SOME REVERSE INEQUALITIES FOR THE NORMALIZED ENTROPIC DETERMINANT OF POSITIVE OPERATORS IN HILBERT SPACES

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Abstract. For positive invertible operators A on a Hilbert space H and a fixed unit vector $x \in H$, define the *normalized entropic determinant* by $\eta_x(A) := \exp[-\langle A \ln Ax, x\rangle]$. In this paper we show among others that, if $A_j > 0$ and $p_j \geq 0$, $j \in \{1, ..., n\}$ with $\sum_{j=1}^n p_j = 1$, then

$$1 \leq \frac{\prod_{j=1}^n [\eta_x(A_j)]^{\frac{p_j}{\langle A_j x,x\rangle}}}{\left(\sum_{j=1}^n \frac{p_j\left\langle A_j^2 x,x\right\rangle}{\langle A_j x,x\rangle}\right)^{-1}} \leq \exp\left(\sum_{j=1}^n \frac{p_j\left\langle A_j^2 x,x\right\rangle}{\langle A_j x,x\rangle}\sum_{j=1}^n \frac{p_j}{\langle A_j x,x\rangle} - 1\right)$$

and

$$1 \leq \frac{\sum_{j=1}^n \frac{p_j}{\langle A_j x,x\rangle}}{\prod_{j=1}^n [\eta_x(A_j)]^{\frac{p_j}{\langle A_j x,x\rangle}}} \leq \exp\left[\sum_{j=1}^n \frac{p_j\left\langle A_j^2 x,x\right\rangle}{\langle A_j x,x\rangle}\sum_{j=1}^n \frac{p_j}{\langle A_j x,x\rangle} - 1\right].$$

1. Introduction

Let $B(H)$ be the space of all bounded linear operators on a Hilbert space H, and I stands for the identity operator on H. An operator A in $B(H)$ is said to be positive (in symbol: $A \geq 0$) if $\langle Ax, x\rangle \geq 0$ for all $x \in H$. In particular, $A > 0$ means that A is positive and invertible. For a pair A, B of selfadjoint operators the order relation $A \geq B$ means as usual that $A - B$ is positive.

In 1998, Fujii et al. [7], [8], introduced the *normalized determinant* $\Delta_x(A)$ for positive invertible operators A on a Hilbert space H and a fixed unit vector $x \in H$, namely $\|x\| = 1$, defined by $\Delta_x(A) := \exp\langle \ln Ax, x\rangle$ and discussed it as a continuous geometric mean and observed some inequalities around the determinant from this point of view.

Some of the fundamental properties of normalized determinant are as follows, [7].

For each unit vector $x \in H$, see also [10], we have:

(i) *continuity*: the map $A \to \Delta_x(A)$ is norm continuous;
(ii) *bounds*: $\left\langle A^{-1}x, x\right\rangle^{-1} \leq \Delta_x(A) \leq \langle Ax, x\rangle$;
(iii) *continuous mean*: $\langle A^p x, x\rangle^{1/p} \downarrow \Delta_x(A)$ for $p \downarrow 0$ and $\langle A^p x, x\rangle^{1/p} \uparrow \Delta_x(A)$ for $p \uparrow 0$;
(iv) *power equality*: $\Delta_x(A^t) = \Delta_x(A)^t$ for all $t > 0$;

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(v) *homogeneity*: $\Delta_x(tA) = t\Delta_x(A)$ and $\Delta_x(tI) = t$ for all $t > 0$;
(vi) *monotonicity*: $0 < A \leq B$ implies $\Delta_x(A) \leq \Delta_x(B)$;
(vii) *multiplicativity*: $\Delta_x(AB) = \Delta_x(A)\Delta_x(B)$ for commuting A and B;
(viii) *Ky Fan type inequality*: $\Delta_x((1-\alpha)A + \alpha B) \geq \Delta_x(A)^{1-\alpha}\Delta_x(B)^{\alpha}$ for $0 < \alpha < 1$.

We define the logarithmic mean of two positive numbers a, b by

$$L(a,b) := \begin{cases} \frac{b-a}{\ln b - \ln a} \text{ if } b \neq a, \\ \\ a \text{ if } b = a. \end{cases} \tag{1.1}$$

In [7] the authors obtained the following additive reverse inequality for the operator A which satisfy the condition $0 < mI \leq A \leq MI$, where m, M are positive numbers,

$$0 \leq \langle Ax, x\rangle - \Delta_x(A) \leq L(m,M)\left[\ln L(m,M) + \frac{M\ln m - m\ln M}{M-m} - 1\right] \tag{1.2}$$

for all $x \in H$, $\|x\| = 1$.

We recall that *Specht's ratio* is defined by [14]

$$S(h) := \begin{cases} \frac{h^{\frac{1}{h-1}}}{e\ln\left(h^{\frac{1}{h-1}}\right)} \text{ if } h \in (0,1)\cup(1,\infty), \\ \\ 1 \text{ if } h = 1. \end{cases} \tag{1.3}$$

It is well known that $\lim_{h\to 1} S(h) = 1$, $S(h) = S\left(\frac{1}{h}\right) > 1$ for $h > 0$, $h \neq 1$. The function is decreasing on $(0,1)$ and increasing on $(1,\infty)$.

In [8], the authors obtained the following multiplicative reverse inequality as well

$$1 \leq \frac{\langle Ax, x\rangle}{\Delta_x(A)} \leq S\left(\frac{M}{m}\right) \tag{1.4}$$

for $0 < mI \leq A \leq MI$ and $x \in H$, $\|x\| = 1$.

For the entropy function $\eta(t) = -t\ln t$, $t > 0$, the operator entropy has the following expression:

$$\eta(A) = -A\ln A$$

for positive A.

For $x \in H$, $\|x\| = 1$, we define the *normalized entropic determinant* $\eta_x(A)$ by

$$\eta_x(A) := \exp(-\langle A\ln Ax, x\rangle) = \exp\langle \eta(A)x, x\rangle. \tag{1.5}$$

Let $x \in H$, $\|x\| = 1$. Observe that the map $A \to \eta_x(A)$ is *norm continuous* and since

$$\begin{aligned} &\exp(-\langle tA\ln(tA)x, x\rangle) \\ &= \exp(-\langle tA(\ln t + \ln A)x, x\rangle) = \exp(-\langle (tA\ln t + tA\ln A)x, x\rangle) \\ &= \exp(-\langle Ax, x\rangle t\ln t)\exp(-t\langle A\ln Ax, x\rangle) \\ &= \exp\ln\left(t^{-\langle Ax,x\rangle t}\right)[\exp(-\langle A\ln Ax, x\rangle)]^{-t}, \end{aligned}$$

hence

$$\eta_x(tA) = t^{-t\langle Ax,x\rangle}[\eta_x(A)]^{-t} \tag{1.6}$$

for $t > 0$ and $A > 0$.

Observe also that

$$\eta_x(I) = 1 \text{ and } \eta_x(tI) = t^{-t} \tag{1.7}$$

for $t > 0$.

In [2] we showed that, if $A, B > 0$, then for all $x \in H$ with $\|x\| = 1$ and $t \in [0, 1]$,

$$\eta_x((1-t)A + tB) \geq (\eta_x(A))^{1-t} (\eta_x(B))^t .$$

We also have the bounds

$$\left(\frac{\langle A^2 x, x \rangle}{\langle Ax, x \rangle} \right)^{-\langle Ax, x \rangle} \leq \eta_x(A) \leq \langle Ax, x \rangle^{-\langle Ax, x \rangle},$$

where $A > 0$ and $x \in H$ with$\|x\| = 1$.

If $y \in H$, $y \neq 0$, then we can extend the definition of the normalized entropic determinant as follows

$$\breve{\eta}_y(A) := \exp(-\langle A \ln Ay, y \rangle) = \exp \langle \eta(A) y, y \rangle .$$

Also we can consider

$$\breve{\Delta}_y(A) := \exp \langle \ln Ay, y \rangle$$

for $y \in H$, $y \neq 0$.

We observe that

$$\breve{\eta}_y(A) = \left[\eta_{\frac{y}{\|y\|}}(A) \right]^{\|y\|^2} \text{ and } \breve{\Delta}_y(A) = \left[\Delta_{\frac{y}{\|y\|}}(A) \right]^{\|y\|^2}$$

for $y \in H$, $y \neq 0$.

Motivated by the above results, in this paper we show among others that, if $A_j > 0$ and $p_j \geq 0$, $j \in \{1, ..., n\}$ with $\sum_{j=1}^n p_j = 1$, then

$$1 \leq \frac{\prod_{j=1}^n [\eta_x(A_j)]^{\frac{p_j}{\langle A_j x, x \rangle}}}{\left(\sum_{j=1}^n \frac{p_j \langle A_j^2 x, x \rangle}{\langle A_j x, x \rangle} \right)^{-1}} \leq \exp \left(\sum_{j=1}^n \frac{p_j \langle A_j^2 x, x \rangle}{\langle A_j x, x \rangle} \sum_{j=1}^n \frac{p_j}{\langle A_j x, x \rangle} - 1 \right)$$

and

$$1 \leq \frac{\sum_{j=1}^n \frac{p_j}{\langle A_j x, x \rangle}}{\prod_{j=1}^n [\eta_x(A_j)]^{\frac{p_j}{\langle A_j x, x \rangle}}} \leq \exp \left[\sum_{j=1}^n \frac{p_j \langle A_j^2 x, x \rangle}{\langle A_j x, x \rangle} \sum_{j=1}^n \frac{p_j}{\langle A_j x, x \rangle} - 1 \right].$$

2. Main Results

We start to the following result:

Theorem 1. *Assume that* $0 < m \le A_j \le M$ *and* $x_j \in H$, $j \in \{1, ..., n\}$ *with* $\sum_{j=1}^{n} \|x_j\|^2 = 1$, *then*

$$
\begin{aligned}
(2.1)\qquad 1 &\le \frac{\left(\sum_{j=1}^{n} \langle A_j x_j, x_j\rangle\right)^{-\sum_{j=1}^{n}\langle A_j x_j, x_j\rangle}}{\prod_{j=1}^{n}\left[\eta_{\frac{x_j}{\|x_j\|}}(A_j)\right]^{\|x_j\|^2}} \\
&\le \frac{\left[\prod_{j=1}^{n}\left[\Delta_{\frac{x_j}{\|x_j\|}}(eA_j)\right]^{\|x_j\|^2}\right]^{-\sum_{j=1}^{n}\langle A_j x_j, x_j\rangle}}{\left(\prod_{j=1}^{n}\left[\eta_{\frac{x_j}{\|x_j\|}}(A_j)\right]^{\|x_j\|^2}\right)^{1/e}} \\
&\le \begin{cases} \exp\Big\{\frac{1}{2}(M-m) \\ \quad\times\left[\sum_{j=1}^{n}\|\ln(eA_j)x_j\|^2 - \left(\sum_{j=1}^{n}\langle \ln(eA_j)x_j, x_j\rangle\right)^2\right]^{1/2}\Big\}, \\ \\ \left(\frac{M}{m}\right)^{\frac{1}{2}\left[\sum_{j=1}^{n}\|A_j x_j\|^2 - \left(\sum_{j=1}^{n}\langle A_j x_j, x_j\rangle\right)^2\right]^{1/2}}, \end{cases} \\
&\le \left(\frac{M}{m}\right)^{\frac{1}{4}(M-m)}.
\end{aligned}
$$

The first two inequalities are also valid for $A_j > 0$, $j \in \{1, ..., n\}$.

Proof. Let I be an interval and $f : I \to \mathbb{R}$ be a convex and differentiable function on $\mathring{I}$ (the interior of I) whose derivative f' is continuous on $\mathring{I}$. In [3] we obtained among others that, if A_j are selfadjoint operators with $\operatorname{Sp}(A_j) \subseteq [m, M] \subset \mathring{I}$, $j \in \{1, ..., n\}$, then

$$
\begin{aligned}
(2.2)\qquad 0 &\le \sum_{j=1}^{n}\langle f(A_j)x_j, x_j\rangle - f\left(\sum_{j=1}^{n}\langle A_j x_j, x_j\rangle\right) \\
&\le \sum_{j=1}^{n}\langle f'(A_j)A_j x_j, x_j\rangle - \sum_{j=1}^{n}\langle A_j x_j, x_j\rangle\sum_{j=1}^{n}\langle f'(A_j)x_j, x_j\rangle \\
&\le \begin{cases} \frac{1}{2}(M-m)\left[\sum_{j=1}^{n}\|f'(A_j)x_j\|^2 - \left(\sum_{j=1}^{n}\langle f'(A_j)x_j, x_j\rangle\right)^2\right]^{1/2}, \\ \\ \frac{1}{2}(f'(M)-f'(m))\left[\sum_{j=1}^{n}\|A_j x_j\|^2 - \left(\sum_{j=1}^{n}\langle A_j x_j, x_j\rangle\right)^2\right]^{1/2}, \end{cases} \\
&\le \frac{1}{4}(M-m)(f'(M)-f'(m)),
\end{aligned}
$$

for any $x_j \in H, j \in \{1, ..., n\}$ with $\sum_{j=1}^{n}\|x_j\|^2 = 1$.

If we write the inequality (2.2) for the convex function $f(t) = t \ln t$, $t > 0$, then we get

$$
\begin{aligned}
(2.3)\quad 0 &\leq \sum_{j=1}^{n} \langle A_j \ln A_j x_j, x_j \rangle - \left(\sum_{j=1}^{n} \langle A_j x_j, x_j \rangle \right) \ln \left(\sum_{j=1}^{n} \langle A_j x_j, x_j \rangle \right) \\
&\leq \sum_{j=1}^{n} \langle A_j \ln (eA_j) x_j, x_j \rangle - \sum_{j=1}^{n} \langle A_j x_j, x_j \rangle \sum_{j=1}^{n} \langle \ln (eA_j) x_j, x_j \rangle \\
&\leq \begin{cases} \frac{1}{2}(M-m) \left[\sum_{j=1}^{n} \|\ln (eA_j) x_j\|^2 - \left(\sum_{j=1}^{n} \langle \ln (eA_j) x_j, x_j \rangle \right)^2 \right]^{1/2}, \\ \ln \left(\frac{M}{m} \right)^{\frac{1}{2}\left[\sum_{j=1}^{n} \|A_j x_j\|^2 - \left(\sum_{j=1}^{n} \langle A_j x_j, x_j \rangle \right)^2 \right]^{1/2}}, \end{cases} \\
&\leq \ln \left(\frac{M}{m} \right)^{\frac{1}{4}(M-m)},
\end{aligned}
$$

for any $x_j \in H$, $j \in \{1, ..., n\}$ with $\sum_{j=1}^{n} \|x_j\|^2 = 1$.

If we take the exponential in (2.3), then we get

$$
\begin{aligned}
(2.4)\quad 1 &\leq \exp \left[\sum_{j=1}^{n} \langle A_j \ln A_j x_j, x_j \rangle - \left(\sum_{j=1}^{n} \langle A_j x_j, x_j \rangle \right) \ln \left(\sum_{j=1}^{n} \langle A_j x_j, x_j \rangle \right) \right] \\
&\leq \exp \left[\sum_{j=1}^{n} \langle A_j \ln (eA_j) x_j, x_j \rangle - \sum_{j=1}^{n} \langle A_j x_j, x_j \rangle \sum_{j=1}^{n} \langle \ln (eA_j) x_j, x_j \rangle \right] \\
&\leq \begin{cases} \exp \left\{ \frac{1}{2}(M-m) \right. \\ \quad \left. \times \left[\sum_{j=1}^{n} \|\ln (eA_j) x_j\|^2 - \left(\sum_{j=1}^{n} \langle \ln (eA_j) x_j, x_j \rangle \right)^2 \right]^{1/2} \right\}, \\ \left(\frac{M}{m} \right)^{\frac{1}{2}\left[\sum_{j=1}^{n} \|A_j x_j\|^2 - \left(\sum_{j=1}^{n} \langle A_j x_j, x_j \rangle \right)^2 \right]^{1/2}} \end{cases} \\
&\leq \left(\frac{M}{m} \right)^{\frac{1}{4}(M-m)},
\end{aligned}
$$

for any $x_j \in H$, $j \in \{1, ..., n\}$ with $\sum_{j=1}^{n} \|x_j\|^2 = 1$.

Observe that

$$\begin{aligned}
&\exp\left[\sum_{j=1}^{n}\langle A_j \ln A_j x_j, x_j\rangle - \left(\sum_{j=1}^{n}\langle A_j x_j, x_j\rangle\right)\ln\left(\sum_{j=1}^{n}\langle A_j x_j, x_j\rangle\right)\right] \\
&= \frac{\exp\left[\ln\left(\sum_{j=1}^{n}\langle A_j x_j, x_j\rangle\right)\right]^{-\sum_{j=1}^{n}\langle A_j x_j, x_j\rangle}}{\exp\left[\sum_{j=1}^{n}\langle -A_j \ln A_j x_j, x_j\rangle\right]} \\
&= \frac{\left(\sum_{j=1}^{n}\langle A_j x_j, x_j\rangle\right)^{-\sum_{j=1}^{n}\langle A_j x_j, x_j\rangle}}{\prod_{j=1}^{n}\breve{\eta}_{x_j}(A_j)} = \frac{\left(\sum_{j=1}^{n}\langle A_j x_j, x_j\rangle\right)^{-\sum_{j=1}^{n}\langle A_j x_j, x_j\rangle}}{\prod_{j=1}^{n}\left[\eta_{\frac{x_j}{\|x_j\|}}(A_j)\right]^{\|x_j\|^2}},
\end{aligned}$$

$$\begin{aligned}
&\exp\left[\sum_{j=1}^{n}\langle A_j \ln(eA_j) x_j, x_j\rangle - \sum_{j=1}^{n}\langle A_j x_j, x_j\rangle \sum_{j=1}^{n}\langle \ln(eA_j) x_j, x_j\rangle\right] \\
&= \exp\left[\frac{1}{e}\sum_{j=1}^{n}\langle eA_j \ln(eA_j) x_j, x_j\rangle - \sum_{j=1}^{n}\langle A_j x_j, x_j\rangle \sum_{j=1}^{n}\langle \ln(eA_j) x_j, x_j\rangle\right] \\
&= \frac{\exp\left(-\sum_{j=1}^{n}\langle A_j x_j, x_j\rangle \sum_{j=1}^{n}\langle \ln(eA_j) x_j, x_j\rangle\right)}{\exp\left(\frac{1}{e}\sum_{j=1}^{n}\langle -eA_j \ln(eA_j) x_j, x_j\rangle\right)} \\
&= \frac{\left[\prod_{j=1}^{n}\exp\left(\langle \ln(eA_j) x_j, x_j\rangle\right)\right]^{-\sum_{j=1}^{n}\langle A_j x_j, x_j\rangle}}{\left[\prod_{j=1}^{n}\exp\left(\langle -eA_j \ln(eA_j) x_j, x_j\rangle\right)\right]^{1/e}} = \frac{\left[\prod_{j=1}^{n}\breve{\Delta}_{x_j}(eA_j)\right]^{-\sum_{j=1}^{n}\langle A_j x_j, x_j\rangle}}{\left[\prod_{j=1}^{n}\breve{\eta}_{x_j}(A_j)\right]^{1/e}} \\
&= \frac{\left[\prod_{j=1}^{n}\left[\Delta_{\frac{x_j}{\|x_j\|}}(eA_j)\right]^{\|x_j\|^2}\right]^{-\sum_{j=1}^{n}\langle A_j x_j, x_j\rangle}}{\left(\prod_{j=1}^{n}\left[\eta_{\frac{x_j}{\|x_j\|}}(A_j)\right]^{\|x_j\|^2}\right)^{1/e}}
\end{aligned}$$

and by (2.4) we derive (2.1). □

Corollary 1. *Assume that* $0 < m \leq A_j \leq M$ *and* $p_j \geq 0$, $j \in \{1, ..., n\}$ *with* $\sum_{j=1}^{n} p_j = 1$, *then for all* $x \in H$ *with* $\|x\| = 1$,

$$
\begin{aligned}
(2.5)\qquad 1 &\leq \frac{\left\langle \sum_{j=1}^{n} p_j A_j x, x \right\rangle^{-\left\langle \sum_{j=1}^{n} p_j A_j x, x \right\rangle}}{\prod_{j=1}^{n} [\eta_x (A_j)]^{p_j}} \\
&\leq \frac{\left[\prod_{j=1}^{n} [\Delta_x (eA_j)]^{p_j} \right]^{-\left\langle \sum_{j=1}^{n} p_j A_j x, x \right\rangle}}{\left(\prod_{j=1}^{n} [\eta_x (A_j)]^{p_j} \right)^{1/e}} \\
&\leq \begin{cases} \exp\left\{ \frac{1}{2} (M - m) \right. \\ \quad \left. \times \left[\sum_{j=1}^{n} p_j \|\ln (eA_j) x\|^2 - \left(\sum_{j=1}^{n} p_j \langle \ln (eA_j) x, x \rangle \right)^2 \right]^{1/2} \right\}, \\ \\ \left(\frac{M}{m} \right)^{\frac{1}{2} \left[\sum_{j=1}^{n} p_j \|A_j x\|^2 - \left\langle \sum_{j=1}^{n} p_j A_j x, x \right\rangle^2 \right]^{1/2}}, \end{cases} \\
&\leq \left(\frac{M}{m} \right)^{\frac{1}{4}(M-m)}.
\end{aligned}
$$

The first two inequalities are also valid for $A_j > 0$, $j \in \{1, ..., n\}$.

Remark 1. *If we take* $n = 1$ *in (2.5), then we get for* $0 < m \leq A \leq M$ *that*

$$
\begin{aligned}
(2.6)\qquad 1 &\leq \frac{\langle Ax, x \rangle^{-\langle Ax, x \rangle}}{\eta_x (A)} \leq \frac{[\Delta_x (eA)]^{-\langle Ax, x \rangle}}{[\eta_x (A)]^{1/e}} \\
&\leq \begin{cases} \exp\left(\frac{1}{2} (M - m) \left[\|\ln (eA) x\|^2 - (\langle \ln (eA) x, x \rangle)^2 \right]^{1/2} \right), \\ \\ \left(\frac{M}{m} \right)^{\frac{1}{2} \left[\|Ax\|^2 - \langle Ax, x \rangle^2 \right]^{1/2}}, \end{cases} \\
&\leq \left(\frac{M}{m} \right)^{\frac{1}{4}(M-m)},
\end{aligned}
$$

for all $x \in H$ *with* $\|x\| = 1$. *The first two inequalities also hold for* $A > 0$.

For $0 < m \le A,\ B \le M$ and $t \in [0,1]$ we also have that

$$1 \le \frac{\langle ((1-t)A + tB)x, x\rangle^{-\langle ((1-t)A+tB)x,x\rangle}}{[\eta_x(A)]^{1-t}[\eta_x(B)]^t} \tag{2.7}$$

$$\le \frac{\left[[\Delta_x(eA)]^{1-t}[\Delta_x(eB)]^t\right]^{-\langle ((1-t)A+tB)x,x\rangle}}{\left([\eta_x(A)]^{1-t}[\eta_x(B)]^t\right)^{1/e}}$$

$$\le \left(\frac{M}{m}\right)^{\frac{1}{2}\left[(1-t)\|Ax\|^2 + t\|Bx\|^2 - \langle ((1-t)A+tB)x,x\rangle^2\right]^{1/2}} \le \left(\frac{M}{m}\right)^{\frac{1}{4}(M-m)},$$

for all $x \in H$ with $\|x\| = 1$.

The first two inequalities also hold for $A,\ B > 0$.

We also have the complementary inequalities:

Theorem 2. *Assume that $0 < m \le A_j \le M$ and $p_j \ge 0,\ j \in \{1, ..., n\}$ with $\sum_{j=1}^n p_j = 1$, then*

$$1 \le \frac{\prod_{j=1}^n [\eta_x(A_j)]^{\frac{p_j}{\langle A_j x,x\rangle}}}{\left(\sum_{j=1}^n \frac{p_j\langle A_j^2 x,x\rangle}{\langle A_j x,x\rangle}\right)^{-1}} \tag{2.8}$$

$$\le \exp\left(\sum_{j=1}^n \frac{p_j\langle A_j^2 x,x\rangle}{\langle A_j x,x\rangle}\sum_{j=1}^n \frac{p_j}{\langle A_j x,x\rangle} - 1\right)$$

$$\le \begin{cases} \exp\left(\frac{1}{2}(M-m)\left[\sum_{j=1}^n \frac{p_j\langle A_j^{-1}x,x\rangle}{\langle A_j x,x\rangle} - \left(\sum_{j=1}^n \frac{p_j}{\langle A_j x,x\rangle}\right)^2\right]^{1/2}\right), \\ \exp\left(\frac{1}{2mM}(M-m)\left[\sum_{j=1}^n \frac{p_j\langle A_j^3 x,x\rangle}{\langle A_j x,x\rangle} - \left(\sum_{j=1}^n \frac{p_j\langle A_j^2 x,x\rangle}{\langle A_j x,x\rangle}\right)^2\right]^{1/2}\right), \end{cases}$$

$$\le \exp\left[\frac{1}{4mM}(M-m)^2\right],$$

for all $x \in H$ with $\|x\| = 1$.

The first two inequalities also hold for $A_j > 0,\ j \in \{1, ..., n\}$.

Proof. Consider

$$x_j := p_j^{1/2}\frac{A_j^{1/2}x}{\left\|A_j^{1/2}x\right\|},\ j \in \{1, ...n\},\ x \in H,\ \ \|x\| = 1.$$

Then

$$\sum_{j=1}^n \|x_j\|^2 = \sum_{j=1}^n p_j\left\|\frac{A_j^{1/2}x}{\left\|A_j^{1/2}x\right\|}\right\|^2 = \sum_{j=1}^n p_j = 1.$$

If we take the convex function $f(t) = -\ln t$, $t > 0$ in (2.2), then we get

$$
\begin{aligned}
0 &\le \ln \left(\sum_{j=1}^{n} \left\langle A_j p_j^{1/2} \frac{A_j^{1/2} x}{\left\| A_j^{1/2} x \right\|}, p_j^{1/2} \frac{A_j^{1/2} x}{\left\| A_j^{1/2} x \right\|} \right\rangle \right) \\
&- \sum_{j=1}^{n} \left\langle \ln A_j p_j^{1/2} \frac{A_j^{1/2} x}{\left\| A_j^{1/2} x \right\|}, p_j^{1/2} \frac{A_j^{1/2} x}{\left\| A_j^{1/2} x \right\|} \right\rangle \\
&\le \sum_{j=1}^{n} \left\langle A_j \frac{p_j^{1/2} A_j^{1/2} x}{\left\| A_j^{1/2} x \right\|}, \frac{p_j^{1/2} A_j^{1/2} x}{\left\| A_j^{1/2} x \right\|} \right\rangle \sum_{j=1}^{n} \left\langle A_j^{-1} \frac{p_j^{1/2} A_j^{1/2} x}{\left\| A_j^{1/2} x \right\|}, \frac{p_j^{1/2} A_j^{1/2} x}{\left\| A_j^{1/2} x \right\|} \right\rangle \\
&- \sum_{j=1}^{n} \left\langle \frac{p_j^{1/2} A_j^{1/2} x}{\left\| A_j^{1/2} x \right\|}, \frac{p_j^{1/2} A_j^{1/2} x}{\left\| A_j^{1/2} x \right\|} \right\rangle \\
&\le \begin{cases}
\frac{1}{2}(M-m) \\
\times \left[\sum_{j=1}^{n} \left\| A_j^{-1} \frac{p_j^{1/2} A_j^{1/2} x}{\left\| A_j^{1/2} x \right\|} \right\|^2 - \left(\sum_{j=1}^{n} \left\langle A_j^{-1} \frac{p_j^{1/2} A_j^{1/2} x}{\left\| A_j^{1/2} x \right\|}, \frac{p_j^{1/2} A_j^{1/2} x}{\left\| A_j^{1/2} x \right\|} \right\rangle \right)^2 \right]^{1/2}, \\
\\
\frac{1}{2mM}(M-m) \\
\times \left[\sum_{j=1}^{n} \left\| A_j \frac{p_j^{1/2} A_j^{1/2} x}{\left\| A_j^{1/2} x \right\|} \right\|^2 - \left(\sum_{j=1}^{n} \left\langle A_j \frac{p_j^{1/2} A_j^{1/2} x}{\left\| A_j^{1/2} x \right\|}, \frac{p_j^{1/2} A_j^{1/2} x}{\left\| A_j^{1/2} x \right\|} \right\rangle \right)^2 \right]^{1/2},
\end{cases} \\
&\le \frac{1}{4mM}(M-m)^2,
\end{aligned}
$$

namely

$$
\begin{aligned}
0 &\le \ln \left(\sum_{j=1}^{n} \frac{p_j}{\langle A_j x, x \rangle} \left\langle A_j^2 x, x \right\rangle \right) - \sum_{j=1}^{n} \frac{p_j}{\langle A_j x, x \rangle} \langle A_j \ln A_j x, x \rangle \\
&\le \sum_{j=1}^{n} \frac{p_j}{\langle A_j x, x \rangle} \left\langle A_j^2 x, x \right\rangle \sum_{j=1}^{n} \frac{p_j}{\langle A_j x, x \rangle} - \sum_{j=1}^{n} \frac{p_j}{\langle A_j x, x \rangle} \langle A_j x, x \rangle \\
&\le \begin{cases}
\frac{1}{2}(M-m) \\
\times \left[\sum_{j=1}^{n} \frac{p_j}{\langle A_j x, x \rangle} \left\| A_j^{-1/2} x \right\|^2 - \left(\sum_{j=1}^{n} \frac{p_j}{\langle A_j x, x \rangle} \right)^2 \right]^{1/2}, \\
\\
\frac{1}{2mM}(M-m) \\
\times \left[\sum_{j=1}^{n} \frac{p_j}{\langle A_j x, x \rangle} \left\| A_j^{3/2} x \right\|^2 - \left(\sum_{j=1}^{n} \frac{p_j}{\langle A_j x, x \rangle} \left\langle A_j^2 x, x \right\rangle \right)^2 \right]^{1/2},
\end{cases} \\
&\le \frac{1}{4mM}(M-m)^2.
\end{aligned}
$$

This can be simplified to

$$
\begin{aligned}
(2.9)\qquad 0 &\le \ln\left(\sum_{j=1}^{n} p_j \frac{\left\langle A_j^2 x, x\right\rangle}{\langle A_j x, x\rangle}\right) - \sum_{j=1}^{n} p_j \frac{\langle A_j \ln A_j x, x\rangle}{\langle A_j x, x\rangle} \\
&\le \sum_{j=1}^{n} p_j \frac{\left\langle A_j^2 x, x\right\rangle}{\langle A_j x, x\rangle} \sum_{j=1}^{n} \frac{p_j}{\langle A_j x, x\rangle} - 1 \\
&\le \begin{cases} \frac{1}{2}(M-m)\left[\sum_{j=1}^{n} p_j \frac{\left\langle A_j^{-1} x, x\right\rangle}{\langle A_j x, x\rangle} - \left(\sum_{j=1}^{n} \frac{p_j}{\langle A_j x, x\rangle}\right)^2\right]^{1/2}, \\ \\ \frac{1}{2mM}(M-m)\left[\sum_{j=1}^{n} \frac{p_j\left\langle A_j^3 x, x\right\rangle}{\langle A_j x, x\rangle} - \left(\sum_{j=1}^{n} \frac{p_j\left\langle A_j^{3/2} x, x\right\rangle}{\langle A_j x, x\rangle}\right)^2\right]^{1/2}, \end{cases} \\
&\le \frac{1}{4mM}(M-m)^2
\end{aligned}
$$

for $x \in H$, $\|x\| = 1$.

If we take the exponential in (2.9), then we get

(2.10)

$$
\begin{aligned}
1 &\le \left(\sum_{j=1}^{n} p_j \frac{\left\langle A_j^2 x, x\right\rangle}{\langle A_j x, x\rangle}\right) \exp\left(-\sum_{j=1}^{n} p_j \frac{\langle A_j \ln A_j x, x\rangle}{\langle A_j x, x\rangle}\right) \\
&\le \exp\left(\sum_{j=1}^{n} p_j \frac{\left\langle A_j^2 x, x\right\rangle}{\langle A_j x, x\rangle} \sum_{j=1}^{n} \frac{p_j}{\langle A_j x, x\rangle} - 1\right) \\
&\le \begin{cases} \exp\left(\frac{1}{2}(M-m)\left[\sum_{j=1}^{n} p_j \frac{\left\langle A_j^{-1} x, x\right\rangle}{\langle A_j x, x\rangle} - \left(\sum_{j=1}^{n} \frac{p_j}{\langle A_j x, x\rangle}\right)^2\right]^{1/2}\right), \\ \\ \exp\left(\frac{1}{2mM}(M-m)\left[\sum_{j=1}^{n} \frac{p_j\left\langle A_j^3 x, x\right\rangle}{\langle A_j x, x\rangle} - \left(\sum_{j=1}^{n} \frac{p_j\left\langle A_j^2 x, x\right\rangle}{\langle A_j x, x\rangle}\right)^2\right]^{1/2}\right), \end{cases} \\
&\le \exp\left[\frac{1}{4mM}(M-m)^2\right].
\end{aligned}
$$

Observe that,

$$
\begin{aligned}
\exp\left(-\sum_{j=1}^{n} p_j \frac{\langle A_j \ln A_j x, x\rangle}{\langle A_j x, x\rangle}\right) &= \exp\left(\sum_{j=1}^{n} p_j \frac{\langle -A_j \ln A_j x, x\rangle}{\langle A_j x, x\rangle}\right) \\
&= \prod_{j=1}^{n} \exp\left(\langle -A_j \ln A_j x, x\rangle\right)^{\frac{p_j}{\langle A_j x, x\rangle}} \\
&= \prod_{j=1}^{n} \left[\eta_x(A_j)\right]^{\frac{p_j}{\langle A_j x, x\rangle}},
\end{aligned}
$$

then by (2.10) we derive (2.8). □

Remark 2. *The case of one operator that satisfies the condition* $0<m\leq A\leq M$ *is as follows:*

$$1\leq\frac{[\eta_x(A)]^{\frac{1}{\langle Ax,x\rangle}}}{\left(\frac{\langle A^2x,x\rangle}{\langle Ax,x\rangle}\right)^{-1}}\leq\exp\left(\frac{\langle A^2x,x\rangle}{\langle Ax,x\rangle^2}-1\right)\tag{2.11}$$

$$\leq\begin{cases}\exp\left(\frac{1}{2}(M-m)\left[\frac{\langle A^{-1}x,x\rangle\langle Ax,x\rangle-1}{\langle Ax,x\rangle^2}\right]^{1/2}\right),\\ \exp\left(\frac{1}{2mM}(M-m)\left[\frac{\langle A^3x,x\rangle\langle Ax,x\rangle-\langle A^2x,x\rangle^2}{\langle Ax,x\rangle^2}\right]^{1/2}\right)\end{cases}$$

$$\leq\exp\left[\frac{1}{4mM}(M-m)^2\right],$$

for all $x\in H$ *with* $\|x\|=1$. *The first two inequalities also hold for* $A>0$.

If $0<m\leq A,\ B\leq M$, *then for* $t\in[0,1]$,

$$1\leq\frac{[\eta_x(A)]^{\frac{1-t}{\langle Ax,x\rangle}}[\eta_x(B)]^{\frac{t}{\langle Bx,x\rangle}}}{\left(\frac{(1-t)\langle A^2x,x\rangle}{\langle Ax,x\rangle}+\frac{t\langle B^2x,x\rangle}{\langle Bx,x\rangle}\right)^{-1}}\tag{2.12}$$

$$\leq\exp\left(\left(\frac{(1-t)\langle A^2x,x\rangle}{\langle Ax,x\rangle}+\frac{t\langle B^2x,x\rangle}{\langle Bx,x\rangle}\right)\left(\frac{1-t}{\langle Ax,x\rangle}+\frac{t}{\langle Bx,x\rangle}\right)-1\right)$$

$$\leq\begin{cases}\exp\left(\frac{1}{2}(M-m)\right.\\ \quad\times\left.\left[\frac{(1-t)\langle A^{-1}x,x\rangle}{\langle Ax,x\rangle}+\frac{t\langle B^{-1}x,x\rangle}{\langle Bx,x\rangle}-\left(\frac{1-t}{\langle Ax,x\rangle}+\frac{t}{\langle Bx,x\rangle}\right)^2\right]^{1/2}\right),\\ \exp\left(\frac{1}{2mM}(M-m)\right.\\ \quad\times\left.\left[\frac{(1-t)\langle A^3x,x\rangle}{\langle Ax,x\rangle}+\frac{t\langle B^3x,x\rangle}{\langle Bx,x\rangle}-\left(\frac{(1-t)\langle A^2x,x\rangle}{\langle Ax,x\rangle}+\frac{t\langle B^2x,x\rangle}{\langle Bx,x\rangle}\right)^2\right]^{1/2}\right),\end{cases}$$

$$\leq\exp\left[\frac{1}{4mM}(M-m)^2\right],$$

for all $x\in H$ *with* $\|x\|=1$. *The first two inequalities also hold for* $A,\ B>0$

3. Related Results

We also have:

Theorem 3. *Assume that* $0<m\leq A_j\leq M$ *and* $p_j\geq0$, $j\in\{1,...,n\}$ *with* $\sum_{j=1}^n p_j=1$, *then*

$$1\leq\frac{\sum_{j=1}^n\frac{p_j}{\langle A_jx,x\rangle}}{\prod_{j=1}^n[\eta_x(A_j)]^{\frac{p_j}{\langle A_jx,x\rangle}}}\tag{3.1}$$

$$\leq\exp\left[\sum_{j=1}^n\frac{p_j\langle A_j^2x,x\rangle}{\langle A_jx,x\rangle}\sum_{j=1}^n\frac{p_j}{\langle A_jx,x\rangle}-1\right]$$

$$\leq \begin{cases} \exp\left(\frac{1}{2}(M-m)\left[\sum_{j=1}^{n}\frac{p_j\left\langle A_j^{-1}x,x\right\rangle}{\langle A_jx,x\rangle}-\left(\sum_{j=1}^{n}\frac{p_j}{\langle A_jx,x\rangle}\right)^2\right]^{1/2}\right), \\ \\ \exp\left(\frac{1}{2mM}(M-m)\left[\sum_{j=1}^{n}\frac{p_j\left\langle A_j^{3}x,x\right\rangle}{\langle A_jx,x\rangle}-\left(\sum_{j=1}^{n}\frac{p_j\left\langle A_j^{2}x,x\right\rangle}{\langle A_jx,x\rangle}\right)^2\right]^{1/2}\right), \end{cases}$$

$$\leq \exp\left[\frac{1}{4mM}(M-m)^2\right].$$

The first two inequalities also hold for $A_j > 0,\ j \in \{1, ..., n\}$.

Proof. Let I be an interval and $f : I \to \mathbb{R}$ be a convex and differentiable function on $\mathring{I}$ (the interior of I) whose derivative f' is continuous on $\mathring{I}$. If $A_j, j \in \{1, ..., n\}$ are selfadjoint operators on the Hilbert space H with $\operatorname{Sp}(A_j) \subseteq [m, M] \subset \mathring{I}$ and

$$\frac{\sum_{j=1}^{n}\langle A_jf'(A_j)x_j,x_j\rangle}{\sum_{j=1}^{n}\langle f'(A_j)x_j,x_j\rangle}\in\mathring{I} \tag{3.2}$$

for each $x_j \in H,\ j \in \{1, ..., n\}$ with $\sum_{j=1}^{n}\|x_j\|^2 = 1$, then we have the following Slater type inequalities [5]:

$$\begin{aligned} 0 &\leq f\left(\frac{\sum_{j=1}^{n}\langle A_jf'(A_j)x_j,x_j\rangle}{\sum_{j=1}^{n}\langle f'(A_j)x_j,x_j\rangle}\right)-\sum_{j=1}^{n}\langle f(A_j)x_j,x_j\rangle \\ &\leq f'\left(\frac{\sum_{j=1}^{n}\langle A_jf'(A_j)x_j,x_j\rangle}{\sum_{j=1}^{n}\langle f'(A_j)x_j,x_j\rangle}\right) \\ &\quad\times\left[\frac{\sum_{j=1}^{n}\langle A_jf'(A_j)x_j,x_j\rangle-\sum_{j=1}^{n}\langle A_jx_j,x_j\rangle\sum_{j=1}^{n}\langle f'(A_j)x_j,x_j\rangle}{\sum_{j=1}^{n}\langle f'(A_j)x_j,x_j\rangle}\right], \end{aligned} \tag{3.3}$$

for each $x_j \in H,\ j \in \{1, ..., n\}$ with $\sum_{j=1}^{n}\|x_j\|^2 = 1$.

If we write the inequality (3.3) for the convex function $f(t) = -\ln t,\ t > 0$ and $A_j,\ j \in \{1, ..., n\}$ are positive definite operators on H, then we obtain

$$\begin{aligned} (0\leq)\sum_{j=1}^{n}\langle \ln A_jx_j,x_j\rangle-\ln\left[\left(\sum_{j=1}^{n}\left\langle A_j^{-1}x_j,x_j\right\rangle\right)^{-1}\right] \\ \leq\sum_{j=1}^{n}\langle A_jx_j,x_j\rangle\sum_{j=1}^{n}\left\langle A_j^{-1}x_j,x_j\right\rangle-1, \end{aligned} \tag{3.4}$$

for each $x_j \in H,\ j \in \{1, ..., n\}$ with $\sum_{j=1}^{n}\|x_j\|^2 = 1$.

Consider

$$x_j := p_j^{1/2}\frac{A_j^{1/2}x}{\left\|A_j^{1/2}x\right\|},\ j\in\{1,....n\},\ x\in H,\ \|x\|=1.$$

Then, as above $\sum_{j=1}^{n} \|x_j\|^2 = 1$ and by (3.4),

$$(0 \leq) \sum_{j=1}^{n} \left\langle \ln A_j p_j^{1/2} \frac{A_j^{1/2} x}{\left\| A_j^{1/2} x \right\|}, p_j^{1/2} \frac{A_j^{1/2} x}{\left\| A_j^{1/2} x \right\|} \right\rangle \tag{3.5}$$
$$- \ln \left[\left(\sum_{j=1}^{n} \left\langle A_j^{-1} p_j^{1/2} \frac{A_j^{1/2} x}{\left\| A_j^{1/2} x \right\|}, p_j^{1/2} \frac{A_j^{1/2} x}{\left\| A_j^{1/2} x \right\|} \right\rangle \right)^{-1} \right]$$
$$\leq \sum_{j=1}^{n} \left\langle A_j p_j^{1/2} \frac{A_j^{1/2} x}{\left\| A_j^{1/2} x \right\|}, p_j^{1/2} \frac{A_j^{1/2} x}{\left\| A_j^{1/2} x \right\|} \right\rangle$$
$$\times \sum_{j=1}^{n} \left\langle A_j^{-1} p_j^{1/2} \frac{A_j^{1/2} x}{\left\| A_j^{1/2} x \right\|}, p_j^{1/2} \frac{A_j^{1/2} x}{\left\| A_j^{1/2} x \right\|} \right\rangle - 1,$$

namely

$$(0 \leq) \ln \left(\sum_{j=1}^{n} \frac{p_j}{\langle A_j x, x \rangle} \right) - \sum_{j=1}^{n} \frac{p_j}{\langle A_j x, x \rangle} \langle -A_j \ln A_j x, x \rangle \tag{3.6}$$
$$\leq \sum_{j=1}^{n} \frac{p_j \langle A_j^2 x, x \rangle}{\langle A_j x, x \rangle} \sum_{j=1}^{n} \frac{p_j}{\langle A_j x, x \rangle} - 1,$$

for $x \in H$, $\|x\| = 1$.

If we take the exponential in (3.6) then we obtain

$$1 \leq \frac{\sum_{j=1}^{n} \frac{p_j}{\langle A_j x, x \rangle}}{\exp \left(\sum_{j=1}^{n} \frac{p_j}{\langle A_j x, x \rangle} \langle -A_j \ln A_j x, x \rangle \right)} \tag{3.7}$$
$$\leq \exp \left[\sum_{j=1}^{n} \frac{p_j \langle A_j^2 x, x \rangle}{\langle A_j x, x \rangle} \sum_{j=1}^{n} \frac{p_j}{\langle A_j x, x \rangle} - 1 \right]$$

for $x \in H$, $\|x\| = 1$.

Observe that,

$$\exp \left(\sum_{j=1}^{n} \frac{p_j}{\langle A_j x, x \rangle} \langle -A_j \ln A_j x, x \rangle \right) = \prod_{j=1}^{n} \exp \left[\frac{p_j}{\langle A_j x, x \rangle} \langle -A_j \ln A_j x, x \rangle \right]$$
$$= \prod_{j=1}^{n} \left(\exp \left[\langle -A_j \ln A_j x, x \rangle \right] \right)^{\frac{p_j}{\langle A_j x, x \rangle}}$$
$$= \prod_{j=1}^{n} \left[\eta_x (A_j) \right]^{\frac{p_j}{\langle A_j x, x \rangle}}$$

and by (3.7) we derive the first two inequalities in (3.1).

The last part follows by Theorem 2. □

Remark 3. *The case of one operator that satisfies the condition* $0 < m \le A \le M$ *is as follows:*

$$1 \le \frac{\frac{1}{\langle Ax,x\rangle}}{\left[\eta_x(A)\right]^{\frac{1}{\langle Ax,x\rangle}}} \tag{3.8}$$
$$\le \begin{cases} \exp\left(\frac{1}{2}(M-m)\left[\frac{\langle A^{-1}x,x\rangle\langle Ax,x\rangle-1}{\langle Ax,x\rangle^2}\right]^{1/2}\right), \\ \\ \exp\left(\frac{1}{2mM}(M-m)\left[\frac{\langle A^3x,x\rangle\langle Ax,x\rangle-\langle A^2x,x\rangle^2}{\langle Ax,x\rangle^2}\right]^{1/2}\right), \end{cases}$$
$$\le \exp\left[\frac{1}{4mM}(M-m)^2\right],$$

for all $x \in H$ *with* $\|x\| = 1$. *The first two inequalities also hold for* $A > 0$.

If $0 < m \le A,\ B \le M$, *then for* $t \in [0,1]$

$$1 \le \frac{\frac{1-t}{\langle Ax,x\rangle}+\frac{t}{\langle Bx,x\rangle}}{\left[\eta_x(A)\right]^{\frac{1-t}{\langle Ax,x\rangle}}\left[\eta_x(B)\right]^{\frac{t}{\langle Bx,x\rangle}}} \tag{3.9}$$
$$\le \exp\left(\left(\frac{(1-t)\langle A^2x,x\rangle}{\langle Ax,x\rangle}+\frac{t\langle B^2x,x\rangle}{\langle Bx,x\rangle}\right)\left(\frac{1-t}{\langle Ax,x\rangle}+\frac{t}{\langle Bx,x\rangle}\right)-1\right)$$
$$\le \begin{cases} \exp\left(\frac{1}{2}(M-m)\right. \\ \quad\left.\times\left[\frac{(1-t)\langle A^{-1}x,x\rangle}{\langle Ax,x\rangle}+\frac{t\langle B^{-1}x,x\rangle}{\langle Bx,x\rangle}-\left(\frac{1-t}{\langle Ax,x\rangle}+\frac{t}{\langle Bx,x\rangle}\right)^2\right]^{1/2}\right), \\ \\ \exp\left(\frac{1}{2mM}(M-m)\right. \\ \quad\left.\times\left[\frac{(1-t)\langle A^3x,x\rangle}{\langle Ax,x\rangle}+\frac{t\langle B^3x,x\rangle}{\langle Bx,x\rangle}-\left(\frac{(1-t)\langle A^2x,x\rangle}{\langle Ax,x\rangle}+\frac{t\langle B^2x,x\rangle}{\langle Bx,x\rangle}\right)^2\right]^{1/2}\right), \end{cases}$$
$$\le \exp\left[\frac{1}{4mM}(M-m)^2\right],$$

for all $x \in H$ *with* $\|x\| = 1$. *The first two inequalities also hold for* $A,\ B > 0$

References

[1] S. S. Dragomir, Čebyšev's type inequalities for functions of selfadjoint operators in Hilbert spaces, *Linear Multilinear Algebra*, **58** (2010), no. 7-8, 805–814; Preprint *RGMIA Res. Rep. Coll.*, **11**(e) (2008), Art. 9. [Online https://rgmia.org/papers/v11e/CebysevOperators.pdf].

[2] S. S. Dragomir, Some basic results for the normalized entropic determinant of positive operators in Hilbert spaces, Preprint RGMIA Res. Rep. Coll. 25 (2022), Art.

[3] S. S. Dragomir, Some Reverses of the Jensen inequality for functions of selfadjoint operators in Hilbert spaces, *Journal of Inequalities and Applications*, Volume **2010**, Article ID 496821, 15 pages doi:10.1155/2010/496821.

[4] S.S. Dragomir, Some inequalities for convex functions of selfadjoint operators in Hilbert spaces, *Filomat* **23** (2009), no. 3, 81–92. Preprint *RGMIA Res. Rep. Coll.* **11** (2008), Suplement, Art. 11. [Online https://rgmia.org/papers/v11e/ConvFuncOp.pdf].

[5] S.S. Dragomir, Some Slater's type inequalities for convex functions of selfadjoint operators in Hilbert spaces, Rev. Un. Mat. Argentina, 52 (2011), no. 1, 109–120; Preprint RGMIA Res. Rep. Coll., 11(e) (2008), Art. 7.

[6] T. Furuta, J. Mićić Hot, J. Pečarić and Y. Seo, *Mond-Pečarić Method in Operator Inequalities. Inequalities for Bounded Selfadjoint Operators on a Hilbert Space*, Element, Zagreb, 2005.
[7] J. I. Fujii and Y. Seo, Determinant for positive operators, *Sci. Math.,* **1** (1998), 153–156.
[8] J. I. Fujii, S. Izumino and Y. Seo, Determinant for positive operators and Specht's Theorem, *Sci. Math.*, **1** (1998), 307–310.
[9] T. Furuta, Precise lower bound of $f(A) - f(B)$ for $A > B > 0$ and non-constant operator monotone function f on $[0, \infty)$. *J. Math. Inequal.* **9** (2015), no. 1, 47–52.
[10] S. Hiramatsu and Y. Seo, Determinant for positive operators and Oppenheim's inequality, *J. Math. Inequal.*, Volume 15 (2021), Number 4, 1637–1645.
[11] J. Mićić, Y. Seo, S.-E. Takahasi and M. Tominaga, Inequalities of Furuta and Mond-Pečarić, *Math. Ineq. Appl.*, **2**(1999), 83-111.
[12] D.S. Mitrinović, J.E. Pečarić and A.M. Fink, *Classical and New Inequalities in Analysis*, Kluwer Academic Publishers, Dordrecht, 1993.
[13] B. Mond and J. Pečarić, Convex inequalities in Hilbert space, *Houston J. Math.*, **19**(1993), 405-420.
[14] W. Specht, Zer Theorie der elementaren Mittel, *Math. Z.* **74** (1960), pp. 91-98.
[15] H. Zuo, G. Duan, Some inequalities of operator monotone functions. *J. Math. Inequal.* **8** (2014), no. 4, 777–781.

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