INEQUALITIES FOR THE NORMALIZED ENTROPIC DETERMINANT OF POSITIVE OPERATORS IN HILBERT SPACES VIA KANTOROVICH CONSTANT

SILVESTRU SEVER DRAGOMIR[1,2]

Abstract. For positive invertible operators A on a Hilbert space H and a fixed unit vector $x \in H$, define the normalized entropic determinant by $\eta_x(A) := \exp[-\langle A \ln Ax, x\rangle]$. In this paper we show among others that, if A satisfies the condition $0 < m \leq A \leq M$, then

$$1 \geq K\left(\frac{M}{m}\right)^{-\left[\frac{1}{2}-\frac{1}{(M-m)\langle Ax,x\rangle}\langle\left|A^2-\frac{1}{2}(m+M)A\right|x,x\rangle\right]}$$
$$\geq \frac{\eta_x(A)}{\left(m^{\frac{M-\frac{\langle A^2x,x\rangle}{\langle Ax,x\rangle}}{M-m}}M^{\frac{\frac{\langle A^2x,x\rangle}{\langle Ax,x\rangle}-m}{M-m}}\right)^{-1}}$$
$$\geq K\left(\frac{M}{m}\right)^{-\left[\frac{1}{2}+\frac{1}{(M-m)\langle Ax,x\rangle}\langle\left|A^2-\frac{1}{2}(m+M)A\right|x,x\rangle\right]} \geq \left[K\left(\frac{M}{m}\right)\right]^{-1}$$

for $x \in H$, $\|x\| = 1$, where $K(h) := \frac{(h+1)^2}{4h}$, $h > 0$ is Kantorovich constant.

1. Introduction

Let $B(H)$ be the space of all bounded linear operators on a Hilbert space H, and I stands for the identity operator on H. An operator A in $B(H)$ is said to be positive (in symbol: $A \geq 0$) if $\langle Ax, x\rangle \geq 0$ for all $x \in H$. In particular, $A > 0$ means that A is positive and invertible. For a pair A, B of selfadjoint operators the order relation $A \geq B$ means as usual that $A - B$ is positive.

In 1998, Fujii et al. [3], [4], introduced the *normalized determinant* $\Delta_x(A)$ for positive invertible operators A on a Hilbert space H and a fixed unit vector $x \in H$, namely $\|x\| = 1$, defined by $\Delta_x(A) := \exp\langle \ln Ax, x\rangle$ and discussed it as a continuous geometric mean and observed some inequalities around the determinant from this point of view.

Some of the fundamental properties of normalized determinant are as follows, [3].

For each unit vector $x \in H$, see also [7], we have:

(i) *continuity*: the map $A \to \Delta_x(A)$ is norm continuous;
(ii) *bounds*: $\langle A^{-1}x, x\rangle^{-1} \leq \Delta_x(A) \leq \langle Ax, x\rangle$;
(iii) *continuous mean*: $\langle A^p x, x\rangle^{1/p} \downarrow \Delta_x(A)$ for $p \downarrow 0$ and $\langle A^p x, x\rangle^{1/p} \uparrow \Delta_x(A)$ for $p \uparrow 0$;
(iv) *power equality:* $\Delta_x(A^t) = \Delta_x(A)^t$ for all $t > 0$;

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(v) *homogeneity*: $\Delta_x(tA) = t\Delta_x(A)$ and $\Delta_x(tI) = t$ for all $t > 0$;
(vi) *monotonicity*: $0 < A \leq B$ implies $\Delta_x(A) \leq \Delta_x(B)$;
(vii) *multiplicativity*: $\Delta_x(AB) = \Delta_x(A)\Delta_x(B)$ for commuting A and B;
(viii) *Ky Fan type inequality*: $\Delta_x((1-\alpha)A + \alpha B) \geq \Delta_x(A)^{1-\alpha}\Delta_x(B)^{\alpha}$ for $0 < \alpha < 1$.

We define the logarithmic mean of two positive numbers a, b by

$$L(a,b) := \begin{cases} \frac{b-a}{\ln b - \ln a} \text{ if } b \neq a, \\ \\ a \text{ if } b = a. \end{cases}$$

In [3] the authors obtained the following additive reverse inequality for the operator A which satisfy the condition $0 < mI \leq A \leq MI$, where m, M are positive numbers,

$$0 \leq \langle Ax, x\rangle - \Delta_x(A) \leq L(m,M)\left[\ln L(m,M) + \frac{M\ln m - m\ln M}{M-m} - 1\right] \tag{1.1}$$

for all $x \in H$, $\|x\| = 1$.

The famous *Young inequality* for scalars says that if $a, b > 0$ and $\nu \in [0,1]$, then

$$a^{1-\nu}b^{\nu} \leq (1-\nu)a + \nu b \tag{1.2}$$

with equality if and only if $a = b$. The inequality (1.2) is also called *ν-weighted arithmetic-geometric mean inequality*.

We recall that *Specht's ratio* is defined by [9]

$$S(h) := \begin{cases} \frac{h^{\frac{1}{h-1}}}{e\ln\left(h^{\frac{1}{h-1}}\right)} \text{ if } h \in (0,1)\cup(1,\infty) \\ \\ 1 \text{ if } h = 1. \end{cases} \tag{1.3}$$

It is well known that $\lim_{h\to 1} S(h) = 1$, $S(h) = S\left(\frac{1}{h}\right) > 1$ for $h > 0$, $h \neq 1$. The function is decreasing on $(0,1)$ and increasing on $(1,\infty)$.

In [4], the authors obtained the following multiplicative reverse inequality as well

$$1 \leq \frac{\langle Ax, x\rangle}{\Delta_x(A)} \leq S\left(\frac{M}{m}\right) \tag{1.4}$$

for $0 < mI \leq A \leq MI$ and $x \in H$, $\|x\| = 1$.

Since $0 < M^{-1}I \leq A^{-1} \leq m^{-1}I$, then by (1.4) for A^{-1} we get

$$1 \leq \frac{\langle A^{-1}x, x\rangle}{\Delta_x(A^{-1})} \leq S\left(\frac{m^{-1}}{M^{-1}}\right) = S\left(\left(\frac{m}{M}\right)^{-1}\right) = S\left(\frac{M}{m}\right),$$

which is equivalent to

$$1 \leq \frac{\Delta_x(A)}{\langle A^{-1}x, x\rangle^{-1}} \leq S\left(\frac{M}{m}\right) \tag{1.5}$$

for $x \in H$, $\|x\| = 1$.

We consider the *Kantorovich's constant* defined by

$$K(h) := \frac{(h+1)^2}{4h},\ h > 0. \tag{1.6}$$

The function K is decreasing on $(0,1)$ and increasing on $[1,\infty)$, $K(h) \geq 1$ for any $h > 0$ and $K(h) = K\left(\frac{1}{h}\right)$ for any $h > 0$.

The following multiplicative refinement and reverse of Young inequality in terms of Kantorovich's constant holds

$$\left(a^{1-\nu}b^{\nu}\leq\right)K^{r}\left(\frac{a}{b}\right)a^{1-\nu}b^{\nu}\leq(1-\nu)\,a+\nu b\leq K^{R}\left(\frac{a}{b}\right)a^{1-\nu}b^{\nu}\tag{1.7}$$

where $a,b>0$, $\nu\in[0,1]$, $r=\min\{1-\nu,\nu\}$ and $R=\max\{1-\nu,\nu\}$.

The first inequality in (1.7) was obtained by Zuo et al. in [12] while the second by Liao et al. [8].

For the entropy function $\eta(t)=-t\ln t$, $t>0$, the operator entropy has the following expression:

$$\eta(A)=-A\ln A$$

for positive A.

For $x\in H$, $\|x\|=1$, we define the *normalized entropic determinant* $\eta_x(A)$ by

$$\eta_x(A):=\exp\left(-\langle A\ln Ax,x\rangle\right)=\exp\langle\eta(A)\,x,x\rangle\,.\tag{1.8}$$

Let $x\in H$, $\|x\|=1$. Observe that the map $A\to\eta_x(A)$ is *norm continuous* and since

$$\begin{aligned}&\exp\left(-\langle tA\ln(tA)\,x,x\rangle\right)\\&=\exp\left(-\langle tA(\ln t+\ln A)\,x,x\rangle\right)=\exp\left(-\langle(tA\ln t+tA\ln A)\,x,x\rangle\right)\\&=\exp\left(-\langle Ax,x\rangle\,t\ln t\right)\exp\left(-t\langle A\ln Ax,x\rangle\right)\\&=\exp\ln\left(t^{-\langle Ax,x\rangle t}\right)\left[\exp\left(-\langle A\ln Ax,x\rangle\right)\right]^{-t},\end{aligned}$$

hence

$$\eta_x(tA)=t^{-t\langle Ax,x\rangle}\left[\eta_x(A)\right]^{-t}\tag{1.9}$$

for $t>0$ and $A>0$.

Observe also that

$$\eta_x(I)=1\text{ and }\eta_x(tI)=t^{-t}\tag{1.10}$$

for $t>0$.

Motivated by the above results, in this paper we provide, among others, some bounds for the quantities

$$\frac{\eta_x(A)}{\left(m^{\frac{M-\frac{\langle A^2x,x\rangle}{\langle Ax,x\rangle}}{M-m}}M^{\frac{\frac{\langle A^2x,x\rangle}{\langle Ax,x\rangle}-m}{M-m}}\right)^{-1}}$$

and

$$\frac{\left[\eta_x(A)\right]^{\langle Ax,x\rangle^{-1}}}{\left(m^{\frac{M-\frac{\langle A^2x,x\rangle}{\langle Ax,x\rangle}}{M-m}}M^{\frac{\frac{\langle A^2x,x\rangle}{\langle Ax,x\rangle}-m}{M-m}}\right)^{-1}},$$

where $0<m\leq A\leq M$ for positive numbers m, M and $x\in H$, $\|x\|=1$.

2. Main Results

Our first main result is as follows:

Theorem 1. *If* $0 < m \leq A \leq M$ *for positive numbers* m, M, *then*

$$1 \geq K\left(\frac{M}{m}\right)^{-\left[\frac{1}{2}-\frac{1}{(M-m)\langle Ax,x\rangle}\left\langle\left|A^2-\frac{1}{2}(m+M)A\right|x,x\right\rangle\right]} \geq \frac{\eta_x(A)}{\left(m^{\frac{M-\frac{\langle A^2x,x\rangle}{\langle Ax,x\rangle}}{M-m}}M^{\frac{\frac{\langle A^2x,x\rangle}{\langle Ax,x\rangle}-m}{M-m}}\right)^{-1}} \geq K\left(\frac{M}{m}\right)^{-\left[\frac{1}{2}+\frac{1}{(M-m)\langle Ax,x\rangle}\left\langle\left|A^2-\frac{1}{2}(m+M)A\right|x,x\right\rangle\right]} \geq \left[K\left(\frac{M}{m}\right)\right]^{-1} \tag{2.1}$$

for $x \in H$, $\|x\| = 1$.

Proof. Assume that $t \in [m, M]$ and consider $\nu = \frac{t-m}{M-m} \in [0,1]$. Then

$$\min\{1-\nu,\nu\} = \frac{1}{2} - \left|\nu - \frac{1}{2}\right| = \frac{1}{2} - \left|\frac{t-m}{M-m} - \frac{1}{2}\right| = \frac{1}{2} - \frac{1}{M-m}\left|t - \frac{1}{2}(m+M)\right|,$$

$$\max\{1-\nu,\nu\} = \frac{1}{2} + \left|\nu - \frac{1}{2}\right| = \frac{1}{2} + \left|\frac{t-m}{M-m} - \frac{1}{2}\right| = \frac{1}{2} + \frac{1}{M-m}\left|t - \frac{1}{2}(m+M)\right|,$$

$$(1-\nu)m + \nu M = \frac{M-t}{M-m}m + \frac{t-m}{M-m}M = t$$

and

$$m^{1-\nu}M^{\nu} = m^{\frac{M-t}{M-m}}M^{\frac{t-m}{M-m}}.$$

By using (1.7) we get

$$m^{\frac{M-t}{M-m}}M^{\frac{t-m}{M-m}} \leq \left[K\left(\frac{M}{m}\right)\right]^{\frac{1}{2}-\frac{1}{M-m}\left|t-\frac{1}{2}(m+M)\right|} m^{\frac{M-t}{M-m}}M^{\frac{t-m}{M-m}} \leq t \leq \left[K\left(\frac{M}{m}\right)\right]^{\frac{1}{2}+\frac{1}{M-m}\left|t-\frac{1}{2}(m+M)\right|} m^{\frac{M-t}{M-m}}M^{\frac{t-m}{M-m}} \tag{2.2}$$

for $t \in [m, M]$.

By taking the log in (2.2) we get

$$\begin{aligned}
&\frac{M-t}{M-m}\ln m+\frac{t-m}{M-m}\ln M \\
&\leq\left[\frac{1}{2}-\frac{1}{M-m}\left|t-\frac{1}{2}(m+M)\right|\right]\ln K\left(\frac{M}{m}\right) \\
&+\frac{M-t}{M-m}\ln m+\frac{t-m}{M-m}\ln M \\
&\leq \ln t\leq\left[\frac{1}{2}+\frac{1}{M-m}\left|t-\frac{1}{2}(m+M)\right|\right]\ln K\left(\frac{M}{m}\right) \\
&+\frac{M-t}{M-m}\ln m+\frac{t-m}{M-m}\ln M \\
&\leq \ln K\left(\frac{M}{m}\right)+\frac{M-t}{M-m}\ln m+\frac{t-m}{M-m}\ln M
\end{aligned}\tag{2.3}$$

for $t\in[m,M]$.

If $0<mI\leq A\leq MI$, then by using the continuous functional calculus for selfadjoint operators we get from (2.3) that

$$\begin{aligned}
&\ln m\frac{MI-A}{M-m}+\ln M\frac{A-mI}{M-m} \\
&\leq\left[\frac{1}{2}I-\frac{1}{M-m}\left|A-\frac{1}{2}(m+M)I\right|\right]\ln K\left(\frac{M}{m}\right) \\
&+\ln m\frac{MI-A}{M-m}+\ln M\frac{A-mI}{M-m} \\
&\leq \ln A\leq\left[\frac{1}{2}I+\frac{1}{M-m}\left|A-\frac{1}{2}(m+M)I\right|\right]\ln K\left(\frac{M}{m}\right) \\
&+\ln m\frac{MI-A}{M-m}+\ln M\frac{A-mI}{M-m} \\
&\leq \ln K\left(\frac{M}{m}\right)I+\ln m\frac{MI-A}{M-m}+\ln M\frac{A-mI}{M-m},
\end{aligned}$$

which is equivalent to

$$\begin{aligned}
&\ln m\frac{M-\langle Ay,y\rangle}{M-m}+\ln M\frac{\langle Ay,y\rangle-m}{M-m} \\
&\leq \ln K\left(\frac{M}{m}\right)\left[\frac{1}{2}-\frac{1}{M-m}\left\langle\left|A-\frac{1}{2}(m+M)I\right|y,y\right\rangle\right] \\
&+\ln m\frac{M-\langle Ay,y\rangle}{M-m}+\ln M\frac{\langle Ay,y\rangle-m}{M-m} \\
&\leq\langle\ln Ay,y\rangle \\
&\leq \ln K\left(\frac{M}{m}\right)\left[\frac{1}{2}+\frac{1}{M-m}\left\langle\left|A-\frac{1}{2}(m+M)I\right|y,y\right\rangle\right] \\
&\ln m\frac{M-\langle Ay,y\rangle}{M-m}+\ln M\frac{\langle Ay,y\rangle-m}{M-m} \\
&\leq \ln K\left(\frac{M}{m}\right)+\ln m\frac{M-\langle Ay,y\rangle}{M-m}+\ln M\frac{\langle Ay,y\rangle-m}{M-m},
\end{aligned}$$

for $y\in H$, $\|y\|=1$.

This inequality can also be written as

$$\begin{aligned}
(2.4)\qquad & \ln\left(m^{\frac{M-\langle Ay,y\rangle}{M-m}}M^{\frac{\langle Ay,y\rangle-m}{M-m}}\right) \\
& \leq \ln\left[K\left(\frac{M}{m}\right)\right]^{\left[\frac{1}{2}-\frac{1}{M-m}\left\langle\left|A-\frac{1}{2}(m+M)I\right|y,y\right\rangle\right]} \\
& + \ln\left(m^{\frac{M-\langle Ay,y\rangle}{M-m}}M^{\frac{\langle Ay,y\rangle-m}{M-m}}\right) \\
& \leq \langle \ln Ay, y\rangle \\
& \leq \ln\left[K\left(\frac{M}{m}\right)\right]^{\left[\frac{1}{2}+\frac{1}{M-m}\left\langle\left|A-\frac{1}{2}(m+M)I\right|y,y\right\rangle\right]} \\
& + \ln\left(m^{\frac{M-\langle Ay,y\rangle}{M-m}}M^{\frac{\langle Ay,y\rangle-m}{M-m}}\right) \\
& \leq \ln K\left(\frac{M}{m}\right) + \ln\left(m^{\frac{M-\langle Ay,y\rangle}{M-m}}M^{\frac{\langle Ay,y\rangle-m}{M-m}}\right)
\end{aligned}$$

for $y \in H$, $\|y\| = 1$.

If we take the exponential in (2.4), then we get

$$\begin{aligned}
(2.5)\qquad & m^{\frac{M-\langle Ay,y\rangle}{M-m}}M^{\frac{\langle Ay,y\rangle-m}{M-m}} \\
& \leq \left(m^{\frac{M-\langle Ay,y\rangle}{M-m}}M^{\frac{\langle Ay,y\rangle-m}{M-m}}\right)K\left(\frac{M}{m}\right)^{\left[\frac{1}{2}-\frac{1}{M-m}\left\langle\left|A-\frac{1}{2}(m+M)I\right|y,y\right\rangle\right]} \\
& \leq \exp\langle \ln Ay, y\rangle \\
& \leq \left(m^{\frac{M-\langle Ay,y\rangle}{M-m}}M^{\frac{\langle Ay,y\rangle-m}{M-m}}\right)\ln\left[K\left(\frac{M}{m}\right)\right]^{\left[\frac{1}{2}+\frac{1}{M-m}\left\langle\left|A-\frac{1}{2}(m+M)I\right|y,y\right\rangle\right]} \\
& \leq \left(m^{\frac{M-\langle Ay,y\rangle}{M-m}}M^{\frac{\langle Ay,y\rangle-m}{M-m}}\right)K\left(\frac{M}{m}\right).
\end{aligned}$$

Let $x \in H$ with $\|x\| = 1$ and take $y = \frac{A^{1/2}x}{\|A^{1/2}x\|}$ in (2.5) to get

$$\begin{aligned}
& m^{\frac{M-\left\langle A\frac{A^{1/2}x}{\|A^{1/2}x\|},\frac{A^{1/2}x}{\|A^{1/2}x\|}\right\rangle}{M-m}}M^{\frac{\left\langle A\frac{A^{1/2}x}{\|A^{1/2}x\|},\frac{A^{1/2}x}{\|A^{1/2}x\|}\right\rangle-m}{M-m}} \\
& \leq \left(m^{\frac{M-\left\langle A\frac{A^{1/2}x}{\|A^{1/2}x\|},\frac{A^{1/2}x}{\|A^{1/2}x\|}\right\rangle}{M-m}}M^{\frac{\left\langle A\frac{A^{1/2}x}{\|A^{1/2}x\|},\frac{A^{1/2}x}{\|A^{1/2}x\|}\right\rangle-m}{M-m}}\right) \\
& \times K\left(\frac{M}{m}\right)^{\left[\frac{1}{2}-\frac{1}{M-m}\left\langle\left|A-\frac{1}{2}(m+M)I\right|\frac{A^{1/2}x}{\|A^{1/2}x\|},\frac{A^{1/2}x}{\|A^{1/2}x\|}\right\rangle\right]} \\
& \leq \exp\left\langle \ln A\frac{A^{1/2}x}{\|A^{1/2}x\|},\frac{A^{1/2}x}{\|A^{1/2}x\|}\right\rangle
\end{aligned}$$

$$\leq \left(m^{\frac{M - \left\langle A \frac{A^{1/2}x}{\left\|A^{1/2}x\right\|}, \frac{A^{1/2}x}{\left\|A^{1/2}x\right\|} \right\rangle}{M-m}} M^{\frac{\left\langle A \frac{A^{1/2}x}{\left\|A^{1/2}x\right\|}, \frac{A^{1/2}x}{\left\|A^{1/2}x\right\|} \right\rangle - m}{M-m}} \right)$$

$$\times K\left(\frac{M}{m}\right)^{\left[\frac{1}{2} + \frac{1}{M-m}\left\langle \left|A - \frac{1}{2}(m+M)I\right| \frac{A^{1/2}x}{\left\|A^{1/2}x\right\|}, \frac{A^{1/2}x}{\left\|A^{1/2}x\right\|} \right\rangle\right]}$$

$$\leq \left(m^{\frac{M - \left\langle A \frac{A^{1/2}x}{\left\|A^{1/2}x\right\|}, \frac{A^{1/2}x}{\left\|A^{1/2}x\right\|} \right\rangle}{M-m}} M^{\frac{\left\langle A \frac{A^{1/2}x}{\left\|A^{1/2}x\right\|}, \frac{A^{1/2}x}{\left\|A^{1/2}x\right\|} \right\rangle - m}{M-m}} \right) K\left(\frac{M}{m}\right),$$

namely

$$m^{\frac{M - \frac{\left\langle A^2x,x\right\rangle}{\langle Ax,x\rangle}}{M-m}} M^{\frac{\frac{\left\langle A^2x,x\right\rangle}{\langle Ax,x\rangle} - m}{M-m}}$$

$$\leq \left(m^{\frac{M - \frac{\left\langle A^2x,x\right\rangle}{\langle Ax,x\rangle}}{M-m}} M^{\frac{\frac{\left\langle A^2x,x\right\rangle}{\langle Ax,x\rangle} - m}{M-m}} \right)$$

$$\times K\left(\frac{M}{m}\right)^{\left[\frac{1}{2} - \frac{1}{(M-m)\langle Ax,x\rangle}\left\langle \left|A^2 - \frac{1}{2}(m+M)A\right| x,x\right\rangle\right]}$$

$$\leq \exp \langle A \ln Ax, x\rangle$$

$$\leq m^{\frac{M - \frac{\left\langle A^2x,x\right\rangle}{\langle Ax,x\rangle}}{M-m}} M^{\frac{\frac{\left\langle A^2x,x\right\rangle}{\langle Ax,x\rangle} - m}{M-m}}$$

$$\times K\left(\frac{M}{m}\right)^{\left[\frac{1}{2} + \frac{1}{(M-m)\langle Ax,x\rangle}\left\langle \left|A^2 - \frac{1}{2}(m+M)A\right| x,x\right\rangle\right]}$$

$$\leq \left(m^{\frac{M - \frac{\left\langle A^2x,x\right\rangle}{\langle Ax,x\rangle}}{M-m}} M^{\frac{\frac{\left\langle A^2x,x\right\rangle}{\langle Ax,x\rangle} - m}{M-m}} \right) K\left(\frac{M}{m}\right),$$

for $x \in H$ with $\|x\| = 1$.

This is equivalent to

$$1 \leq K\left(\frac{M}{m}\right)^{\left[\frac{1}{2} - \frac{1}{(M-m)\langle Ax,x\rangle}\left\langle \left|A^2 - \frac{1}{2}(m+M)A\right| x,x\right\rangle\right]}$$

$$\leq \frac{[\eta_x(A)]^{-1}}{m^{\frac{M - \frac{\left\langle A^2x,x\right\rangle}{\langle Ax,x\rangle}}{M-m}} M^{\frac{\frac{\left\langle A^2x,x\right\rangle}{\langle Ax,x\rangle} - m}{M-m}}}$$

$$\leq K\left(\frac{M}{m}\right)^{\left[\frac{1}{2} + \frac{1}{(M-m)\langle Ax,x\rangle}\left\langle \left|A^2 - \frac{1}{2}(m+M)A\right| x,x\right\rangle\right]} \leq K\left(\frac{M}{m}\right)$$

and the theorem is proved. □

Corollary 1. *With the assumption of Theorem 1, we have the alternative inequality*

$$
\begin{aligned}
(2.6)\qquad 1 &\leq K\left(\frac{M}{m}\right)^{\left[\frac{1}{2}-\frac{1}{(m^{-1}-M^{-1})\langle Ax,x\rangle}\left\langle\left|1-\frac{1}{2}\left(M^{-1}+m^{-1}\right)A\right|x,x\right\rangle\right]} \\
&\leq \frac{[\eta_x(A)]^{\langle Ax,x\rangle^{-1}}}{\left(M^{\frac{m^{-1}-\langle Ax,x\rangle^{-1}}{m^{-1}-M^{-1}}}m^{\frac{\langle Ax,x\rangle^{-1}-M^{-1}}{m^{-1}-M^{-1}}}\right)^{-1}} \\
&\leq \left[K\left(\frac{M}{m}\right)\right]^{\left[\frac{1}{2}+\frac{1}{(m^{-1}-M^{-1})\langle Ax,x\rangle}\left\langle\left|1-\frac{1}{2}\left(M^{-1}+m^{-1}\right)A\right|x,x\right\rangle\right]} \leq K\left(\frac{M}{m}\right)
\end{aligned}
$$

for $x \in H$, $\|x\| = 1$.

Proof. If we write the inequality (2.5) for A^{-1} that satisfies the condition $0 < M^{-1} \leq A^{-1} \leq m^{-1}$, then for $y \in H$ with $\|y\| = 1$,

$$
\begin{aligned}
1 &\leq K\left(\frac{m^{-1}}{M^{-1}}\right)^{\left[\frac{1}{2}-\frac{1}{m^{-1}-M^{-1}}\left\langle\left|A^{-1}-\frac{1}{2}\left(M^{-1}+m^{-1}\right)I\right|y,y\right\rangle\right]} \\
&\leq \frac{\exp\left\langle \ln A^{-1}y,y\right\rangle}{M^{-\frac{m^{-1}-\langle A^{-1}y,y\rangle}{m^{-1}-M^{-1}}}m^{-\frac{\langle A^{-1}y,y\rangle-M^{-1}}{m^{-1}-M^{-1}}}} \\
&\leq \left[K\left(\frac{m^{-1}}{M^{-1}}\right)\right]^{\left[\frac{1}{2}+\frac{1}{m^{-1}-M^{-1}}\left\langle\left|A^{-1}-\frac{1}{2}\left(M^{-1}+m^{-1}\right)I\right|y,y\right\rangle\right]} \\
&\leq K\left(\frac{m^{-1}}{M^{-1}}\right),
\end{aligned}
$$

namely

$$
\begin{aligned}
1 &\leq K\left(\frac{M}{m}\right)^{\left[\frac{1}{2}-\frac{1}{m^{-1}-M^{-1}}\left\langle\left|A^{-1}-\frac{1}{2}\left(M^{-1}+m^{-1}\right)I\right|y,y\right\rangle\right]} \\
&\leq \frac{\exp\left\langle -\ln Ay,y\right\rangle}{\left(M^{\frac{m^{-1}-\langle A^{-1}y,y\rangle}{m^{-1}-M^{-1}}}m^{\frac{\langle A^{-1}y,y\rangle-M^{-1}}{m^{-1}-M^{-1}}}\right)^{-1}} \\
&\leq \left[K\left(\frac{M}{m}\right)\right]^{\left[\frac{1}{2}+\frac{1}{m^{-1}-M^{-1}}\left\langle\left|A^{-1}-\frac{1}{2}\left(M^{-1}+m^{-1}\right)I\right|y,y\right\rangle\right]} \leq K\left(\frac{M}{m}\right).
\end{aligned}
$$

Let $x \in H$ with $\|x\| = 1$ and take $y = \frac{A^{1/2}x}{\|A^{1/2}x\|}$ to get

$$
1 \leq K\left(\frac{M}{m}\right)^{\left[\frac{1}{2}-\frac{1}{m^{-1}-M^{-1}}\left\langle\left|A^{-1}-\frac{1}{2}\left(M^{-1}+m^{-1}\right)I\right|\frac{A^{1/2}x}{\|A^{1/2}x\|},\frac{A^{1/2}x}{\|A^{1/2}x\|}\right\rangle\right]}
$$

$$\leq \frac{\exp\left\langle -\ln A \frac{A^{1/2}x}{\|A^{1/2}x\|}, \frac{A^{1/2}x}{\|A^{1/2}x\|}\right\rangle}{\left(M^{\frac{m^{-1}-\left\langle A^{-1}\frac{A^{1/2}x}{\|A^{1/2}x\|}, \frac{A^{1/2}x}{\|A^{1/2}x\|}\right\rangle}{m^{-1}-M^{-1}}} m^{\frac{\left\langle A^{-1}\frac{A^{1/2}x}{\|A^{1/2}x\|}, \frac{A^{1/2}x}{\|A^{1/2}x\|}\right\rangle - M^{-1}}{m^{-1}-M^{-1}}}\right)^{-1}}$$

$$\leq \left[K\left(\frac{M}{m}\right)\right]^{\left[\frac{1}{2}+\frac{1}{m^{-1}-M^{-1}}\left\langle \left|A^{-1}-\frac{1}{2}\left(M^{-1}+m^{-1}\right)I\right|\frac{A^{1/2}x}{\|A^{1/2}x\|}, \frac{A^{1/2}x}{\|A^{1/2}x\|}\right\rangle\right]} \leq K\left(\frac{M}{m}\right),$$

namely

$$\begin{aligned}
1 &\leq K\left(\frac{M}{m}\right)^{\left[\frac{1}{2}-\frac{1}{(m^{-1}-M^{-1})\langle Ax,x\rangle}\left\langle \left|1-\frac{1}{2}\left(M^{-1}+m^{-1}\right)A\right|x,x\right\rangle\right]} \\
&\leq \frac{\exp \frac{1}{\langle Ax,x\rangle}\langle -A\ln Ax, x\rangle}{\left(M^{\frac{m^{-1}-\langle Ax,x\rangle^{-1}}{m^{-1}-M^{-1}}} m^{\frac{\langle Ax,x\rangle^{-1}-M^{-1}}{m^{-1}-M^{-1}}}\right)^{-1}} \\
&\leq \left[K\left(\frac{M}{m}\right)\right]^{\left[\frac{1}{2}+\frac{1}{(m^{-1}-M^{-1})\langle Ax,x\rangle}\left\langle \left|1-\frac{1}{2}\left(M^{-1}+m^{-1}\right)A\right|x,x\right\rangle\right]} \\
&\leq K\left(\frac{M}{m}\right),
\end{aligned}$$

which proves (2.6). □

3. Related Results

We also have:

Theorem 2. *Assume that* $0 < mI \leq A \leq MI$ *and* $x \in H$, $\|x\| = 1$, *then*

$$\begin{aligned}
1 &\geq \frac{[\eta_x(A)]^{\langle Ax,x\rangle^{-1}}}{\left(m^{\frac{M-\frac{\langle A^2x,x\rangle}{\langle Ax,x\rangle}}{M-m}} M^{\frac{\frac{\langle A^2x,x\rangle}{\langle Ax,x\rangle}-m}{M-m}}\right)^{-1}} \\
&\geq \exp\left[\frac{-1}{Mm\langle Ax,x\rangle}\langle A(MI-A)(A-mI)x,x\rangle\right] \\
&\geq \exp\left[\frac{-1}{Mm}\left(M-\frac{\langle A^2x,x\rangle}{\langle Ax,x\rangle}\right)\left(\frac{\langle A^2x,x\rangle}{\langle Ax,x\rangle}-m\right)\right] \\
&\geq \exp\left[-K\left(\frac{M}{m}\right)\right].
\end{aligned} \tag{3.1}$$

Proof. In [1] we obtained the following reverses of Young's inequality:

$$1 \leq \frac{(1-\nu)a+\nu b}{a^{1-\nu}b^{\nu}} \leq \exp\left[4\nu(1-\nu)\left(K\left(\frac{a}{b}\right)-1\right)\right],$$

where a, $b > 0$, $\nu \in [0,1]$.

This is equivalent, by taking the logarithm, with

$$0 \le \ln\left(\left(1-\nu\right)a+\nu b\right)-\left(1-\nu\right)\ln a-\nu\ln b \le \nu\left(1-\nu\right)\frac{\left(b-a\right)^2}{ba}$$

where $a,\ b>0,\ \nu\in[0,1]$.

If we take $a=m,\ b=M,\ t\in[m,M]$ and $\nu=\frac{t-m}{M-m}\in[0,1]$, then we get

$$0 \le \ln t-\frac{M-t}{M-m}\ln m-\frac{t-m}{M-m}\ln M \le \frac{\left(M-t\right)\left(t-m\right)}{\left(M-m\right)^2}\frac{\left(M-m\right)^2}{Mm} = \frac{\left(M-t\right)\left(t-m\right)}{Mm}.$$

Using the continuous functional calculus for selfadjoint operators, we have

$$0 \le \ln A-\frac{MI-A}{M-m}\ln m-\frac{AI-m}{M-m}\ln M \le \frac{\left(MI-A\right)\left(A-mI\right)}{Mm},$$

which is equivalent to

$$0 \le \left\langle \ln Ay,y\right\rangle-\frac{M-\left\langle Ay,y\right\rangle}{M-m}\ln m-\frac{\left\langle Ay,y\right\rangle-m}{M-m}\ln M \le \frac{1}{Mm}\left\langle \left(MI-A\right)\left(A-mI\right)y,y\right\rangle,$$

for all $y\in H$, $\|y\|=1$.

If we take the exponential, then we get

$$1 \le \frac{\exp\left\langle \ln Ay,y\right\rangle}{\exp\left[\frac{M-\left\langle Ay,y\right\rangle}{M-m}\ln m+\frac{\left\langle Ay,y\right\rangle-m}{M-m}\ln M\right]} \le \exp\left[\frac{1}{Mm}\left\langle \left(MI-A\right)\left(A-mI\right)y,y\right\rangle\right], \tag{3.2}$$

for all $y\in H$, $\|y\|=1$.

Observe that

$$\exp\left[\frac{M-\left\langle Ay,y\right\rangle}{M-m}\ln m+\frac{\left\langle Ay,y\right\rangle-m}{M-m}\ln M\right]=\exp\left[\ln\left(m^{\frac{M-\left\langle Ay,y\right\rangle}{M-m}}M^{\frac{\left\langle Ay,y\right\rangle-m}{M-m}}\right)\right] = m^{\frac{M-\left\langle Ay,y\right\rangle}{M-m}}M^{\frac{\left\langle Ay,y\right\rangle-m}{M-m}}$$

and by (3.2) we obtain the first inequality in (3.3).

The function $g\left(t\right)=\left(M-t\right)\left(t-m\right)$ is concave on $[m,M]$ and by Jensen's inequality

$$\left\langle g\left(A\right)y,y\right\rangle \le g\left(\left\langle Ay,y\right\rangle\right),\ \ y\in H,\|y\|=1$$

we have

$$\left\langle \left(MI-A\right)\left(A-mI\right)y,y\right\rangle \le \left(\left(M-\left\langle Ay,y\right\rangle\right)\left(\left\langle Ay,y\right\rangle-m\right)\right)$$

for all $y\in H,\|y\|=1$, which proves that

$$\begin{aligned} 1 &\le \frac{\exp\left\langle \ln Ay,y\right\rangle}{m^{\frac{M-\left\langle Ay,y\right\rangle}{M-m}}M^{\frac{\left\langle Ay,y\right\rangle-m}{M-m}}} \le \exp\left[\frac{1}{Mm}\left\langle \left(MI-A\right)\left(A-mI\right)y,y\right\rangle\right] \\ &\le \exp\left[\frac{1}{Mm}\left(M-\left\langle Ay,y\right\rangle\right)\left(\left\langle Ay,y\right\rangle-m\right)\right] \\ &\le \exp\left[\frac{1}{4Mm}\left(M-m\right)^2\right]. \end{aligned} \tag{3.3}$$

Let $x \in H$ with $\|x\| = 1$ and take $y = \frac{A^{1/2}x}{\|A^{1/2}x\|}$ to get

$$\begin{aligned}
1 &\leq \frac{\exp \left\langle \ln A \frac{A^{1/2}x}{\|A^{1/2}x\|}, \frac{A^{1/2}x}{\|A^{1/2}x\|} \right\rangle}{m^{\frac{M - \left\langle A \frac{A^{1/2}x}{\|A^{1/2}x\|}, \frac{A^{1/2}x}{\|A^{1/2}x\|} \right\rangle}{M-m}} M^{\frac{\left\langle A \frac{A^{1/2}x}{\|A^{1/2}x\|}, \frac{A^{1/2}x}{\|A^{1/2}x\|} \right\rangle - m}{M-m}}} \\
&\leq \exp \left[\frac{1}{Mm} \left\langle (MI - A)(A - mI) \frac{A^{1/2}x}{\|A^{1/2}x\|}, \frac{A^{1/2}x}{\|A^{1/2}x\|} \right\rangle \right] \\
&\leq \exp \left[\frac{1}{Mm} \left(M - \left\langle A \frac{A^{1/2}x}{\|A^{1/2}x\|}, \frac{A^{1/2}x}{\|A^{1/2}x\|} \right\rangle \right) \right. \\
&\quad \times \left. \left(\left\langle A \frac{A^{1/2}x}{\|A^{1/2}x\|}, \frac{A^{1/2}x}{\|A^{1/2}x\|} \right\rangle - m \right) \right] \\
&\leq \exp \left[\frac{1}{4Mm} (M - m)^2 \right],
\end{aligned}$$

namely

$$\begin{aligned}
1 &\leq \frac{\exp \frac{1}{\langle Ax, x\rangle} \langle A \ln Ax, x\rangle}{m^{\frac{M - \frac{\langle A^2x, x\rangle}{\langle Ax, x\rangle}}{M-m}} M^{\frac{\frac{\langle A^2x, x\rangle}{\langle Ax, x\rangle} - m}{M-m}}} \\
&\leq \exp \left[\frac{1}{Mm \langle Ax, x\rangle} \langle A (MI - A)(A - mI) x, x\rangle \right] \\
&\leq \exp \left[\frac{1}{Mm} \left(M - \frac{\langle A^2x, x\rangle}{\langle Ax, x\rangle} \right) \left(\frac{\langle A^2x, x\rangle}{\langle Ax, x\rangle} - m \right) \right] \\
&\leq \exp \left[K \left(\frac{M}{m} \right) \right],
\end{aligned}$$

and by taking the power -1, we derive (3.1). □

Corollary 2. *With the assumptions of Theorem 2,*

$$\begin{aligned}
1 &\leq \frac{[\eta_x(A)]^{\langle Ax, x\rangle^{-1}}}{\left(M^{\frac{m^{-1} - \langle Ax, x\rangle^{-1}}{m^{-1} - M^{-1}}} m^{\frac{\langle Ax, x\rangle^{-1} - M^{-1}}{m^{-1} - M^{-1}}} \right)^{-1}} \qquad (3.4) \\
&\leq \exp \left[\frac{1}{m^{-1}M^{-1} \langle Ax, x\rangle} \left\langle A \left(m^{-1}I - A^{-1} \right) \left(A^{-1} - M^{-1}I \right) x, x \right\rangle \right] \\
&\leq \exp \left[\frac{1}{m^{-1}M^{-1}} \left(m^{-1} - \langle Ax, x\rangle^{-1} \right) \left(\langle Ax, x\rangle^{-1} - M^{-1} \right) \right] \\
&\leq \exp \left[K \left(\frac{M}{m} \right) \right]
\end{aligned}$$

for $x \in H$ with $\|x\| = 1$.

Proof. We have from (3.3) for A^{-1} that satisfies the condition $0 < M^{-1} \leq A^{-1} \leq m^{-1}$,

$$\begin{aligned}
1 &\leq \frac{\exp \left\langle \ln A^{-1} y, y \right\rangle}{M^{-\frac{m^{-1} - \left\langle A^{-1} y, y \right\rangle}{m^{-1} - M^{-1}}} m^{-\frac{\left\langle A^{-1} y, y \right\rangle - M^{-1}}{m^{-1} - M^{-1}}}} \\
&\leq \exp \left[\frac{1}{m^{-1} M^{-1}} \left\langle \left(m^{-1} I - A^{-1} \right) \left(A^{-1} - M^{-1} I \right) y, y \right\rangle \right] \\
&\leq \exp \left[\frac{1}{m^{-1} M^{-1}} \left(m^{-1} - \left\langle A^{-1} y, y \right\rangle \right) \left(\left\langle A^{-1} y, y \right\rangle - M^{-1} \right) \right] \\
&\leq \exp \left[K \left(\frac{M}{m} \right) \right]
\end{aligned}$$

for $y \in H$, $\|y\| = 1$.

Let $x \in H$ with $\|x\| = 1$ and take $y = \frac{A^{1/2} x}{\|A^{1/2} x\|}$ to get

$$\begin{aligned}
1 &\leq \frac{\exp \frac{1}{\langle Ax, x \rangle} \langle -A \ln Ax, x \rangle}{\left(M^{\frac{m^{-1} - \langle Ax, x \rangle^{-1}}{m^{-1} - M^{-1}}} m^{\frac{\langle Ax, x \rangle^{-1} - M^{-1}}{m^{-1} - M^{-1}}} \right)^{-1}} \\
&\leq \exp \left[\frac{1}{m^{-1} M^{-1} \langle Ax, x \rangle} \left\langle A \left(m^{-1} I - A^{-1} \right) \left(A^{-1} - M^{-1} I \right) x, x \right\rangle \right] \\
&\leq \exp \left[\frac{1}{m^{-1} M^{-1}} \left(m^{-1} - \langle Ax, x \rangle^{-1} \right) \left(\langle Ax, x \rangle^{-1} - M^{-1} \right) \right] \\
&\leq \exp \left[K \left(\frac{M}{m} \right) \right]
\end{aligned}$$

□

In [2] we obtained the following refinement and reverse of Young's inequality:

$$\begin{aligned}
&\exp \left[\frac{1}{2} \nu (1 - \nu) \left(1 - \frac{\min \{a, b\}}{\max \{a, b\}} \right)^2 \right] \\
&\leq \frac{(1 - \nu) a + \nu b}{a^{1-\nu} b^{\nu}} \\
&\leq \exp \left[\frac{1}{2} \nu (1 - \nu) \left(\frac{\max \{a, b\}}{\min \{a, b\}} - 1 \right)^2 \right],
\end{aligned} \tag{3.5}$$

for any a, $b > 0$ and $\nu \in [0, 1]$.

Theorem 3. *Assume that* $0 < mI \leq A \leq MI$ *and* $x \in H$, $\|x\| = 1$, *then*

$$\begin{aligned}
1 &\geq \exp \left[\frac{-1}{2M^2 \langle Ax, x \rangle} \langle A (M - A)(A - m) x, x \rangle \right] \\
&\geq \frac{\left[\eta_x (A) \right]^{\langle Ax, x \rangle^{-1}}}{\left(m^{\frac{M - \frac{\langle A^2 x, x \rangle}{\langle Ax, x \rangle}}{M - m}} M^{\frac{\frac{\langle A^2 x, x \rangle}{\langle Ax, x \rangle} - m}{M - m}} \right)^{-1}}
\end{aligned} \tag{3.6}$$

$$\geq \exp\left[\frac{-1}{2m^2\langle Ax,x\rangle}\langle A(M-A)(A-m)x,x\rangle\right]$$
$$\geq \exp\left[\frac{-1}{2m^2}\left(M-\frac{\langle A^2x,x\rangle}{\langle Ax,x\rangle}\right)\left(\frac{\langle A^2x,x\rangle}{\langle Ax,x\rangle}-m\right)\right]$$
$$\geq \exp\left[-\frac{1}{8}\left(\frac{M}{m}-1\right)^2\right].$$

Proof. From (3.5) we have

$$\exp\left[\frac{1}{2}\nu(1-\nu)\left(1-\frac{m}{M}\right)^2\right]$$
$$\leq \frac{(1-\nu)m+\nu M}{m^{1-\nu}M^{\nu}} \leq \exp\left[\frac{1}{2}\nu(1-\nu)\left(\frac{M}{m}-1\right)^2\right],$$

for $\nu \in [0,1]$.

By taking the logarithm, we obtain

$$\begin{aligned}(3.7)\qquad &\frac{1}{2}\nu(1-\nu)\left(1-\frac{m}{M}\right)^2\\ &\leq \ln((1-\nu)m+\nu M)-(1-\nu)\ln m-\nu\ln M\\ &\leq \frac{1}{2}\nu(1-\nu)\left(\frac{M}{m}-1\right)^2,\end{aligned}$$

for $\nu \in [0,1]$.

If we take $a=m$, $b=M$, $t\in[m,M]$ and $\nu=\frac{t-m}{M-m}\in[0,1]$, then we get

$$\frac{(M-t)(t-m)}{2M^2} \leq \ln t-\frac{M-t}{M-m}\ln m-\frac{t-m}{M-m}\ln M \leq \frac{(M-t)(t-m)}{2m^2}$$

$t\in[m,M]$.

As above, we get the vector inequality

$$\frac{1}{2M^2}\langle(MI-A)(A-mI)y,y\rangle$$
$$\leq \langle \ln Ay,y\rangle-\frac{M-\langle Ay,y\rangle}{M-m}\ln m-\frac{\langle Ay,y\rangle-m}{M-m}\ln M$$
$$\leq \frac{1}{2m^2}\langle(MI-A)(A-mI)y,y\rangle,$$

for $y\in H$, $\|y\|=1$.

If we take the exponential, then we derive

$$\exp\left[\frac{1}{2M^2}\langle(MI-A)(A-mI)y,y\rangle\right]$$
$$\leq \frac{\exp\langle \ln Ay,y\rangle}{\exp\left[\frac{M-\langle Ay,y\rangle}{M-m}\ln m+\frac{\langle Ay,y\rangle-m}{M-m}\ln M\right]}$$
$$\leq \exp\left[\frac{1}{2m^2}\langle(MI-A)(A-mI)y,y\rangle\right],$$

for all $y \in H$, $\|y\| = 1$, which proves that

$$
\begin{aligned}
1 &\leq \exp\left[\frac{1}{2M^2}\left\langle (MI - A)(A - mI) y, y\right\rangle\right] \\
&\leq \frac{\exp\langle \ln Ay, y\rangle}{m^{\frac{M - \langle Ay, y\rangle}{M-m}} M^{\frac{\langle Ay, y\rangle - m}{M-m}}} \\
&\leq \exp\left[\frac{1}{2m^2}\left\langle (MI - A)(A - mI) y, y\right\rangle\right] \\
&\leq \exp\left[\frac{1}{2m^2}\left(M - \langle Ay, y\rangle\right)\left(\langle Ay, y\rangle - m\right)\right] \\
&\leq \exp\left[\frac{1}{8}\left(\frac{M}{m} - 1\right)^2\right].
\end{aligned}
\tag{3.8}
$$

Let $x \in H$ with $\|x\| = 1$ and take $y = \frac{A^{1/2}x}{\|A^{1/2}x\|}$ to get

$$
\begin{aligned}
1 &\leq \exp\left[\frac{1}{2M^2}\left\langle (M - A)(A - m)\frac{A^{1/2}x}{\|A^{1/2}x\|}, \frac{A^{1/2}x}{\|A^{1/2}x\|}\right\rangle\right] \\
&\leq \frac{\exp\left\langle \ln A\frac{A^{1/2}x}{\|A^{1/2}x\|}, \frac{A^{1/2}x}{\|A^{1/2}x\|}\right\rangle}{m^{\frac{M - \left\langle A\frac{A^{1/2}x}{\|A^{1/2}x\|}, \frac{A^{1/2}x}{\|A^{1/2}x\|}\right\rangle}{M-m}} M^{\frac{\left\langle A\frac{A^{1/2}x}{\|A^{1/2}x\|}, \frac{A^{1/2}x}{\|A^{1/2}x\|}\right\rangle - m}{M-m}}} \\
&\leq \exp\left[\frac{1}{2m^2}\left\langle (M - A)(A - m)\frac{A^{1/2}x}{\|A^{1/2}x\|}, \frac{A^{1/2}x}{\|A^{1/2}x\|}\right\rangle\right] \\
&\leq \exp\left[\frac{1}{2m^2}\left(M - \left\langle A\frac{A^{1/2}x}{\|A^{1/2}x\|}, \frac{A^{1/2}x}{\|A^{1/2}x\|}\right\rangle\right)\right. \\
&\quad \times \left.\left(\left\langle A\frac{A^{1/2}x}{\|A^{1/2}x\|}, \frac{A^{1/2}x}{\|A^{1/2}x\|}\right\rangle - m\right)\right] \\
&\leq \exp\left[\frac{1}{8}\left(\frac{M}{m} - 1\right)^2\right],
\end{aligned}
$$

namely

$$
\begin{aligned}
1 &\leq \exp\left[\frac{1}{2M^2\langle Ax, x\rangle}\left\langle A(M - A)(A - m)x, x\right\rangle\right] \\
&\leq \frac{\exp\langle A\ln Ax, x\rangle}{m^{\frac{M - \frac{\langle A^2x, x\rangle}{\langle Ax, x\rangle}}{M-m}} M^{\frac{\frac{\langle A^2x, x\rangle}{\langle Ax, x\rangle} - m}{M-m}}} \\
&\leq \exp\left[\frac{1}{2m^2\langle Ax, x\rangle}\left\langle A(M - A)(A - m)x, x\right\rangle\right] \\
&\leq \exp\left[\frac{1}{2m^2}\left(M - \frac{\langle A^2x, x\rangle}{\langle Ax, x\rangle}\right)\left(\frac{\langle A^2x, x\rangle}{\langle Ax, x\rangle} - m\right)\right] \leq \exp\left[\frac{1}{8}\left(\frac{M}{m} - 1\right)^2\right],
\end{aligned}
$$

which is equivalent to (3.6). □

Finally, we have:

Corollary 3. *With the assumptions of Theorem 2,*

$$
\begin{aligned}
(3.9)\qquad 1 &\leq \exp\left[\frac{1}{2m^{-2}\langle Ax,x\rangle}\left\langle A\left(m^{-1}I-A^{-1}\right)\left(A^{-1}-M^{-1}I\right)x,x\right\rangle\right] \\
&\leq \frac{\left[\eta_x(A)\right]^{\langle Ax,x\rangle^{-1}}}{\left(M^{\frac{m^{-1}-\langle Ax,x\rangle^{-1}}{m^{-1}-M^{-1}}}m^{\frac{\langle Ax,x\rangle^{-1}-M^{-1}}{m^{-1}-M^{-1}}}\right)^{-1}} \\
&\leq \exp\left[\frac{1}{2M^{-2}\langle Ax,x\rangle}\left\langle A\left(m^{-1}I-A^{-1}\right)\left(A^{-1}-M^{-1}I\right)x,x\right\rangle\right] \\
&\leq \exp\left[\frac{1}{2M^{-2}}\left(m^{-1}-\langle Ax,x\rangle^{-1}\right)\left(\langle Ax,x\rangle^{-1}-M^{-1}\right)\right] \\
&\leq \exp\left[\frac{1}{8}\left(\frac{M}{m}-1\right)^2\right]
\end{aligned}
$$

for $x \in H$ *with* $\|x\| = 1$.

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[1]Mathematics, College of Engineering & Science, Victoria University, PO Box 14428, Melbourne City, MC 8001, Australia.

E-mail address: sever.dragomir@vu.edu.au

URL: http://rgmia.org/dragomir

[2]DST-NRF Centre of Excellence in the Mathematical, and Statistical Sciences, School of Computer Science, & Applied Mathematics, University of the Witwatersrand,, Private Bag 3, Johannesburg 2050, South Africa