VARIOUS BOUNDS FOR THE NORMALIZED ENTROPIC DETERMINANT OF POSITIVE OPERATORS IN HILBERT SPACES

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Abstract. For positive invertible operators A on a Hilbert space H and a fixed unit vector $x \in H$, define the normalized entropic determinant by $\eta_x(A) := \exp[-\langle A \ln Ax, x\rangle]$. In this paper we show among others that, if A satisfies the condition $0 < m \le A \le M$, then

$$
\begin{aligned}
1 &\le \exp\left[\frac{1}{2M}\langle (A-m)(M-A)x, x\rangle\right] \\
&\le \frac{\eta_x(A)}{\left[m^{\frac{m(M-\langle Ax,x\rangle)}{M-m}} M^{\frac{M(\langle Ax,x\rangle - m)}{M-m}}\right]^{-1}} \\
&\le \exp\left[\frac{1}{2m}\langle (A-m)(M-A)x, x\rangle\right] \\
&\le \exp\left[\frac{1}{2m}(M-\langle Ax,x\rangle)(\langle Ax,x\rangle - m)\right] \le \exp\left[\frac{1}{8m}(M-m)^2\right]
\end{aligned}
$$

for $x \in H$, $\|x\| = 1$.

1. Introduction

Let $B(H)$ be the space of all bounded linear operators on a Hilbert space H, and I stands for the identity operator on H. An operator A in $B(H)$ is said to be positive (in symbol: $A \ge 0$) if $\langle Ax, x\rangle \ge 0$ for all $x \in H$. In particular, $A > 0$ means that A is positive and invertible. For a pair A, B of selfadjoint operators the order relation $A \ge B$ means as usual that $A - B$ is positive.

In 1998, Fujii et al. [3], [4], introduced the *normalized determinant* $\Delta_x(A)$ for positive invertible operators A on a Hilbert space H and a fixed unit vector $x \in H$, namely $\|x\| = 1$, defined by $\Delta_x(A) := \exp\langle \ln Ax, x\rangle$ and discussed it as a continuous geometric mean and observed some inequalities around the determinant from this point of view.

Some of the fundamental properties of normalized determinant are as follows, [3].

For each unit vector $x \in H$, see also [7], we have:

(i) *continuity*: the map $A \to \Delta_x(A)$ is norm continuous;
(ii) *bounds*: $\langle A^{-1}x, x\rangle^{-1} \le \Delta_x(A) \le \langle Ax, x\rangle$;
(iii) *continuous mean*: $\langle A^p x, x\rangle^{1/p} \downarrow \Delta_x(A)$ for $p \downarrow 0$ and $\langle A^p x, x\rangle^{1/p} \uparrow \Delta_x(A)$ for $p \uparrow 0$;
(iv) *power equality:* $\Delta_x(A^t) = \Delta_x(A)^t$ for all $t > 0$;

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(v) *homogeneity*: $\Delta_x(tA) = t\Delta_x(A)$ and $\Delta_x(tI) = t$ for all $t > 0$;
(vi) *monotonicity*: $0 < A \le B$ implies $\Delta_x(A) \le \Delta_x(B)$;
(vii) *multiplicativity*: $\Delta_x(AB) = \Delta_x(A)\Delta_x(B)$ for commuting A and B;
(viii) *Ky Fan type inequality*: $\Delta_x((1-\alpha)A + \alpha B) \ge \Delta_x(A)^{1-\alpha}\Delta_x(B)^{\alpha}$ for $0 < \alpha < 1$.

We define the *logarithmic mean* of two positive numbers a, b by

$$L(a,b) := \begin{cases} \frac{b-a}{\ln b - \ln a} \text{ if } b \neq a, \\ \\ a \text{ if } b = a. \end{cases}$$

In [3] the authors obtained the following additive reverse inequality for the operator A which satisfy the condition $0 < m \le A \le M$, where m, M are positive numbers,

$$0 \le \langle Ax, x\rangle - \Delta_x(A) \le L(m,M)\left[\ln L(m,M) + \frac{M\ln m - m\ln M}{M-m} - 1\right] \tag{1.1}$$

for all $x \in H$, $\|x\| = 1$.

The famous *Young inequality* for scalars says that if $a, b > 0$ and $\nu \in [0,1]$, then

$$a^{1-\nu}b^{\nu} \le (1-\nu)a + \nu b \tag{1.2}$$

with equality if and only if $a = b$. The inequality (1.2) is also called *ν-weighted arithmetic-geometric mean inequality*.

We recall that *Specht's ratio* is defined by [8]

$$S(h) := \begin{cases} \frac{h^{\frac{1}{h-1}}}{e\ln\left(h^{\frac{1}{h-1}}\right)} \text{ if } h \in (0,1)\cup(1,\infty) \\ \\ 1 \text{ if } h = 1. \end{cases} \tag{1.3}$$

It is well known that $\lim_{h\to 1} S(h) = 1$, $S(h) = S\left(\frac{1}{h}\right) > 1$ for $h > 0$, $h \neq 1$. The function is decreasing on $(0,1)$ and increasing on $(1,\infty)$.

In [4], the authors obtained the following multiplicative reverse inequality as well

$$1 \le \frac{\langle Ax, x\rangle}{\Delta_x(A)} \le S\left(\frac{M}{m}\right) \tag{1.4}$$

for $0 < mI \le A \le MI$ and $x \in H$, $\|x\| = 1$.

Since $0 < M^{-1}I \le A^{-1} \le m^{-1}I$, then by (1.4) for A^{-1} we get

$$1 \le \frac{\langle A^{-1}x, x\rangle}{\Delta_x(A^{-1})} \le S\left(\frac{m^{-1}}{M^{-1}}\right) = S\left(\left(\frac{m}{M}\right)^{-1}\right) = S\left(\frac{M}{m}\right),$$

which is equivalent to

$$1 \le \frac{\Delta_x(A)}{\langle A^{-1}x, x\rangle^{-1}} \le S\left(\frac{M}{m}\right) \tag{1.5}$$

for $x \in H$, $\|x\| = 1$.

For the entropy function $\eta(t) = -t\ln t$, $t > 0$, the operator entropy has the following expression:

$$\eta(A) = -A\ln A$$

for positive A.

For $x \in H$, $\|x\| = 1$, we define the *normalized entropic determinant* $\eta_x(A)$ by

$$\eta_x(A) := \exp(-\langle A \ln Ax, x\rangle) = \exp\langle \eta(A)x, x\rangle. \tag{1.6}$$

Let $x \in H$, $\|x\| = 1$. Observe that the map $A \to \eta_x(A)$ is *norm continuous* and since

$$\begin{aligned}
&\exp(-\langle tA \ln(tA)x, x\rangle) \\
&= \exp(-\langle tA(\ln t + \ln A)x, x\rangle) = \exp(-\langle (tA\ln t + tA\ln A)x, x\rangle) \\
&= \exp(-\langle Ax, x\rangle t \ln t)\exp(-t\langle A\ln Ax, x\rangle) \\
&= \exp\ln\left(t^{-\langle Ax,x\rangle t}\right)[\exp(-\langle A\ln Ax, x\rangle)]^{-t},
\end{aligned}$$

hence

$$\eta_x(tA) = t^{-t\langle Ax,x\rangle}[\eta_x(A)]^{-t} \tag{1.7}$$

for $t > 0$ and $A > 0$.

Observe also that

$$\eta_x(I) = 1 \text{ and } \eta_x(tI) = t^{-t} \tag{1.8}$$

for $t > 0$.

Motivated by the above results, in this paper we show among others that, if A satisfies the condition $0 < m \le A \le M$, then

$$\begin{aligned}
1 &\le \exp\left[\frac{1}{2M}\langle (A-m)(M-A)x, x\rangle\right] \\
&\le \frac{\eta_x(A)}{\left[m^{\frac{m(M-\langle Ax,x\rangle)}{M-m}} M^{\frac{M(\langle Ax,x\rangle - m)}{M-m}}\right]^{-1}} \\
&\le \exp\left[\frac{1}{2m}\langle (A-m)(M-A)x, x\rangle\right] \\
&\le \exp\left[\frac{1}{2m}(M-\langle Ax,x\rangle)(\langle Ax,x\rangle - m)\right] \le \exp\left[\frac{1}{8m}(M-m)^2\right]
\end{aligned}$$

for $x \in H$, $\|x\| = 1$.

2. Main Results

We start with the following result:

Theorem 1. *Assume that A satisfies the condition $0 < m \le A \le M$, where m, M are positive numbers, then*

$$\begin{aligned}
1 &\le \frac{\eta_x(A)}{\left[m^{\frac{m(M-\langle Ax,x\rangle)}{M-m}} M^{\frac{M(\langle Ax,x\rangle - m)}{M-m}}\right]^{-1}} \le \left(\frac{M}{m}\right)^{\frac{\langle (A-m)(M-A)x,x\rangle}{M-m}} \\
&\le \left(\frac{M}{m}\right)^{\frac{(M-\langle Ax,x\rangle)(\langle Ax,x\rangle - m)}{M-m}} \le \left(\frac{M}{m}\right)^{\frac{1}{4}(M-m)}
\end{aligned} \tag{2.1}$$

for $x \in H$, $\|x\| = 1$

Proof. In [1] we obtained the following result: if the function $f : I \subset \mathbb{R} \to \mathbb{R}$ is a differentiable convex function on $\mathring{I}$, then for any $a,\ b \in \mathring{I}$ and $\nu \in [0,1]$ we have

$$0 \le (1-\nu) f(a) + \nu f(b) - f((1-\nu) a + \nu b) \le \nu (1-\nu)(b-a)[f'(b) - f'(a)]. \tag{2.2}$$

If we write the inequality (2.2) for the convex function $f(t) = t \ln t$, $t > 0$, then for all $a, b > 0$ and $\nu \in [0,1]$

$$0 \le (1-\nu) a \ln a + \nu b \ln b - ((1-\nu) a + \nu b) \ln ((1-\nu) a + \nu b) \le \nu (1-\nu)(b-a)(\ln b - \ln a). \tag{2.3}$$

If $m < M$ and by taking $a = m$, $b = M$ and putting

$$\nu = \frac{t-m}{M-m},$$

then we get for $t \in [m, M]$ that $\nu \in [0,1]$ and

$$\begin{aligned} 0 &\le \frac{M-t}{M-m} m \ln m + \frac{t-m}{M-m} M \ln M - t \ln t \\ &\le \frac{M-t}{M-m}\frac{t-m}{M-m}(M-m)\ln\left(\frac{M}{m}\right) \\ &= \frac{\ln\left(\frac{M}{m}\right)}{M-m}(t-m)(M-t). \end{aligned} \tag{2.4}$$

If we use the continuous functional calculus for selfadjoint operator A satisfying the condition $0 < m \le A \le M$, we obtain from (2.4) that

$$\begin{aligned} 0 &\le m \ln m \frac{M-A}{M-m} + M \ln M \frac{A-m}{M-m} - A \ln A \\ &\le \frac{\ln\left(\frac{M}{m}\right)}{M-m}(A-m)(M-A). \end{aligned} \tag{2.5}$$

If we take the inner product for $x \in H$, $\|x\| = 1$ in (2.5), then we get

$$\begin{aligned} 0 &\le m \ln m \frac{M - \langle Ax, x\rangle}{M-m} + M \ln M \frac{\langle Ax, x\rangle - m}{M-m} - \langle A \ln A x, x\rangle \\ &\le \frac{\ln\left(\frac{M}{m}\right)}{M-m}\langle (A-m)(M-A) x, x\rangle. \end{aligned} \tag{2.6}$$

The function $g(t) = (M-t)(t-m)$ is concave on $[m, M]$ and by Jensen's inequality for concave function

$$\langle g(A) x, x\rangle \le g(\langle Ax, x\rangle) \text{ for } x \in H, \ \|x\| = 1$$

we get

$$\begin{aligned} \langle (A-m)(M-A) x, x\rangle &\le (M - \langle Ax, x\rangle)(\langle Ax, x\rangle - m) \\ &\le \frac{1}{4}(M-m)^2 \end{aligned} \tag{2.7}$$

for $x \in H$, $\|x\| = 1$.

Therefore, by (2.6) and (2.7) we derive

$$\begin{aligned} 0 &\le m\ln m\frac{M-\langle Ax,x\rangle}{M-m}+M\ln M\frac{\langle Ax,x\rangle-m}{M-m}-\langle A\ln Ax,x\rangle \\ &\le \frac{\ln\left(\frac{M}{m}\right)}{M-m}\langle (A-m)(M-A)x,x\rangle \\ &\le \frac{\ln\left(\frac{M}{m}\right)}{M-m}(M-\langle Ax,x\rangle)(\langle Ax,x\rangle-m) \\ &\le \frac{1}{4}(M-m)\ln\left(\frac{M}{m}\right) \end{aligned}$$

namely

$$\begin{aligned} 0 &\le \ln\left[(m^m)^{\frac{M-\langle Ax,x\rangle}{M-m}}\left(M^M\right)^{\frac{\langle Ax,x\rangle-m}{M-m}}\right]-\langle A\ln Ax,x\rangle \\ &\le \ln\left(\frac{M}{m}\right)^{\frac{\langle (A-m)(M-A)x,x\rangle}{M-m}} \le \ln\left(\frac{M}{m}\right)^{\frac{(M-\langle Ax,x\rangle)(\langle Ax,x\rangle-m)}{M-m}} \\ &\le \ln\left(\frac{M}{m}\right)^{\frac{1}{4}(M-m)} \end{aligned}$$

for $x\in H$, $\|x\|=1$.

By taking the exponential in (2.8) we get

$$\begin{aligned} 1 &\le \left[(m^m)^{\frac{M-\langle Ax,x\rangle}{M-m}}\left(M^M\right)^{\frac{\langle Ax,x\rangle-m}{M-m}}\right]\exp\left(-\langle A\ln Ax,x\rangle\right) \\ &\le \left(\frac{M}{m}\right)^{\frac{\langle (A-m)(M-A)x,x\rangle}{M-m}} \le \left(\frac{M}{m}\right)^{\frac{(M-\langle Ax,x\rangle)(\langle Ax,x\rangle-m)}{M-m}} \le \left(\frac{M}{m}\right)^{\frac{1}{4}(M-m)} \end{aligned}$$

for $x\in H$, $\|x\|=1$, which is equivalent to (2.1). □

Corollary 1. *With the assumptions of Theorem 1, we also have*

$$\begin{aligned} 1 &\ge \frac{\eta_x(A)}{\left[M^{\frac{\left(m^{-1}\langle A^2x,x\rangle-\langle Ax,x\rangle\right)M^{-1}}{m^{-1}-M^{-1}}}m^{\frac{\left(\langle Ax,x\rangle-M^{-1}\langle A^2x,x\rangle\right)m^{-1}}{m^{-1}-M^{-1}}}\right]^{-1}} \\ &\ge \left(\frac{M}{m}\right)^{-\frac{\left\langle\left(1-M^{-1}A\right)\left(m^{-1}A-1\right)x,x\right\rangle}{m^{-1}-M^{-1}}} \\ &\ge \left(\frac{M}{m}\right)^{-\frac{\left(m^{-1}\left\langle A^2x,x\right\rangle-\langle Ax,x\rangle\right)\left(\langle Ax,x\rangle-M^{-1}\left\langle A^2x,x\right\rangle\right)}{\left(m^{-1}-M^{-1}\right)\left\langle A^2x,x\right\rangle}} \\ &\ge \left(\frac{M}{m}\right)^{-\frac{1}{4}\left(m^{-1}-M^{-1}\right)\left\langle A^2x,x\right\rangle} \ge \left(\frac{M}{m}\right)^{-\frac{1}{4}\left(m^{-1}-M^{-1}\right)M^2} \end{aligned} \tag{2.8}$$

for $x\in H$, $\|x\|=1$.

Proof. Observe that

$$\begin{aligned}
&\eta_{\frac{Ax}{\|Ax\|}}\left(A^{-1}\right)\\
&=\exp\left(-\left\langle A^{-1}\left(\ln A^{-1}\right)\frac{Ax}{\|Ax\|},\frac{Ax}{\|Ax\|}\right\rangle\right)\\
&=\exp\left(\frac{1}{\|Ax\|^{2}}\langle A\ln Ax,x\rangle\right)=\exp\left(\frac{-1}{\|Ax\|^{2}}\langle -A\ln Ax,x\rangle\right)\\
&=\left[\eta_{x}(A)\right]^{-\frac{1}{\|Ax\|^{2}}}=\left[\eta_{x}(A)\right]^{-\frac{1}{\langle A^{2}x,x\rangle}}
\end{aligned}$$

for $x\in H$, $\|x\|=1$, which gives that

$$\eta_{x}(A)=\left[\eta_{\frac{Ax}{\|Ax\|}}\left(A^{-1}\right)\right]^{-\left\langle A^{2}x,x\right\rangle} \tag{2.9}$$

for $x\in H$, $\|x\|=1$.

Since $0<M^{-1}\leq A^{-1}\leq m^{-1}$, hence by (2.1) written for A^{-1} we get

$$\begin{aligned}
1&\leq\frac{\eta_{\frac{Ax}{\|Ax\|}}\left(A^{-1}\right)}{\left[\left(M^{M^{-1}}\right)^{-\frac{m^{-1}-\left\langle A^{-1}\frac{Ax}{\|Ax\|},\frac{Ax}{\|Ax\|}\right\rangle}{m^{-1}-M^{-1}}}\left(m^{m^{-1}}\right)^{-\frac{\left\langle A^{-1}\frac{Ax}{\|Ax\|},\frac{Ax}{\|Ax\|}\right\rangle-M^{-1}}{m^{-1}-M^{-1}}}\right]^{-1}}\\
&\leq\left(\frac{m^{-1}}{M^{-1}}\right)^{\frac{\left\langle\left(A^{-1}-M^{-1}\right)\left(m^{-1}-A^{-1}\right)\frac{Ax}{\|Ax\|},\frac{Ax}{\|Ax\|}\right\rangle}{m^{-1}-M^{-1}}}\\
&\leq\left(\frac{m^{-1}}{M^{-1}}\right)^{\frac{\left(m^{-1}-\left\langle A^{-1}\frac{Ax}{\|Ax\|},\frac{Ax}{\|Ax\|}\right\rangle\right)\left(\left\langle A^{-1}\frac{Ax}{\|Ax\|},\frac{Ax}{\|Ax\|}\right\rangle-M^{-1}\right)}{m^{-1}-M^{-1}}}\leq\left(\frac{m^{-1}}{M^{-1}}\right)^{\frac{1}{4}\left(m^{-1}-M^{-1}\right)}
\end{aligned}$$

for $x\in H$, $\|x\|=1$, which is equivalent to

$$\begin{aligned}
1&\leq\frac{\eta_{\frac{Ax}{\|Ax\|}}\left(A^{-1}\right)}{(M)^{\frac{\left(m^{-1}\left\langle A^{2}x,x\right\rangle-\langle Ax,x\rangle\right)M^{-1}}{\left(m^{-1}-M^{-1}\right)\left\langle A^{2}x,x\right\rangle}}(m)^{\frac{\left(\langle Ax,x\rangle-M^{-1}\left\langle A^{2}x,x\right\rangle\right)m^{-1}}{\left(m^{-1}-M^{-1}\right)\left\langle A^{2}x,x\right\rangle}}}\\
&\leq\left(\frac{M}{m}\right)^{\frac{\left\langle\left(1-M^{-1}A\right)\left(m^{-1}A-1\right)x,x\right\rangle}{\left\langle A^{2}x,x\right\rangle\left(m^{-1}-M^{-1}\right)}}\\
&\leq\left(\frac{M}{m}\right)^{\frac{\left(m^{-1}\left\langle A^{2}x,x\right\rangle-\langle Ax,x\rangle\right)\left(\langle Ax,x\rangle-M^{-1}\left\langle A^{2}x,x\right\rangle\right)}{\left(m^{-1}-M^{-1}\right)\left\langle A^{2}x,x\right\rangle^{2}}}\\
&\leq\left(\frac{M}{m}\right)^{\frac{1}{4}\left(m^{-1}-M^{-1}\right)}
\end{aligned}$$

for $x\in H$, $\|x\|=1$.

By taking the power $-\left\langle A^2x,x\right\rangle < 0$, we get

$$
\begin{aligned}
1 &\geq \frac{\left[\eta_{\frac{Ax}{\|Ax\|}}\left(A^{-1}\right)\right]^{-\left\langle A^2x,x\right\rangle}}{M^{-\frac{\left(m^{-1}\left\langle A^2x,x\right\rangle-\left\langle Ax,x\right\rangle\right)M^{-1}}{m^{-1}-M^{-1}}}m^{-\frac{\left(\left\langle Ax,x\right\rangle-M^{-1}\left\langle A^2x,x\right\rangle\right)m^{-1}}{m^{-1}-M^{-1}}}} \\
&\geq \left(\frac{M}{m}\right)^{-\frac{\left\langle\left(1-M^{-1}A\right)\left(m^{-1}A-1\right)x,x\right\rangle}{m^{-1}-M^{-1}}} \\
&\geq \left(\frac{M}{m}\right)^{-\frac{\left(m^{-1}\left\langle A^2x,x\right\rangle-\left\langle Ax,x\right\rangle\right)\left(\left\langle Ax,x\right\rangle-M^{-1}\left\langle A^2x,x\right\rangle\right)}{\left(m^{-1}-M^{-1}\right)\left\langle A^2x,x\right\rangle}} \geq \left(\frac{M}{m}\right)^{-\frac{1}{4}\left(m^{-1}-M^{-1}\right)\left\langle A^2x,x\right\rangle} \\
&\geq \left(\frac{M}{m}\right)^{-\frac{1}{4}\left(m^{-1}-M^{-1}\right)M^2}
\end{aligned}
$$

for $x \in H$, $\|x\| = 1$.

By utilizing (2.9) we derive (2.8). □

We also have:

Theorem 2. *Assume that A satisfies the condition $0 < m \leq A \leq M$, where m, M are positive numbers, then*

$$
\begin{aligned}
1 &\leq \exp\left[\frac{1}{2M}\left\langle\left(A-m\right)\left(M-A\right)x,x\right\rangle\right] \\
&\leq \frac{\eta_x(A)}{\left[m^{\frac{m(M-\langle Ax,x\rangle)}{M-m}}M^{\frac{M(\langle Ax,x\rangle-m)}{M-m}}\right]^{-1}} \\
&\leq \exp\left[\frac{1}{2m}\left\langle\left(A-m\right)\left(M-A\right)x,x\right\rangle\right] \\
&\leq \exp\left[\frac{1}{2m}\left(M-\langle Ax,x\rangle\right)\left(\langle Ax,x\rangle-m\right)\right] \leq \exp\left[\frac{1}{8m}\left(M-m\right)^2\right]
\end{aligned} \tag{2.10}
$$

for $x \in H$, $\|x\| = 1$.

Proof. In [2] we obtained the following result as well: let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a twice differentiable function on the interval $\mathring{I}$, the interior of I. If there exists the constants d, D such that

$$d \leq f''(t) \leq D \text{ for any } t \in \mathring{I},$$

then

$$
\begin{aligned}
\frac{1}{2}\nu\left(1-\nu\right)d\left(b-a\right)^2 &\leq \left(1-\nu\right)f\left(a\right)+\nu f\left(b\right)-f\left(\left(1-\nu\right)a+\nu b\right) \\
&\leq \frac{1}{2}\nu\left(1-\nu\right)D\left(b-a\right)^2
\end{aligned} \tag{2.11}
$$

for any a, $b \in \mathring{I}$ and $\nu \in [0,1]$.

Consider the convex function $f(t) = t\ln t$, $t > 0$. Then $f''(t) = \frac{1}{t}$, $t > 0$ and

$$\frac{1}{M} \leq f''(t) \leq \frac{1}{m} \text{ for } t \in [m, M] \subset (0,\infty).$$

Now, by writing the inequality (2.11) for this function, $a = m$ and $b = M$, we get

$$\begin{aligned}
0 &\leq \frac{1}{2M}(M-m)^2 \nu(1-\nu) \\
&\leq (1-\nu) m \ln m + \nu M \ln M - ((1-\nu) m + \nu M) \ln ((1-\nu) m + \nu M) \\
&\leq \frac{1}{2m}(M-m)^2 \nu(1-\nu)
\end{aligned}$$

for all $\nu \in [0,1]$ and $[m, M] \subset (0, \infty)$.

By putting

$$\nu = \frac{t-m}{M-m},$$

then we get for $t \in [m, M]$ that $\nu \in [0,1]$ and

$$\begin{aligned}
0 &\leq \frac{1}{2M}(t-m)(M-t) \\
&\leq \frac{M-t}{M-m} m \ln m + \frac{t-m}{M-m} M \ln M - t \ln t \leq \frac{1}{2m}(t-m)(M-t)
\end{aligned}$$

for all $t \in [m, M]$.

If we use the continuous functional calculus for selfadjoint operator A satisfying the condition $0 < m \leq A \leq M$, we obtain

$$\begin{aligned}
0 &\leq \frac{1}{2M}(A-m)(M-A) \\
&\leq \frac{M-A}{M-m} m \ln m + \frac{A-m}{M-m} M \ln M - A \ln A \\
&\leq \frac{1}{2m}(A-m)(M-A).
\end{aligned}$$

If we take the inner product for $x \in H$, $\|x\| = 1$ in (2.5), then we get

$$\begin{aligned}
0 &\leq \frac{1}{2M}\langle (A-m)(M-A) x, x\rangle \\
&\leq m \ln m \frac{M - \langle Ax, x\rangle}{M-m} + M \ln M \frac{\langle Ax, x\rangle - m}{M-m} - \langle A \ln Ax, x\rangle \\
&\leq \frac{1}{2m}\langle (A-m)(M-A) x, x\rangle \leq \frac{1}{2m}(M - \langle Ax, x\rangle)(\langle Ax, x\rangle - m) \\
&\leq \frac{1}{8m}(M-m)^2
\end{aligned}$$

and by taking the exponential, we derive

$$\begin{aligned}
1 &\leq \exp\left[\frac{1}{2M}\langle (A-m)(M-A) x, x\rangle\right] \\
&\leq \exp\left[m \ln m \frac{M - \langle Ax, x\rangle}{M-m} + M \ln M \frac{\langle Ax, x\rangle - m}{M-m} - \langle A \ln Ax, x\rangle\right] \\
&\leq \exp\left[\frac{1}{2m}\langle (A-m)(M-A) x, x\rangle\right] \\
&\leq \exp\left[\frac{1}{2m}(M - \langle Ax, x\rangle)(\langle Ax, x\rangle - m)\right] \leq \exp\left[\frac{1}{8m}(M-m)^2\right],
\end{aligned}$$

which is equivalent to (2.10). □

Corollary 2. *With the assumptions of Theorem 2 we have*

$$1 \geq \exp\left[-\frac{\left\langle\left(1-M^{-1}A\right)\left(m^{-1}A-1\right)x,x\right\rangle}{2M\left(m^{-1}-M^{-1}\right)}\right] \tag{2.12}$$
$$\geq \frac{\eta_x(A)}{\left[m^{\frac{m(M-\langle Ax,x\rangle)}{M-m}}M^{\frac{M(\langle Ax,x\rangle-m)}{M-m}}\right]^{-1}}$$
$$\geq \exp\left[-\frac{\left\langle\left(1-M^{-1}A\right)\left(m^{-1}A-1\right)x,x\right\rangle}{2m\left(m^{-1}-M^{-1}\right)}\right]$$
$$\geq \exp\left[-\frac{\left(m^{-1}\left\langle A^2x,x\right\rangle-\langle Ax,x\rangle\right)\left(\langle Ax,x\rangle-M^{-1}\left\langle A^2x,x\right\rangle\right)}{2m\left(m^{-1}-M^{-1}\right)\left\langle A^2x,x\right\rangle}\right]$$
$$\geq \exp\left[-\frac{1}{8m}\left(M-m\right)^2\left\langle A^2x,x\right\rangle\right] \geq \exp\left[-\frac{1}{8m}\left(M-m\right)^2 M^2\right]$$

for $x \in H$, $\|x\| = 1$.

3. Related Results

From a different perspective, we can state the following result as well:

Theorem 3. *Assume that* A *satisfies the condition* $0 < m \leq A \leq M$, *where* m, M *are positive numbers, then*

$$1 \leq \left[\frac{\sqrt{m^m M^M}}{\left(\frac{m+M}{2}\right)^{\frac{m+M}{2}}}\right]^{2\left(\frac{1}{2}-\left\langle\left|\frac{A-m}{M-m}-\frac{1}{2}\right|x,x\right\rangle\right)} \tag{3.1}$$
$$\leq \frac{\eta_x(A)}{\left[m^{\frac{m(M-\langle Ax,x\rangle)}{M-m}}M^{\frac{M(\langle Ax,x\rangle-m)}{M-m}}\right]^{-1}}$$
$$\leq \left[\frac{\sqrt{m^m M^M}}{\left(\frac{m+M}{2}\right)^{\frac{m+M}{2}}}\right]^{2\left(\frac{1}{2}+\left\langle\left|\frac{A-m}{M-m}-\frac{1}{2}\right|x,x\right\rangle\right)} \leq \left[\frac{\sqrt{m^m M^M}}{\left(\frac{m+M}{2}\right)^{\frac{m+M}{2}}}\right]^{2}$$

for $x \in H$, $\|x\| = 1$.

Proof. We use the following result: let $f : [a, b] \to \mathbb{R}$ be a convex function on $[a, b]$ and $t \in [0, 1]$, then

$$0 \leq 2\min\{t, 1-t\}\left[\frac{f(a)+f(b)}{2} - f\left(\frac{a+b}{2}\right)\right] \tag{3.2}$$
$$\leq (1-t)f(a) + tf(b) - f((1-t)a + tb)$$
$$\leq 2\max\{t, 1-t\}\left[\frac{f(a)+f(b)}{2} - f\left(\frac{a+b}{2}\right)\right].$$

The proof follows, for instance, by Corollary 1 from [**?**] for $n = 2$, $p_1 = 1-t$, $p_2 = t$, $t \in [0, 1]$ and $x_1 = a$, $x_2 = b$.

Observe that

$$\min\{t, 1-t\} = \frac{1}{2} - \left|t - \frac{1}{2}\right|$$

and

$$\max\{t, 1-t\} = \frac{1}{2} + \left|t - \frac{1}{2}\right|$$

and if we take $a = m < M = b$ and $t = \frac{u-m}{M-m}$ with $u \in [m, M]$, then by (3.2) we get

$$\begin{aligned} 0 &\le 2\left(\frac{1}{2} - \left|\frac{u-m}{M-m} - \frac{1}{2}\right|\right)\left[\frac{f(m)+f(M)}{2} - f\left(\frac{m+M}{2}\right)\right] \\ &\le \frac{M-u}{M-m}f(m) + \frac{u-m}{M-m}f(M) - f(u) \\ &\le 2\left(\frac{1}{2} + \left|\frac{u-m}{M-m} - \frac{1}{2}\right|\right)\left[\frac{f(m)+f(M)}{2} - f\left(\frac{m+M}{2}\right)\right] \end{aligned} \tag{3.3}$$

for $u \in [m, M]$.

If we apply the inequality (3.3) for the convex function $f(t) = t\ln t$, $t > 0$, then we get

$$\begin{aligned} 0 &\le 2\left(\frac{1}{2} - \left|\frac{u-m}{M-m} - \frac{1}{2}\right|\right) \\ &\quad \times \left[\frac{m\ln m + M\ln M}{2} - \left(\frac{m+M}{2}\right)\ln\left(\frac{m+M}{2}\right)\right] \\ &\le \frac{M-u}{M-m}m\ln m + \frac{u-m}{M-m}M\ln M - u\ln u \\ &\le 2\left(\frac{1}{2} + \left|\frac{u-m}{M-m} - \frac{1}{2}\right|\right) \\ &\quad \times \left[\frac{m\ln m + M\ln M}{2} - \left(\frac{m+M}{2}\right)\ln\left(\frac{m+M}{2}\right)\right], \end{aligned}$$

namely

$$\begin{aligned} 0 &\le 2\left(\frac{1}{2} - \left|\frac{u-m}{M-m} - \frac{1}{2}\right|\right)\ln\left[\frac{\sqrt{m^m M^M}}{\left(\frac{m+M}{2}\right)^{\frac{m+M}{2}}}\right] \\ &\le \frac{M-u}{M-m}m\ln m + \frac{u-m}{M-m}M\ln M - u\ln u \\ &\le 2\left(\frac{1}{2} + \left|\frac{u-m}{M-m} - \frac{1}{2}\right|\right)\ln\left[\frac{\sqrt{m^m M^M}}{\left(\frac{m+M}{2}\right)^{\frac{m+M}{2}}}\right] \end{aligned}$$

for $u \in [m, M]$.

Using a similar argument as above, we deduce for $0 < m \leq A \leq M$, that

$$
\begin{aligned}
0 &\leq 2\left(\frac{1}{2} - \left\langle \left| \frac{A-m}{M-m} - \frac{1}{2} \right| x, x \right\rangle\right) \ln\left[\frac{\sqrt{m^m M^M}}{\left(\frac{m+M}{2}\right)^{\frac{m+M}{2}}}\right] \\
&\leq m \ln m \frac{M - \langle Ax, x\rangle}{M-m} + M \ln M \frac{\langle Ax, x\rangle - m}{M-m} - \langle A \ln Ax, x\rangle \\
&\leq 2\left(\frac{1}{2} + \left\langle \left| \frac{A-m}{M-m} - \frac{1}{2} \right| x, x \right\rangle\right) \ln\left[\frac{\sqrt{m^m M^M}}{\left(\frac{m+M}{2}\right)^{\frac{m+M}{2}}}\right] \\
&\leq \ln\left[\frac{\sqrt{m^m M^M}}{\left(\frac{m+M}{2}\right)^{\frac{m+M}{2}}}\right]^2
\end{aligned}
$$

for $x \in H$, $\|x\| = 1$.

By taking the exponential we derive the desired result (3.1). $\square$

Corollary 3. *With the assumptions of Theorem 3, we have*

$$
\begin{aligned}
(3.4)\qquad 1 &\geq \left[\frac{\sqrt{m^{-m^{-1}} M^{-M^{-1}}}}{\left(\frac{m^{-1}+M^{-1}}{2}\right)^{\frac{m^{-1}+M^{-1}}{2}}}\right]^{-2\left(\frac{1}{2}\left\langle A^2 x, x\right\rangle - \left\langle \left| \frac{A - M^{-1}A^2}{m^{-1}-M^{-1}} - \frac{1}{2}A^2 \right| x, x \right\rangle\right)} \\
&\geq \frac{\eta_x(A)}{\left[M^{\frac{\left(m^{-1}\left\langle A^2 x, x\right\rangle - \langle Ax, x\rangle\right)M^{-1}}{m^{-1}-M^{-1}}} m^{\frac{\left(\langle Ax, x\rangle - M^{-1}\left\langle A^2 x, x\right\rangle\right)m^{-1}}{m^{-1}-M^{-1}}}\right]^{-1}} \\
&\geq \left[\frac{\sqrt{m^{-m^{-1}} M^{-M^{-1}}}}{\left(\frac{m^{-1}+M^{-1}}{2}\right)^{\frac{m^{-1}+M^{-1}}{2}}}\right]^{-2\left(\frac{1}{2}\left\langle A^2 x, x\right\rangle + \left\langle \left| \frac{A - M^{-1}A^2}{m^{-1}-M^{-1}} - \frac{1}{2}A^2 \right| x, x \right\rangle\right)} \\
&\geq \left[\frac{\sqrt{m^{-m^{-1}} M^{-M^{-1}}}}{\left(\frac{m^{-1}+M^{-1}}{2}\right)^{\frac{m^{-1}+M^{-1}}{2}}}\right]^{-2\left\langle A^2 x, x\right\rangle} \geq \left[\frac{\sqrt{m^{-m^{-1}} M^{-M^{-1}}}}{\left(\frac{m^{-1}+M^{-1}}{2}\right)^{\frac{m^{-1}+M^{-1}}{2}}}\right]^{-2M^2}
\end{aligned}
$$

for $x \in H$, $\|x\| = 1$.

Proof. If we write the inequality for A^{-1} and $\frac{Ax}{\|Ax\|}$, we get

$$\begin{aligned}
1 &\leq \left[\frac{\sqrt{m^{-m^{-1}}M^{-M^{-1}}}}{\left(\frac{m^{-1}+M^{-1}}{2}\right)^{\frac{m^{-1}+M^{-1}}{2}}}\right]^{2\left(\frac{1}{2}-\frac{1}{\langle A^2x,x\rangle}\left\langle\left|\frac{A-M^{-1}A^2}{m^{-1}-M^{-1}}-\frac{1}{2}A^2\right|x,x\right\rangle\right)} \\
&\leq \frac{\eta_{\frac{Ax}{\|Ax\|}}(A^{-1})}{(M)^{\frac{(m^{-1}\langle A^2x,x\rangle-\langle Ax,x\rangle)M^{-1}}{(m^{-1}-M^{-1})\langle A^2x,x\rangle}}(m)^{\frac{(\langle Ax,x\rangle-M^{-1}\langle A^2x,x\rangle)m^{-1}}{(m^{-1}-M^{-1})\langle A^2x,x\rangle}}} \\
&\leq \left[\frac{\sqrt{m^{-m^{-1}}M^{-M^{-1}}}}{\left(\frac{m^{-1}+M^{-1}}{2}\right)^{\frac{m^{-1}+M^{-1}}{2}}}\right]^{2\left(\frac{1}{2}+\frac{1}{\langle A^2x,x\rangle}\left\langle\left|\frac{A-M^{-1}A^2}{m^{-1}-M^{-1}}-\frac{1}{2}A^2\right|x,x\right\rangle\right)} \\
&\leq \left[\frac{\sqrt{m^{-m^{-1}}M^{-M^{-1}}}}{\left(\frac{m^{-1}+M^{-1}}{2}\right)^{\frac{m^{-1}+M^{-1}}{2}}}\right]^{2}
\end{aligned}$$

for $x \in H$, $\|x\| = 1$.

By taking the power $-\langle A^2x, x\rangle < 0$, we get

$$\begin{aligned}
1 &\geq \left[\frac{\sqrt{m^{-m^{-1}}M^{-M^{-1}}}}{\left(\frac{m^{-1}+M^{-1}}{2}\right)^{\frac{m^{-1}+M^{-1}}{2}}}\right]^{-2\left(\frac{1}{2}\langle A^2x,x\rangle-\left\langle\left|\frac{A-M^{-1}A^2}{m^{-1}-M^{-1}}-\frac{1}{2}A^2\right|x,x\right\rangle\right)} \\
&\geq \frac{\left[\eta_{\frac{Ax}{\|Ax\|}}(A^{-1})\right]^{-\langle A^2x,x\rangle}}{M^{-\frac{(m^{-1}\langle A^2x,x\rangle-\langle Ax,x\rangle)M^{-1}}{m^{-1}-M^{-1}}}m^{-\frac{(\langle Ax,x\rangle-M^{-1}\langle A^2x,x\rangle)m^{-1}}{m^{-1}-M^{-1}}}} \\
&\geq \left[\frac{\sqrt{m^{-m^{-1}}M^{-M^{-1}}}}{\left(\frac{m^{-1}+M^{-1}}{2}\right)^{\frac{m^{-1}+M^{-1}}{2}}}\right]^{-2\left(\frac{1}{2}\langle A^2x,x\rangle+\left\langle\left|\frac{A-M^{-1}A^2}{m^{-1}-M^{-1}}-\frac{1}{2}A^2\right|x,x\right\rangle\right)} \\
&\geq \left[\frac{\sqrt{m^{-m^{-1}}M^{-M^{-1}}}}{\left(\frac{m^{-1}+M^{-1}}{2}\right)^{\frac{m^{-1}+M^{-1}}{2}}}\right]^{-2\langle A^2x,x\rangle} \geq \left[\frac{\sqrt{m^{-m^{-1}}M^{-M^{-1}}}}{\left(\frac{m^{-1}+M^{-1}}{2}\right)^{\frac{m^{-1}+M^{-1}}{2}}}\right]^{-2M^2}
\end{aligned}$$

for $x \in H$, $\|x\| = 1$. □

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