\langle A\frac{A^{1/2}x}{\|A^{1/2}x\|},\frac{A^{1/2}x}{\|A^{1/2}x\|}\right\rangle^{-\langle Ax,x\rangle}&=\left[\frac{1}{\|A^{1/2}x\|^2}\left\langle A^2x,x\right\rangle\right]^{-\langle Ax,x\rangle}\\&=\left[\frac{1}{\langle Ax,x\rangle}\left\langle A^2x,x\right\rangle\right]^{-\langle Ax,x\rangle}\\&=\langle Ax,x\rangle^{-\langle Ax,x\rangle}\left[\frac{\langle A^2x,x\rangle}{\langle Ax,x\rangle^2}\right]^{-\langle Ax,x\rangle}\end{aligned} \tag{2.4}$$

and

$$\left\langle A^{-1}\frac{A^{1/2}x}{\|A^{1/2}x\|},\frac{A^{1/2}x}{\|A^{1/2}x\|}\right\rangle^{\langle Ax,x\rangle}=\left[\frac{1}{\langle Ax,x\rangle}\right]^{\langle Ax,x\rangle}=\langle Ax,x\rangle^{-\langle Ax,x\rangle} \tag{2.5}$$

for all $x\in H$, $\|x\|=1$.

By making use of (2.4) and (2.5) we derive

$$\langle Ax,x\rangle^{-\langle Ax,x\rangle}\left[\frac{\langle A^2x,x\rangle}{\langle Ax,x\rangle^2}\right]^{-\langle Ax,x\rangle}\le\eta_x(A)\le\langle Ax,x\rangle^{-\langle Ax,x\rangle} \tag{2.6}$$

for all $x\in H$, $\|x\|=1$, which is equivalent to (2.1). □

Corollary 1. *Assume that A satisfies the condition $0<m\le A\le M$, then we have the simpler bounds*

$$\begin{aligned}1\le\frac{\langle Ax,x\rangle^{-\langle Ax,x\rangle}}{\eta_x(A)}&\le\left[\frac{\langle A^2x,x\rangle}{\langle Ax,x\rangle^2}\right]^{\langle Ax,x\rangle}\le\left(\frac{m+M}{2\sqrt{mM}}\right)^{2\langle Ax,x\rangle}\\&\le\left(\frac{m+M}{2\sqrt{mM}}\right)^{2M}\end{aligned} \tag{2.7}$$

for $x\in H$, $\|x\|=1$.

The proof of the last part follows by Kantorovich inequality

$$\frac{\langle A^2x,x\rangle}{\langle Ax,x\rangle^2} \le \frac{1}{4mM}(m+M)^2$$

that holds for A that satisfies the condition $0<m\le A\le M$ and $x\in H$, $\|x\|=1$.

Theorem 2. *Assume that A satisfies the condition $0<m\le A\le M$, then*

$$0 \le \frac{\langle A^2x,x\rangle}{\langle Ax,x\rangle} - [\eta_x(A)]^{-\frac{1}{\langle Ax,x\rangle}} \le L(m,M)\left[\ln L(m,M) + \frac{M\ln m - m\ln M}{M-m} - 1\right], \tag{2.8}$$

$$\left[S\left(\frac{M}{m}\right)\right]^{-M} \le \left[S\left(\frac{M}{m}\right)\right]^{-\langle Ax,x\rangle} \le \frac{\left(\frac{\langle A^2x,x\rangle}{\langle Ax,x\rangle^2}\right)^{-\langle Ax,x\rangle}\langle Ax,x\rangle^{\langle Ax,x\rangle}}{\eta_x(A)} \le 1 \tag{2.9}$$

and

$$\left[S\left(\frac{M}{m}\right)\right]^{-M} \le \left[S\left(\frac{M}{m}\right)\right]^{-\langle Ax,x\rangle} \le \frac{\eta_x(A)}{\langle Ax,x\rangle^{-\langle Ax,x\rangle}} \le 1 \tag{2.10}$$

for $x\in H$, $\|x\|=1$.

Proof. From (1.1) we get

$$0 \le \left\langle A\frac{A^{1/2}x}{\|A^{1/2}x\|}, \frac{A^{1/2}x}{\|A^{1/2}x\|}\right\rangle - \Delta_{\frac{A^{1/2}x}{\|A^{1/2}x\|}}(A) \le L(m,M)\left[\ln L(m,M) + \frac{M\ln m - m\ln M}{M-m} - 1\right],$$

namely

$$0 \le \frac{\langle A^2x,x\rangle}{\langle Ax,x\rangle} - \Delta_{\frac{A^{1/2}x}{\|A^{1/2}x\|}}(A) \le L(m,M)\left[\ln L(m,M) + \frac{M\ln m - m\ln M}{M-m} - 1\right], \tag{2.11}$$

for $x\in H$, $\|x\|=1$.

By (2.2) we derive

$$\Delta_{\frac{A^{1/2}x}{\|A^{1/2}x\|}}(A) = [\eta_x(A)]^{-\frac{1}{\langle Ax,x\rangle}}$$

and by (2.11) we obtain (2.8).

From (1.4) we have

$$1 \le \frac{\left\langle A\frac{A^{1/2}x}{\|A^{1/2}x\|}, \frac{A^{1/2}x}{\|A^{1/2}x\|}\right\rangle}{\Delta_{\frac{A^{1/2}x}{\|A^{1/2}x\|}}(A)} \le S\left(\frac{M}{m}\right),$$

namely

$$1 \le \frac{\frac{\langle A^2x,x\rangle}{\langle Ax,x\rangle}}{\Delta_{\frac{A^{1/2}x}{\|A^{1/2}x\|}}(A)} \le S\left(\frac{M}{m}\right)$$

and by taking the power $-\langle Ax,x\rangle$ we get

$$1 \ge \frac{\left(\frac{\langle A^2x,x\rangle}{\langle Ax,x\rangle}\right)^{-\langle Ax,x\rangle}}{\left[\Delta_{\frac{A^{1/2}x}{\|A^{1/2}x\|}}(A)\right]^{-\langle Ax,x\rangle}} \ge \left[S\left(\frac{M}{m}\right)\right]^{-\langle Ax,x\rangle} \ge \left[S\left(\frac{M}{m}\right)\right]^{M}$$

which gives (2.9).

From (1.5) we derive

$$1 \le \frac{\Delta_{\frac{A^{1/2}x}{\|A^{1/2}x\|}}(A)}{\left\langle A^{-1}\frac{A^{1/2}x}{\|A^{1/2}x\|}, \frac{A^{1/2}x}{\|A^{1/2}x\|}\right\rangle^{-1}} \le S\left(\frac{M}{m}\right),$$

namely

$$1 \le \frac{\Delta_{\frac{A^{1/2}x}{\|A^{1/2}x\|}}(A)}{\langle Ax,x\rangle} \le S\left(\frac{M}{m}\right),$$

for $x \in H$, $\|x\| = 1$.

By taking the power $-\langle Ax,x\rangle$ we get

$$1 \ge \frac{\left[\Delta_{\frac{A^{1/2}x}{\|A^{1/2}x\|}}(A)\right]^{-\langle Ax,x\rangle}}{\langle Ax,x\rangle^{-\langle Ax,x\rangle}} \ge \left[S\left(\frac{M}{m}\right)\right]^{-\langle Ax,x\rangle}$$

and the inequality (2.10). □

3. Further Lower and Upper Bounds

We start to the following logarithmic inequalities:

Lemma 1. *For any a, $b > 0$ we have*

$$\frac{1}{2}\left(1-\frac{\min\{a,b\}}{\max\{a,b\}}\right)^2 = \frac{1}{2}\frac{(b-a)^2}{\max^2\{a,b\}} \le \frac{b-a}{a} - \ln b + \ln a \le \frac{1}{2}\frac{(b-a)^2}{\min^2\{a,b\}} = \frac{1}{2}\left(\frac{\max\{a,b\}}{\min\{a,b\}}-1\right)^2. \tag{3.1}$$

Proof. It is easy to see that

$$\int_a^b \frac{b-t}{t^2}dt = \frac{b-a}{a} - \ln b + \ln a \tag{3.2}$$

for any a, $b > 0$.

If $b > a$, then

$$\frac{1}{2}\frac{(b-a)^2}{a^2} \geq \int_a^b \frac{b-t}{t^2}dt \geq \frac{1}{2}\frac{(b-a)^2}{b^2}. \tag{3.3}$$

If $a > b$ then

$$\int_a^b \frac{b-t}{t^2}dt = -\int_b^a \frac{b-t}{t^2}dt = \int_b^a \frac{t-b}{t^2}dt$$

and

$$\frac{1}{2}\frac{(b-a)^2}{b^2} \geq \int_b^a \frac{t-b}{t^2}dt \geq \frac{1}{2}\frac{(b-a)^2}{a^2}. \tag{3.4}$$

Therefore, by (3.3) and (3.4) we have for any a, $b > 0$ that

$$\int_a^b \frac{b-t}{t^2}dt \geq \frac{1}{2}\frac{(b-a)^2}{\max^2\{a,b\}} = \frac{1}{2}\left(\frac{\min\{a,b\}}{\max\{a,b\}} - 1\right)^2$$

and

$$\int_a^b \frac{b-t}{t^2}dt \leq \frac{1}{2}\frac{(b-a)^2}{\min^2\{a,b\}} = \frac{1}{2}\left(\frac{\max\{a,b\}}{\min\{a,b\}} - 1\right)^2.$$

By the representation (3.2) we then get the desired result (3.1). □

When some bounds for a, b are provided, then we have:

Corollary 2. *Assume that a, $b \in [m, M] \subset (0,\infty)$, then we have the local bounds*

$$\frac{1}{2}\frac{(b-a)^2}{M^2} \leq \frac{b-a}{a} - \ln b + \ln a \leq \frac{1}{2}\frac{(b-a)^2}{m^2} \tag{3.5}$$

and

$$\frac{1}{2}\frac{(b-a)^2}{M^2} \leq \ln b - \ln a - \frac{b-a}{b} \leq \frac{1}{2}\frac{(b-a)^2}{m^2}. \tag{3.6}$$

Remark 1. *If we take in (3.1) $a = 1$ and $b = u \in (0,\infty)$, then we get*

$$\begin{aligned} \frac{1}{2}\left(1 - \frac{\min\{1,u\}}{\max\{1,u\}}\right)^2 &= \frac{1}{2}\frac{(u-1)^2}{\max^2\{1,u\}} \\ &\leq u - 1 - \ln u \\ &\leq \frac{1}{2}\frac{(u-1)^2}{\min^2\{1,u\}} = \frac{1}{2}\left(\frac{\max\{1,u\}}{\min\{1,u\}} - 1\right)^2 \end{aligned} \tag{3.7}$$

and if we take $a = u$ and $b = 1$, then we also get

$$\begin{aligned} \frac{1}{2}\left(1 - \frac{\min\{1,u\}}{\max\{1,u\}}\right)^2 &= \frac{1}{2}\frac{(u-1)^2}{\max^2\{1,u\}} \\ &\leq \ln u - \frac{u-1}{u} \\ &\leq \frac{1}{2}\frac{(u-1)^2}{\min^2\{1,u\}} = \frac{1}{2}\left(\frac{\max\{1,u\}}{\min\{1,u\}} - 1\right)^2. \end{aligned} \tag{3.8}$$

If $u \in [k, K] \subset (0,\infty)$, then by analyzing all possible locations of the interval $[k, K]$ and 1 we have

$$\min\{1,k\} \leq \min\{1,u\} \leq \min\{1,K\}$$

and

$$\max\{1,k\}\leq\max\{1,u\}\leq\max\{1,K\}.$$

By (3.7) and (3.8) we get the *local bounds*

$$\frac{1}{2}\frac{(u-1)^2}{\max^2\{1,K\}}\leq u-1-\ln u\leq\frac{1}{2}\frac{(u-1)^2}{\min^2\{1,k\}} \tag{3.9}$$

and

$$\frac{1}{2}\frac{(u-1)^2}{\max^2\{1,K\}}\leq\ln u-\frac{u-1}{u}\leq\frac{1}{2}\frac{(u-1)^2}{\min^2\{1,k\}} \tag{3.10}$$

for any $u\in[k,K]$.

Theorem 3. *Assume that* $0<m\leq A\leq M$*, then for all* $x\in H$, $\|x\|=1$

$$\begin{aligned} 1 &\leq \exp\left[\frac{1}{2}\frac{m\left\langle (A-1)^2x,x\right\rangle}{\max^2\{1,M\}}\right] \\ &\leq \exp\left[\frac{1}{2}\frac{\left\langle A(A-1)^2x,x\right\rangle}{\max^2\{1,M\}}\right]\leq\frac{\eta_x(A)}{\exp\left[\langle Ax,x\rangle-\langle A^2x,x\rangle\right]} \\ &\leq \exp\left[\frac{1}{2}\frac{\left\langle A(A-1)^2x,x\right\rangle}{\min^2\{1,m\}}\right]\leq\exp\left[\frac{1}{2}\frac{M\left\langle (A-1)^2x,x\right\rangle}{\min^2\{1,m\}}\right] \end{aligned} \tag{3.11}$$

and

$$\begin{aligned} 1 &\leq \exp\left[\frac{1}{2}\frac{m\left\langle (A-1)^2x,x\right\rangle}{\max^2\{1,M\}}\right] \\ &\leq \exp\left[\frac{1}{2}\frac{\left\langle A(A-1)^2x,x\right\rangle}{\max^2\{1,M\}}\right]\leq\frac{\exp\left(-\langle Ax,x\rangle+1\right)}{\eta_x(A)} \\ &\leq \exp\left[\frac{1}{2}\frac{\left\langle A(A-1)^2x,x\right\rangle}{\min^2\{1,m\}}\right]\leq\exp\left[\frac{1}{2}\frac{M\left\langle (A-1)^2x,x\right\rangle}{\min^2\{1,m\}}\right]. \end{aligned} \tag{3.12}$$

Proof. If we multiply by $u\in[k,K]\subset(0,\infty)$ in (3.9) and (3.10), then we get

$$\begin{aligned} \frac{1}{2}\frac{k(u-1)^2}{\max^2\{1,K\}} &\leq \frac{1}{2}\frac{u(u-1)^2}{\max^2\{1,K\}}\leq u^2-u-u\ln u \\ &\leq \frac{1}{2}\frac{u(u-1)^2}{\min^2\{1,k\}}\leq\frac{1}{2}\frac{K(u-1)^2}{\min^2\{1,k\}} \end{aligned}$$

and

$$\begin{aligned} \frac{1}{2}\frac{k(u-1)^2}{\max^2\{1,K\}} &\leq \frac{1}{2}\frac{u(u-1)^2}{\max^2\{1,K\}}\leq u\ln u-u+1 \\ &\leq \frac{1}{2}\frac{u(u-1)^2}{\min^2\{1,k\}}\leq\frac{1}{2}\frac{K(u-1)^2}{\min^2\{1,k\}} \end{aligned}$$

for any $u\in[k,K]$.

Using the continuous functional calculus for selfadjoint operators A with $\mathrm{Sp}(A) \subseteq [m, M]$, we get

$$\frac{1}{2}\frac{m(A-1)^2}{\max^2\{1,M\}} \le \frac{1}{2}\frac{A(A-1)^2}{\max^2\{1,M\}} \le A^2 - A - A\ln A \le \frac{1}{2}\frac{A(A-1)^2}{\min^2\{1,m\}} \le \frac{1}{2}\frac{M(A-1)^2}{\min^2\{1,m\}}$$

and

$$\frac{1}{2}\frac{m(A-1)^2}{\max^2\{1,M\}} \le \frac{1}{2}\frac{A(A-1)^2}{\max^2\{1,M\}} \le A\ln A - A + 1 \le \frac{1}{2}\frac{A(A-1)^2}{\min^2\{1,m\}} \le \frac{1}{2}\frac{M(A-1)^2}{\min^2\{1,m\}}.$$

If we take the inner product over $x \in H$, $\|x\| = 1$, then we get

$$\begin{aligned}\frac{1}{2}\frac{m\left\langle (A-1)^2 x, x\right\rangle}{\max^2\{1,M\}} &\le \frac{1}{2}\frac{\left\langle A(A-1)^2 x, x\right\rangle}{\max^2\{1,M\}} \\ &\le \left\langle A^2x, x\right\rangle - \langle Ax, x\rangle - \langle A\ln Ax, x\rangle \\ &\le \frac{1}{2}\frac{\left\langle A(A-1)^2 x, x\right\rangle}{\min^2\{1,m\}} \le \frac{1}{2}\frac{M\left\langle (A-1)^2 x, x\right\rangle}{\min^2\{1,m\}}\end{aligned} \tag{3.13}$$

and

$$\begin{aligned}\frac{1}{2}\frac{m\left\langle (A-1)^2 x, x\right\rangle}{\max^2\{1,M\}} &\le \frac{1}{2}\frac{\left\langle A(A-1)^2 x, x\right\rangle}{\max^2\{1,M\}} \\ &\le \langle A\ln Ax, x\rangle - \langle Ax, x\rangle + 1 \\ &\le \frac{1}{2}\frac{\left\langle A(A-1)^2 x, x\right\rangle}{\min^2\{1,m\}} \le \frac{1}{2}\frac{M\left\langle (A-1)^2 x, x\right\rangle}{\min^2\{1,m\}}.\end{aligned} \tag{3.14}$$

By taking the exponential in (3.13) and (3.14) we obtain

$$\begin{aligned}&\exp\left[\frac{1}{2}\frac{m\left\langle (A-1)^2 x, x\right\rangle}{\max^2\{1,M\}}\right] \\ &\le \exp\left[\frac{1}{2}\frac{\left\langle A(A-1)^2 x, x\right\rangle}{\max^2\{1,M\}}\right] \le \frac{\exp\left[-\langle A\ln Ax, x\rangle\right]}{\exp\left[\langle Ax, x\rangle - \langle A^2x, x\rangle\right]} \\ &\le \exp\left[\frac{1}{2}\frac{\left\langle A(A-1)^2 x, x\right\rangle}{\min^2\{1,m\}}\right] \le \exp\left[\frac{1}{2}\frac{M\left\langle (A-1)^2 x, x\right\rangle}{\min^2\{1,m\}}\right]\end{aligned}$$

and

$$\begin{aligned}
&\exp\left[\frac{1}{2}\frac{m\left\langle (A-1)^2 x,x\right\rangle}{\max^2\{1,M\}}\right] \\
&\leq \exp\left[\frac{1}{2}\frac{\left\langle A(A-1)^2 x,x\right\rangle}{\max^2\{1,M\}}\right] \leq \frac{\exp\left(-\langle Ax,x\rangle+1\right)}{\exp\left[-\langle A\ln Ax,x\rangle\right]} \\
&\leq \exp\left[\frac{1}{2}\frac{\left\langle A(A-1)^2 x,x\right\rangle}{\min^2\{1,m\}}\right] \leq \exp\left[\frac{1}{2}\frac{M\left\langle (A-1)^2 x,x\right\rangle}{\min^2\{1,m\}}\right],
\end{aligned}$$

and the theorem is proved. □

Theorem 4. *With the assumptions of Theorem 3, we have for all $a\in[m,M]$ that*

$$\begin{aligned}
1 &\leq \exp\left[\frac{m}{2M^2}\left\langle (A-a)^2 x,x\right\rangle\right] \\
&\leq \exp\left[\frac{1}{2M^2}\left\langle A(A-a)^2 x,x\right\rangle\right] \\
&\leq \frac{\eta_x(A)}{\exp\left[(1-\ln a)\langle Ax,x\rangle-\frac{1}{a}\langle A^2x,x\rangle\right]} \\
&\leq \exp\left[\frac{1}{2m^2}\left\langle A(A-a)^2 x,x\right\rangle\right] \leq \exp\left[\frac{M}{2m^2}\left\langle (A-a)^2 x,x\right\rangle\right],
\end{aligned} \tag{3.15}$$

where $x\in H$, $\|x\|=1$.

Proof. From (3.5) and using the functional calculus, we get

$$\frac{1}{2}\frac{(A-a)^2}{M^2}\leq\frac{A-a}{a}-\ln A+\ln a\leq\frac{1}{2}\frac{(A-a)^2}{m^2}$$

for all operator A with $0<m\leq A\leq M$ and all real number $a\in[m,M]$.

By taking the inner product over $y\in H$, $\|y\|=1$, we get

$$\begin{aligned}
\frac{1}{2M^2}\left\langle (A-a)^2 y,y\right\rangle &\leq \frac{1}{a}\langle Ay,y\rangle-\langle\ln Ay,y\rangle+\ln a-1 \\
&\leq \frac{1}{2m^2}\left\langle (A-a)^2 y,y\right\rangle.
\end{aligned}$$

Further, if we take $y=\frac{A^{1/2}x}{\|A^{1/2}x\|}$, $x\in H$, $\|x\|=1$, then we get

$$\begin{aligned}
&\frac{1}{2M^2}\left\langle (A-a)^2\frac{A^{1/2}x}{\|A^{1/2}x\|},\frac{A^{1/2}x}{\|A^{1/2}x\|}\right\rangle \\
&\leq \frac{1}{a}\left\langle A\frac{A^{1/2}x}{\|A^{1/2}x\|},\frac{A^{1/2}x}{\|A^{1/2}x\|}\right\rangle-\left\langle \ln A\frac{A^{1/2}x}{\|A^{1/2}x\|},\frac{A^{1/2}x}{\|A^{1/2}x\|}\right\rangle+\ln a-1 \\
&\leq \frac{1}{2m^2}\left\langle (A-a)^2\frac{A^{1/2}x}{\|A^{1/2}x\|},\frac{A^{1/2}x}{\|A^{1/2}x\|}\right\rangle,
\end{aligned}$$

namely

$$\begin{aligned}&\frac{1}{2M^2\langle Ax,x\rangle}\left\langle A(A-a)^2x,x\right\rangle\\&\le\frac{1}{a\langle Ax,x\rangle}\left\langle A^2x,x\right\rangle-\frac{1}{\langle Ax,x\rangle}\langle A\ln Ax,x\rangle+\ln a-1\\&\le\frac{1}{2m^2\langle Ax,x\rangle}\left\langle A(A-a)^2x,x\right\rangle.\end{aligned}$$

Since

$$\left\langle A(A-a)^2x,x\right\rangle\ge m\left\langle (A-a)^2x,x\right\rangle$$

and

$$\left\langle A(A-a)^2x,x\right\rangle\le M\left\langle (A-a)^2x,x\right\rangle,$$

hence we get the following chain of inequalities

$$\begin{aligned}&\frac{m}{2M^2}\left\langle (A-a)^2x,x\right\rangle\\&\le\frac{1}{2M^2}\left\langle A(A-a)^2x,x\right\rangle\\&\le\frac{1}{a}\left\langle A^2x,x\right\rangle-\langle A\ln Ax,x\rangle+(\ln a-1)\langle Ax,x\rangle\\&\le\frac{1}{2m^2}\left\langle A(A-a)^2x,x\right\rangle\le\frac{M}{2m^2}\left\langle A(A-a)^2x,x\right\rangle.\end{aligned}$$

By taking the exponential, we derive

$$\begin{aligned}&\exp\left[\frac{m}{2M^2}\left\langle (A-a)^2x,x\right\rangle\right]\\&\le\exp\left[\frac{1}{2M^2}\left\langle A(A-a)^2x,x\right\rangle\right]\\&\le\exp\left[\frac{1}{a}\left\langle A^2x,x\right\rangle-\langle A\ln Ax,x\rangle+(\ln a-1)\langle Ax,x\rangle\right]\\&\le\exp\left[\frac{1}{2m^2}\left\langle A(A-a)^2x,x\right\rangle\right]\le\exp\left[\frac{M}{2m^2}\left\langle (A-a)^2x,x\right\rangle\right]\end{aligned}$$

and the inequality (3.15) is proved. □

Corollary 3. *With the assumptions of Theorem 3 we have*

$$\begin{aligned}1&\le\exp\left[\frac{m}{2M^2}\left(\left\langle A^2x,x\right\rangle-\langle Ax,x\rangle^2\right)\right]\\&\le\exp\left[\frac{1}{2M^2}\left\langle A(A-\langle Ax,x\rangle)^2x,x\right\rangle\right]\\&\le\frac{\eta_x(A)}{\exp\left[(1-\ln\langle Ax,x\rangle)\langle Ax,x\rangle-\frac{\langle A^2x,x\rangle}{\langle Ax,x\rangle}\right]}\\&\le\exp\left[\frac{1}{2m^2}\left\langle A(A-\langle Ax,x\rangle)^2x,x\right\rangle\right]\\&\le\exp\left[\frac{M}{2m^2}\left(\left\langle A^2x,x\right\rangle-\langle Ax,x\rangle^2\right)\right]\le\exp\left[\frac{1}{8}M\left(\frac{M}{m}-1\right)^2\right]\end{aligned}\tag{3.16}$$

for $x\in H$, $\|x\|=1$.

Proof. It follows by taking $a = \langle Ax, x\rangle \in [m, M]$ for $x \in H$, $\|x\| = 1$ in (3.15) and observing that

$$\left\langle (A - \langle Ax, x\rangle)^2 x, x\right\rangle = \|Ax - \langle Ax, x\rangle x\|^2 = \|Ax\|^2 - \langle Ax, x\rangle^2 = \left\langle A^2 x, x\right\rangle - \langle Ax, x\rangle^2 .$$

Also, by using the Schwarz's reverse inequality

$$\left\langle A^2 x, x\right\rangle - \langle Ax, x\rangle^2 \le \frac{1}{4}(M - m)^2 ,$$

we obtain the last part of (3.16). □

Corollary 4. *With the assumptions of Theorem 3 we have*

$$\begin{aligned}
1 &\le \exp\left[\frac{m}{2M^2}\left\langle \left(A - \frac{m+M}{2}\right)^2 x, x\right\rangle\right] \\
&\le \exp\left[\frac{1}{2M^2}\left\langle A\left(A - \frac{m+M}{2}\right)^2 x, x\right\rangle\right] \\
&\le \frac{\eta_x(A)}{\exp\left[\left(1 - \ln\frac{m+M}{2}\right)\langle Ax, x\rangle - \frac{2}{m+M}\left\langle A^2 x, x\right\rangle\right]} \\
&\le \exp\left[\frac{1}{2m^2}\left\langle A\left(A - \frac{m+M}{2}\right)^2 x, x\right\rangle\right] \\
&\le \exp\left[\frac{M}{2m^2}\left\langle \left(A - \frac{m+M}{2}\right)^2 x, x\right\rangle\right] \le \exp\left[\frac{1}{8}M\left(\frac{M}{m} - 1\right)^2\right]
\end{aligned} \tag{3.17}$$

for $x \in H$, $\|x\| = 1$.

Finally, we have

Theorem 5. *Assume that $0 < m \le A \le M$, then for all $a \in [m, M]$ and $x \in H$, $\|x\| = 1$*

$$\begin{aligned}
1 &\le \exp\left[\frac{m}{2M^3}\left\langle (A - a)^2 x, x\right\rangle\right] \\
&\le \exp\left[\frac{1}{2M^2\langle Ax, x\rangle}\left\langle A(A - a)^2 x, x\right\rangle\right] \le \frac{\exp\left(\frac{a - \langle Ax, x\rangle}{\langle Ax, x\rangle}\right)}{a\eta_x(A)} \\
&\le \exp\left[\frac{1}{2m^2\langle Ax, x\rangle}\left\langle A(A - a)^2 x, x\right\rangle\right] \le \exp\left[\frac{M}{2m^3}\left\langle (A - a)^2 x, x\right\rangle\right].
\end{aligned} \tag{3.18}$$

Proof. From (3.6) and using the functional calculus, we get

$$\frac{1}{2}\frac{(A - a)^2}{M^2} \le \ln A + aA^{-1} - \ln a - 1 \le \frac{1}{2}\frac{(A - a)^2}{m^2}$$

for all operator A with $0 < m \le A \le M$ and all real number $a \in [m, M]$.

By taking the inner product over $y \in H$, $\|y\| = 1$, we get

$$\frac{1}{2M^2}\left\langle (A - a)^2 y, y\right\rangle \le \langle \ln Ay, y\rangle + a\left\langle A^{-1} y, y\right\rangle - \ln a - 1 \le \frac{1}{2m^2}\left\langle (A - a)^2 y, y\right\rangle .$$

Further, if we take $y = \frac{A^{1/2}x}{\|A^{1/2}x\|}$, $x \in H$, $\|x\| = 1$, then we get

$$\begin{aligned}
&\frac{1}{2M^2}\left\langle (A-a)^2 \frac{A^{1/2}x}{\|A^{1/2}x\|}, \frac{A^{1/2}x}{\|A^{1/2}x\|}\right\rangle \\
&\leq \left\langle \ln A \frac{A^{1/2}x}{\|A^{1/2}x\|}, \frac{A^{1/2}x}{\|A^{1/2}x\|}\right\rangle + a\left\langle A^{-1} \frac{A^{1/2}x}{\|A^{1/2}x\|}, \frac{A^{1/2}x}{\|A^{1/2}x\|}\right\rangle - \ln a - 1 \\
&\leq \frac{1}{2m^2}\left\langle (A-a)^2 \frac{A^{1/2}x}{\|A^{1/2}x\|}, \frac{A^{1/2}x}{\|A^{1/2}x\|}\right\rangle
\end{aligned}$$

namely

$$\begin{aligned}
&\frac{1}{2M^2\langle Ax,x\rangle}\left\langle A(A-a)^2 x, x\right\rangle \\
&\leq \langle A\ln Ax, x\rangle + \frac{a}{\langle Ax,x\rangle} - \ln a - 1 \leq \frac{1}{2m^2\langle Ax,x\rangle}\left\langle A(A-a)^2 x, x\right\rangle
\end{aligned} \tag{3.19}$$

for $x \in H$, $\|x\| = 1$.

By taking the exponential in (3.19) we get

$$\begin{aligned}
&\exp\left[\frac{1}{2M^2\langle Ax,x\rangle}\left\langle A(A-a)^2 x, x\right\rangle\right] \\
&\leq \frac{\exp\left(\frac{a}{\langle Ax,x\rangle} - \ln a - 1\right)}{\exp\left[-\langle A\ln Ax, x\rangle\right]} \leq \exp\left[\frac{1}{2m^2\langle Ax,x\rangle}\left\langle A(A-a)^2 x, x\right\rangle\right]
\end{aligned}$$

namely

$$\begin{aligned}
&\exp\left[\frac{1}{2M^2\langle Ax,x\rangle}\left\langle A(A-a)^2 x, x\right\rangle\right] \\
&\leq \frac{\exp\left(\frac{a-\langle Ax,x\rangle}{\langle Ax,x\rangle}\right)}{a\exp\left[-\langle A\ln Ax, x\rangle\right]} \leq \exp\left[\frac{1}{2m^2\langle Ax,x\rangle}\left\langle A(A-a)^2 x, x\right\rangle\right]
\end{aligned}$$

for $x \in H$, $\|x\| = 1$.

Observe that

$$\exp\left[\frac{m}{2M^3}\left\langle (A-a)^2 x, x\right\rangle\right] \leq \exp\left[\frac{1}{2M^2\langle Ax,x\rangle}\left\langle A(A-a)^2 x, x\right\rangle\right]$$

and

$$\exp\left[\frac{1}{2m^2\langle Ax,x\rangle}\left\langle A(A-a)^2 x, x\right\rangle\right] \leq \exp\left[\frac{M}{2m^3}\left\langle (A-a)^2 x, x\right\rangle\right],$$

which proves (3.18). □

Corollary 5. *Assume that* $0 < m \leq A \leq M$*, then*

$$\begin{aligned}
1 &\leq \exp\left[\frac{m}{2M^3}\left(\left\langle A^2x,x\right\rangle - \langle Ax,x\rangle^2\right)\right] \\
&\leq \exp\left[\frac{1}{2M^2\langle Ax,x\rangle}\left\langle A\left(A-\langle Ax,x\rangle\right)^2 x,x\right\rangle\right] \\
&\leq \frac{\langle Ax,x\rangle^{-1}}{\eta_x(A)} \\
&\leq \exp\left[\frac{1}{2m^2\langle Ax,x\rangle}\left\langle A\left(A-\langle Ax,x\rangle\right)^2 x,x\right\rangle\right] \\
&\leq \exp\left[\frac{M}{2m^3}\left(\left\langle A^2x,x\right\rangle - \langle Ax,x\rangle^2\right)\right] \leq \exp\left[\frac{1}{8}\frac{M}{m}\left(\frac{M}{m}-1\right)^2\right]
\end{aligned} \tag{3.20}$$

for all $x \in H$, $\|x\| = 1$.

The proof follows by taking $a = \langle Ax,x\rangle$ in (3.18).

References

[1] T. Furuta, J. Mićić Hot, J. Pečarić and Y. Seo, *Mond-Pečarić Method in Operator Inequalities. Inequalities for Bounded Selfadjoint Operators on a Hilbert Space*, Element, Zagreb, 2005.

[2] J. I. Fujii and Y. Seo, Determinant for positive operators, *Sci. Math.,* **1** (1998), 153–156.

[3] J. I. Fujii, S. Izumino and Y. Seo, Determinant for positive operators and Specht's Theorem, *Sci. Math.*, **1** (1998), 307–310.

[4] T. Furuta, Precise lower bound of $f(A) - f(B)$ for $A > B > 0$ and non-constant operator monotone function f on $[0,\infty)$. *J. Math. Inequal.* **9** (2015), no. 1, 47–52.

[5] S. Hiramatsu and Y. Seo, Determinant for positive operators and Oppenheim's inequality, *J. Math. Inequal.*, Volume 15 (2021), Number 4, 1637–1645.

[6] J. Mićić, Y. Seo, S.-E. Takahasi and M. Tominaga, Inequalities of Furuta and Mond-Pečarić, *Math. Ineq. Appl.*, **2**(1999), 83-111.

[7] W. Specht, Zer Theorie der elementaren Mittel, *Math. Z.* **74** (1960), pp. 91-98.

[8] H. Zuo, G. Duan, Some inequalities of operator monotone functions. *J. Math. Inequal.* **8** (2014), no. 4, 777–781.

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