FUNCTIONAL PROPERTIES FOR THE NORMALIZED ENTROPIC DETERMINANT OF SEQUENCES OF POSITIVE OPERATORS IN HILBERT SPACES

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Abstract. For positive invertible operators A on a Hilbert space H and a fixed unit vector $x \in H$, define the *normalized entropic determinant* $\eta_x(A)$ by

$$\eta_x(A) := \exp\left(-\langle A \ln Ax, x\rangle\right).$$

We consider the functional

$$N_n(\mathbf{p}; \mathbf{A}, x) := \frac{\left[\eta_x\left(\frac{1}{P_n}\sum_{j=1}^n p_j A_j\right)\right]^{P_n}}{\prod_{i=1}^n \left[\eta_x(A_j)\right]^{p_i}},$$

where $\mathbf{A} = (A_1, ..., A_n)$ is an n-tuple of selfadjoint positive operators, $\mathbf{p} \in \mathcal{P}_n^+$ the set of nonnegative n-tuples and $x \in H$, $\|x\| = 1$.

In this paper we show among others that, for any $\mathbf{p}, \mathbf{q} \in \mathcal{P}_n^+$ and $x \in H$, $\|x\| = 1$ we have

$$N_n(\mathbf{p} + \mathbf{q}; \mathbf{A}, x) \geq N_n(\mathbf{p}; \mathbf{A}, x)\, N_n(\mathbf{q}; \mathbf{A}, x) \geq 1.$$

Moreover, if $\mathbf{p}, \mathbf{q} \in \mathcal{P}_n^+$ with $\mathbf{p} \geq \mathbf{q}$, then also

$$N_n(\mathbf{p}; \mathbf{A}, x) \geq D_n(\mathbf{q}; \mathbf{A}, x) \geq 1$$

for all $x \in H$, $\|x\| = 1$. Some upper bounds for $N_n(\mathbf{p}; \mathbf{A}, x)$ under boundedness assumptions for $\mathbf{A}$ are also provided.

1. Introduction

Let $B(H)$ be the space of all bounded linear operators on a Hilbert space H, and I stands for the identity operator on H. An operator A in $B(H)$ is said to be positive (in symbol: $A \geq 0$) if $\langle Ax, x\rangle \geq 0$ for all $x \in H$. In particular, $A > 0$ means that A is positive and invertible. For a pair A, B of selfadjoint operators the order relation $A \geq B$ means as usual that $A - B$ is positive.

In 1998, Fujii et al. [8], [9], introduced the *normalized determinant* $\Delta_x(A)$ for positive invertible operators A on a Hilbert space H and a fixed unit vector $x \in H$, namely $\|x\| = 1$, defined by $\Delta_x(A) := \exp\langle \ln Ax, x\rangle$ and discussed it as a continuous geometric mean and observed some inequalities around the determinant from this point of view.

Some of the fundamental properties of normalized determinant are as follows, [8].

For each unit vector $x \in H$, see also [10], we have:

(i) *continuity*: the map $A \to \Delta_x(A)$ is norm continuous;
(ii) *bounds*: $\left\langle A^{-1}x, x\right\rangle^{-1} \leq \Delta_x(A) \leq \langle Ax, x\rangle$;

1991 *Mathematics Subject Classification.* 47A63, 26D15, 46C05.

Key words and phrases. Positive operators, Normalized determinants, Inequalities.

RGMIA Res. Rep. Coll. 25 (2022), Art. 41, 14 pp. Received 06/04/22

(iii) *continuous mean*: $\langle A^p x, x\rangle^{1/p} \downarrow \Delta_x(A)$ for $p \downarrow 0$ and $\langle A^p x, x\rangle^{1/p} \uparrow \Delta_x(A)$ for $p \uparrow 0$;
(iv) *power equality:* $\Delta_x(A^t) = \Delta_x(A)^t$ for all $t > 0$;
(v) *homogeneity*: $\Delta_x(tA) = t\Delta_x(A)$ and $\Delta_x(tI) = t$ for all $t > 0$;
(vi) *monotonicity*: $0 < A \leq B$ implies $\Delta_x(A) \leq \Delta_x(B)$;
(vii) *multiplicativity*: $\Delta_x(AB) = \Delta_x(A)\Delta_x(B)$ for commuting A and B;
(viii) *Ky Fan type inequality*: $\Delta_x((1-\alpha)A + \alpha B) \geq \Delta_x(A)^{1-\alpha}\Delta_x(B)^{\alpha}$ for $0 < \alpha < 1$.

We define the logarithmic mean of two positive numbers a, b by

$$L(a,b) := \begin{cases} \frac{b-a}{\ln b - \ln a} \text{ if } b \neq a, \\ \\ a \text{ if } b = a. \end{cases} \tag{1.1}$$

In [8] the authors obtained the following additive reverse inequality for the operator A which satisfy the condition $0 < m \leq A \leq M$, where m, M are positive numbers,

$$0 \leq \langle Ax, x\rangle - \Delta_x(A) \leq L(m,M)\left[\ln L(m,M) + \frac{M\ln m - m\ln M}{M-m} - 1\right] \tag{1.2}$$

for all $x \in H$, $\|x\| = 1$.

We recall that *Specht's ratio* is defined by [14]

$$S(h) := \begin{cases} \frac{h^{\frac{1}{h-1}}}{e\ln\left(h^{\frac{1}{h-1}}\right)} \text{ if } h \in (0,1) \cup (1,\infty), \\ \\ 1 \text{ if } h = 1. \end{cases} \tag{1.3}$$

It is well known that $\lim_{h\to 1} S(h) = 1$, $S(h) = S\left(\frac{1}{h}\right) > 1$ for $h > 0$, $h \neq 1$. The function is decreasing on $(0,1)$ and increasing on $(1,\infty)$.

In [9], the authors obtained the following multiplicative reverse inequality as well

$$1 \leq \frac{\langle Ax, x\rangle}{\Delta_x(A)} \leq S\left(\frac{M}{m}\right) \tag{1.4}$$

for $0 < m \leq A \leq M$ and $x \in H$, $\|x\| = 1$.

For the entropy function $\eta(t) = -t\ln t$, $t > 0$, the operator entropy has the following expression:

$$\eta(A) = -A\ln A$$

for positive A.

For $x \in H$, $\|x\| = 1$, we define the *normalized entropic determinant* $\eta_x(A)$ by

$$\eta_x(A) := \exp(-\langle A\ln Ax, x\rangle) = \exp\langle \eta(A)x, x\rangle.$$

Let $x \in H$, $\|x\| = 1$. Observe that the map $A \to \eta_x(A)$ is *norm continuous* and since

$$\begin{aligned} &\exp(-\langle tA\ln(tA)x, x\rangle) \\ &= \exp(-\langle tA(\ln t + \ln A)x, x\rangle) = \exp(-\langle (tA\ln t + tA\ln A)x, x\rangle) \\ &= \exp(-\langle Ax, x\rangle t\ln t)\exp(-t\langle A\ln Ax, x\rangle) \\ &= \exp\ln\left(t^{-\langle Ax,x\rangle t}\right)[\exp(-\langle A\ln Ax, x\rangle)]^{-t}, \end{aligned}$$

hence

$$\eta_x(tA) = t^{-t\langle Ax,x\rangle}\left[\eta_x(A)\right]^{-t} \tag{1.5}$$

for $t > 0$ and $A > 0$.

Observe also that

$$\eta_x(I) = 1 \text{ and } \eta_x(tI) = t^{-t} \tag{1.6}$$

for $t > 0$.

For the entropy function $\eta(t) = -t \ln t$, $t > 0$, the operator entropy has the following expression:

$$\eta(A) = -A \ln A$$

for positive A.

For $x \in H$, $\|x\| = 1$, we define the *normalized entropic determinant* $\eta_x(A)$ by

$$\eta_x(A) := \exp\left(-\langle A \ln Ax, x\rangle\right) = \exp\langle \eta(A)x, x\rangle . \tag{1.7}$$

Let $x \in H$, $\|x\| = 1$. Observe that the map $A \to \eta_x(A)$ is *norm continuous* and since

$$\begin{aligned}
&\exp\left(-\langle tA \ln(tA)x, x\rangle\right) \\
&= \exp\left(-\langle tA(\ln t + \ln A)x, x\rangle\right) = \exp\left(-\langle (tA \ln t + tA \ln A)x, x\rangle\right) \\
&= \exp\left(-\langle Ax, x\rangle t \ln t\right) \exp\left(-t\langle A \ln Ax, x\rangle\right) \\
&= \exp\ln\left(t^{-\langle Ax,x\rangle t}\right)\left[\exp\left(-\langle A \ln Ax, x\rangle\right)\right]^{-t},
\end{aligned}$$

hence

$$\eta_x(tA) = t^{-t\langle Ax,x\rangle}\left[\eta_x(A)\right]^{-t} \tag{1.8}$$

for $t > 0$ and $A > 0$.

Observe also that

$$\eta_x(I) = 1 \text{ and } \eta_x(tI) = t^{-t} \tag{1.9}$$

for $t > 0$.

In the recent paper [6] we obtained among others that, if A, $B > 0$, then for all $x \in H, \|x\| = 1$ and $t \in [0, 1]$,

$$\eta_x((1-t)A + tB) \ge (\eta_x(A))^{1-t}(\eta_x(B))^t . \tag{1.10}$$

Also we have the bounds

$$\left(\frac{\langle A^2x, x\rangle}{\langle Ax, x\rangle}\right)^{-\langle Ax,x\rangle} \le \eta_x(A) \le \langle Ax, x\rangle^{-\langle Ax,x\rangle},$$

where $A > 0$ and $x \in H, \|x\| = 1$.

We consider the functional

$$N_n(\mathbf{p}; \mathbf{A}, x) := \frac{\left[\eta_x\left(\frac{1}{P_n}\sum_{j=1}^n p_j A_j\right)\right]^{P_n}}{\prod_{i=1}^n \left[\eta_x(A_j)\right]^{p_i}} \ge 1,$$

where $\mathbf{A} = (A_1, ..., A_n)$ is an n-tuple of selfadjoint positive operators, $\mathbf{p} \in \mathcal{P}_n^+$ the set of nonnegative n-tuples and $x \in H$, $\|x\| = 1$.

In this paper we show among others that, for any $\mathbf{p},\mathbf{q}\in\mathcal{P}_n^+$ and $x\in H$, $\|x\|=1$ we have

$$N_n(\mathbf{p}+\mathbf{q};\mathbf{A},x)\geq N_n(\mathbf{p};\mathbf{A},x)\,N_n(\mathbf{q};\mathbf{A},x)\geq 1.$$

Moreover, if $\mathbf{p},\mathbf{q}\in\mathcal{P}_n^+$ with $\mathbf{p}\geq\mathbf{q}$, then also

$$N_n(\mathbf{p};\mathbf{A},x)\geq D_n(\mathbf{q};\mathbf{A},x)\geq 1$$

for all $x\in H$, $\|x\|=1$. Some upper bounds for $N_n(\mathbf{p};\mathbf{A},x)$ under boundedness assumptions for $\mathbf{A}$ are also provided.

2. Main Results

We consider the functional

$$J_n(\mathbf{p};\mathbf{A},f,I):=\sum_{j=1}^n p_j f(A_j)-P_n f\left(\frac{1}{P_n}\sum_{j=1}^n p_jA_j\right) \tag{2.1}$$

where $\mathbf{p}=(p_1,...,p_n)$, $p_j\geq 0$ with $j\in\{1,...,n\}$ and $P_n>0$, $\mathbf{A}=(A_1,...,A_n)$ is an n-tuple of selfadjoint operators with $\operatorname{Sp}(A_j)\subseteq I$ for $j\in\{1,...,n\}$ and $f:I\to\mathbb{R}$ is a operator convex function defined on the interval I.

We denote by $\mathcal{P}_n^+$ the set of all n-tuples $\mathbf{p}=(p_1,...,p_n)$, $p_j\geq 0$ with $j\in\{1,...,n\}$ and $P_n>0$. For $\mathbf{p},\mathbf{q}\in\mathcal{P}_n^+$ we denote $\mathbf{p}\geq\mathbf{q}$ if $p_j\geq q_j$ for any $j\in\{1,...,n\}$.

In [4] we obtained the following result:

Lemma 1. *Assume that $f:I\to\mathbb{R}$ is an operator convex function and $\mathbf{A}=(A_1,...,A_n)$ an n-tuple of selfadjoint operators with $\operatorname{Sp}(A_j)\subseteq I$, then for any $\mathbf{p},\mathbf{q}\in\mathcal{P}_n^+$ we have*

$$J_n(\mathbf{p}+\mathbf{q};\mathbf{A},f,I)\geq J_n(\mathbf{p};\mathbf{A},f,I)+J_n(\mathbf{q};\mathbf{A},f,I)\geq 0, \tag{2.2}$$

i.e., $J_n(\cdot;\mathbf{A},f,I)$ is a super-additive functional in the operator order.

Moreover, if $\mathbf{p},\mathbf{q}\in\mathcal{P}_n^+$ with $\mathbf{p}\geq\mathbf{q}$, then also

$$J_n(\mathbf{p};\mathbf{A},f,I)\geq J_n(\mathbf{q};\mathbf{A},f,I)\geq 0, \tag{2.3}$$

i.e., $J_n(\cdot;\mathbf{A},f,I)$ is a monotonic functional in the operator order.

Corollary 1. *Assume that the function $f:I\to\mathbb{R}$ is operator convex and the n-tuple of selfadjoint operators $(A_1,...,A_n)$ satisfies the condition $\operatorname{Sp}(A_j)\subseteq I$ for any $j\in\{1,...,n\}$. If $\mathbf{p},\mathbf{q}\in\mathcal{P}_n^+$ and there exists the positive constants m, M such that*

$$m\mathbf{q}\leq\mathbf{p}\leq M\mathbf{q}, \tag{2.4}$$

then

$$mJ_n(\mathbf{q};\mathbf{A},f,I)\leq J_n(\mathbf{p};\mathbf{A},f,I)\leq MJ_n(\mathbf{q};\mathbf{A},f,I) \tag{2.5}$$

in the operator order.

Remark 1. *We observe that if all $q_j>0$ then we have the inequality*

$$\min_{j\in\{1,...,n\}}\left\{\frac{p_j}{q_j}\right\}J_n(\mathbf{q};\mathbf{A},f,I)\leq J_n(\mathbf{p};\mathbf{A},f,I)\leq\max_{j\in\{1,...,n\}}\left\{\frac{p_j}{q_j}\right\}J_n(\mathbf{q};\mathbf{A},f,I) \tag{2.6}$$

in the operator order.

In particular, if $\mathbf{q}$ is the uniform distribution, i.e., $q_j=\frac{1}{n}, j\in\{1,...,n\}$, then we have the inequalities

$$n\min_{j\in\{1,...,n\}}\{p_j\}J_n(\mathbf{A},f,I)\leq J_n(\mathbf{p};\mathbf{A},f,I)\leq n\max_{j\in\{1,...,n\}}\{p_j\}J_n(\mathbf{A},f,I), \tag{2.7}$$

where

$$J_n(\mathbf{A}, f, I) := \frac{1}{n}\sum_{j=1}^{n} f(A_j) - f\left(\frac{1}{n}\sum_{j=1}^{n} A_j\right). \tag{2.8}$$

For $n = 2$ and by choosing $p_1 = \alpha$, $p_2 = 1 - \alpha$ with $\alpha \in [0, 1]$, we get from (2.7) the inequality

$$\begin{aligned} &2\min\{\alpha, 1-\alpha\}\left[\frac{f(A)+f(B)}{2} - f\left(\frac{A+B}{2}\right)\right] \\ &\leq (1-\alpha) f(A) + \alpha f(B) - f((1-\alpha)A + \alpha B) \\ &\leq 2\max\{\alpha, 1-\alpha\}\left[\frac{f(A)+f(B)}{2} - f\left(\frac{A+B}{2}\right)\right], \end{aligned} \tag{2.9}$$

in the operator order, where $f : I \to \mathbb{R}$ is an operator convex function and A and B are two bounded selfadjoint operators on the complex Hilbert space H with $\operatorname{Sp}(A)$, $\operatorname{Sp}(B) \subseteq I$.

Our first main results is as follows:

Theorem 1. *Assume that $\mathbf{A} = (A_1, ..., A_n)$ is an n-tuple of selfadjoint positive operators, then for any $\mathbf{p}, \mathbf{q} \in \mathcal{P}_n^+$ and $x \in H$, $\|x\| = 1$ we have*

$$N_n(\mathbf{p} + \mathbf{q}; \mathbf{A}, x) \geq N_n(\mathbf{p}; \mathbf{A}, x) N_n(\mathbf{q}; \mathbf{A}, x) \geq 1, \tag{2.10}$$

i.e., $N_n(\cdot; \mathbf{A}, x)$ is a super-multiplicative functional.

Moreover, if $\mathbf{p}, \mathbf{q} \in \mathcal{P}_n^+$ with $\mathbf{p} \geq \mathbf{q}$, then also

$$N_n(\mathbf{p}; \mathbf{A}, x) \geq N_n(\mathbf{q}; \mathbf{A}, x) \geq 1 \tag{2.11}$$

for all $x \in H$, $\|x\| = 1$, i.e., $N_n(\cdot; \mathbf{A}, x)$ is a monotonic non-decreasing functional.

Proof. For the operator convex entropy function $\eta(t) = t \ln t$, $t > 0$, we have

$$\begin{aligned} J_n(\mathbf{p}; \mathbf{A}, \eta) &:= J_n(\mathbf{p}; \mathbf{A}, \eta, (0, \infty)) \\ &= \sum_{j=1}^{n} p_j A_j \ln A_j - P_n\left(\frac{1}{P_n}\sum_{j=1}^{n} p_j A_j\right) \ln\left(\frac{1}{P_n}\sum_{j=1}^{n} p_j A_j\right). \end{aligned}$$

For $x \in H$, $\|x\| = 1$ we have

$$\begin{aligned} \langle J_n(\mathbf{p}; \mathbf{A}, \eta) x, x\rangle &= \sum_{j=1}^{n} p_j \langle A_j \ln A_j x, x\rangle \\ &\quad - P_n\left\langle \left(\frac{1}{P_n}\sum_{j=1}^{n} p_j A_j\right) \ln\left(\frac{1}{P_n}\sum_{j=1}^{n} p_j A_j\right) x, x\right\rangle. \end{aligned}$$

If we take the exponential, then we get

$$\begin{aligned}
&\exp\left\langle J_n\left(\mathbf{p};\mathbf{A},\eta\right)x,x\right\rangle \\
&=\exp\left(\sum_{j=1}^{n}p_j\left\langle A_j\ln A_jx,x\right\rangle\right) \\
&\times\exp\left[-P_n\left\langle\left(\frac{1}{P_n}\sum_{j=1}^{n}p_jA_j\right)\ln\left(\frac{1}{P_n}\sum_{j=1}^{n}p_jA_j\right)x,x\right\rangle\right] \\
&=\frac{\left(\exp\left[-\left\langle\left(\frac{1}{P_n}\sum_{j=1}^{n}p_jA_j\right)\ln\left(\frac{1}{P_n}\sum_{j=1}^{n}p_jA_j\right)x,x\right\rangle\right]\right)^{P_n}}{\prod_{j=1}^{n}\left[\exp\left(-\left\langle A_j\ln A_jx,x\right\rangle\right)\right]^{p_j}} \\
&=\frac{\left[\eta_x\left(\frac{1}{P_n}\sum_{j=1}^{n}p_jA_j\right)\right]^{P_n}}{\prod_{j=1}^{n}\left[\eta_x\left(A_j\right)\right]^{p_j}}=N_n\left(\mathbf{p};\mathbf{A},x\right).
\end{aligned}$$

For $\mathbf{p},\mathbf{q}\in\mathcal{P}_n^+$ we have by (**??**) and Lemma 1 that

$$\begin{aligned}
N_n\left(\mathbf{p}+\mathbf{q};\mathbf{A},x\right)&=\exp\left[\left\langle J_n\left(\mathbf{p}+\mathbf{q};\mathbf{A},\eta\right)x,x\right\rangle\right] \\
&\geq\exp\left[\left\langle J_n\left(\mathbf{p};\mathbf{A},\eta\right)x,x\right\rangle+\left\langle J_n\left(\mathbf{q};\mathbf{A},\eta\right)x,x\right\rangle\right] \\
&=\exp\left\langle J_n\left(\mathbf{p};\mathbf{A},\eta\right)x,x\right\rangle\exp\left\langle J_n\left(\mathbf{q};\mathbf{A},\eta\right)x,x\right\rangle \\
&=D_n\left(\mathbf{p};\mathbf{A},x\right)D_n\left(\mathbf{q};\mathbf{A},x\right),
\end{aligned}$$

for all $x\in H$, $\|x\|=1$, which proves (2.10).

The property (2.11) follows in a similar way by (2.3). □

Corollary 2. *With the assumptions of Theorem 1, if $\mathbf{p},\mathbf{q}\in\mathcal{P}_n^+$ and there exists the positive constants m, M such that $m\mathbf{q}\leq\mathbf{p}\leq M\mathbf{q}$, then*

$$1\leq\left[N_n\left(\mathbf{q};\mathbf{A},x\right)\right]^m\leq N_n\left(\mathbf{p};\mathbf{A},x\right)\leq\left[N_n\left(\mathbf{q};\mathbf{A},x\right)\right]^M \tag{2.12}$$

for all $x\in H$, $\|x\|=1$.

Remark 2. *We observe that if all $q_j>0$ then we have the inequality*

$$\begin{aligned}
1&\leq\left[N_n\left(\mathbf{q};\mathbf{A},x\right)\right]^{\min_{j\in\{1,\dots,n\}}\left\{\frac{p_j}{q_j}\right\}} \\
&\leq N_n\left(\mathbf{p};\mathbf{A},x\right)\leq\left[N_n\left(\mathbf{q};\mathbf{A},x\right)\right]^{\max_{j\in\{1,\dots,n\}}\left\{\frac{p_j}{q_j}\right\}}
\end{aligned} \tag{2.13}$$

for all $x\in H$, $\|x\|=1$.

In particular, if $\mathbf{q}$ is the uniform distribution, i.e., $q_j=\frac{1}{n}$, $j\in\{1,...,n\}$, then we have the inequalities

$$\begin{aligned}
1&\leq\left[N_n\left(\mathbf{A},x\right)\right]^{n\min_{j\in\{1,\dots,n\}}\{p_j\}} \\
&\leq N_n\left(\mathbf{p};\mathbf{A},x\right)\leq\left[N_n\left(\mathbf{A},x\right)\right]^{n\max_{j\in\{1,\dots,n\}}\{p_j\}}
\end{aligned} \tag{2.14}$$

for all $x \in H$, $\|x\| = 1$, *where*

$$N_n(\mathbf{A},x) := \frac{\eta_x\left(\frac{1}{n}\sum_{j=1}^{n} A_j\right)}{\left[\prod_{j=1}^{n}\eta_x(A_j)\right]^{1/n}}.$$

For $n = 2$ *and by choosing* $p_1 = \alpha$, $p_2 = 1-\alpha$ *with* $\alpha \in [0,1]$, *we get from (2.14) the inequality for two positive operators* A, B

$$1 \le \left(\frac{\eta_x\left(\frac{A+B}{2}\right)}{[\eta_x(A)]^{1/2}[\eta_x(B)]^{1/2}}\right)^{2\min\{\alpha,1-\alpha\}} \le \frac{\eta_x((1-\alpha)A+\alpha B)}{[\eta_x(A)]^{1-\alpha}[\eta_x(B)]^{\alpha}} \le \left(\frac{\eta_x\left(\frac{A+B}{2}\right)}{[\eta_x(A)]^{1/2}[\eta_x(B)]^{1/2}}\right)^{2\max\{\alpha,1-\alpha\}}, \tag{2.15}$$

which provides a refinement and a reverse of the Ky Fan type inequality (1.10).

Let $\mathcal{P}_f(\mathbb{N})$ be the family of finite parts of the set of natural numbers $\mathbb{N}$, $\mathcal{A}(H)$ the linear space of all sequences of selfadjoint operators defined on the complex Hilbert space, i.e.,

$$\mathcal{A}(H) = \left\{\mathbf{A} = (A_k)_{k\in\mathbb{N}} \mid A_k \text{ are selfadjoint operators on } H \text{ for all } k \in \mathbb{N}\right\}$$

and $\mathcal{S}_+(\mathbb{R})$ the family of nonnegative real sequences.

We consider the functional

$$J(K,\mathbf{p};\mathbf{A},f,I) := \sum_{k\in K} p_k f(A_k) - P_K f\left(\frac{1}{P_K}\sum_{k\in K} p_k A_k\right), \tag{2.16}$$

where $K \in \mathcal{P}_f(\mathbb{N})$, $\mathbf{p} \in \mathcal{S}_+(\mathbb{R})$, $\mathbf{A} \in \mathcal{A}(H)$ with $P_K := \sum_{k\in K} p_k > 0$ and $f : I \to \mathbb{R}$ is an operator convex function on the interval I.

In [4] we obtained the following result as well:

Lemma 2. *Let* $f : I \to \mathbb{R}$ *be an operator convex function on the interval* I *and* $\mathbf{p} \in \mathcal{S}_+(\mathbb{R})$, $\mathbf{A} \in \mathcal{A}(H)$. *Assume that* $\operatorname{Sp}(A_k) \subseteq I$ *for any* $k \in \mathbb{N}$.

If K, $L \in \mathcal{P}_f(\mathbb{N}) \setminus \{\emptyset\}$ *with* $K \cap L = \emptyset$ *and* P_K, $P_L > 0$, *then we have the inequality*

$$J(K\cup L,\mathbf{p};\mathbf{A},f,I) \ge J(K,\mathbf{p};\mathbf{A},f,I) + J(L,\mathbf{p};\mathbf{A},f,I) \ge 0, \tag{2.17}$$

i.e., $J(\cdot,\mathbf{p};\mathbf{A},f,I)$ *is super-additive as an index set functional in the operator order.*

If $\emptyset \ne K \subset L$, *then we have*

$$J(L,\mathbf{p};\mathbf{A},f,I) \ge J(K,\mathbf{p};\mathbf{A},f,I) \ge 0, \tag{2.18}$$

i.e., $J(\cdot,\mathbf{p};\mathbf{A},f,I)$ *is monotonic as an index set functional in the operator order.*

In particular, we have:

Corollary 3. *Let* $f : I \to \mathbb{R}$ *be an operator convex function on the interval* I *and* $\mathbf{p} = (p_1,...,p_n)$, $\mathbf{A} = (A_1,...,A_n)$ *with* $p_k > 0$, A_k *selfadjoint operators and such that* $\operatorname{Sp}(A_k) \subseteq I$ *for any* $k \in \{1,...,n\}$, $n \ge 2$. *Then we have the inequality*

$$J_k(\mathbf{p};\mathbf{A},f,I) \ge J_{k-1}(\mathbf{p};\mathbf{A},f,I) \ge 0 \tag{2.19}$$

for any $k \in \{1,...,n\}$ *with* $n \ge k \ge 2$.

We also have that

$$J_n(\mathbf{p};\mathbf{A},f,I) \geq p_j f(A_j) + p_k f(A_k) - (p_j+p_k) f\left(\frac{p_j A_j + p_k A_k}{p_j+p_k}\right) \geq 0 \tag{2.20}$$

for any $k,\ j \in \{1,...,n\}$ *in the operator order.*

We define the functional

$$N_K(\mathbf{p};\mathbf{A},x) := \frac{\left[\Delta_x\left(\frac{1}{P_K}\sum_{k\in K} p_k A_k\right)\right]^{P_K}}{\prod_{k\in K}\left[\Delta_x(A_k)\right]^{p_k}} \geq 1, \tag{2.21}$$

where $\mathbf{A} \in \mathcal{A}(H)$ is a sequence of selfadjoint positive operators, $K \in \mathcal{P}_f(\mathbb{N})$, $\mathbf{p} \in \mathcal{S}_+(\mathbb{R})$ and $x \in H$, $\|x\| = 1$.

Theorem 2. *Assume that* $\mathbf{A} \in \mathcal{A}(H)$ *is a sequence of selfadjoint positive operators,* $\mathbf{p} \in \mathcal{S}_+(\mathbb{R})$ *and* $x \in H$ *with* $\|x\| = 1$. *If* $K,\ L \in \mathcal{P}_f(\mathbb{N}) \setminus \{\emptyset\}$ *with* $K \cap L = \emptyset$ *and* $P_K,\ P_L > 0$, *then we have the inequality*

$$N_{K\cup L}(\mathbf{p};\mathbf{A},x) \geq N_K(\mathbf{p};\mathbf{A},x)\, N_L(\mathbf{p};\mathbf{A},x) \geq 1, \tag{2.22}$$

i.e., $N_{\cdot}(\mathbf{p};\mathbf{A},x)$ *is super-multiplicative as an index set functional.*
If $\emptyset \neq K \subset L$, *then we have*

$$N_L(\mathbf{p};\mathbf{A},x) \geq N_K(\mathbf{p};\mathbf{A},x) \geq 1 \tag{2.23}$$

i.e., $N_{\cdot}(\mathbf{p};\mathbf{A},x)$ *is monotonic as an index set functional.*

The proof is similar to the one in Theorem 1 by employing the inequalities in Lemma 2.

Corollary 4. *Assume that* $\mathbf{A} = (A_1,...,A_n)$ *is an* n*-tuple of selfadjoint positive operators,* $\mathbf{p} \in \mathcal{P}_n^+$ *and* $x \in H$ *with* $\|x\| = 1$. *Then we have the inequality*

$$N_k(\mathbf{p};\mathbf{A},x) \geq N_{k-1}(\mathbf{p};\mathbf{A},x) \geq 1 \tag{2.24}$$

for any $k \in \{1,...,n\}$ *with* $n \geq k \geq 2$.
Also, we have

$$N_n(\mathbf{p};\mathbf{A},x) \geq \max_{k,j\in\{1,...,n\}} \frac{\left[\eta_x\left(\frac{p_k A_k + p_j A_j}{p_j+p_k}\right)\right]^{p_j+p_k}}{\left[\eta_x(A_k)\right]^{p_k}\left[\eta_x(A_j)\right]^{p_j}} \geq 1. \tag{2.25}$$

3. Related Results

In [4] we also obtained the following result:

Lemma 3. *If the function* $f : [m,M] \to \mathbb{R}$ *is operator convex and if the* n*-tuple of selfadjoint operators* $(A_1,...,A_n)$ *has the property that* $\operatorname{Sp}(A_j) \subseteq [m,M]$ *for any* $j \in \{1,...,n\}$, *then for any* $p_j \geq 0$ *with* $j \in \{1,...,n\}$ *and* $P_n := \sum_{j=1}^n p_j > 0$ *we* *have*

$$0 \leq \frac{1}{P_n}\sum_{j=1}^{n} p_j f(A_j) - f\left(\frac{1}{P_n}\sum_{j=1}^{n} p_j A_j\right) \tag{3.1}$$
$$\leq \frac{2}{M-m}\left[\frac{f(m)+f(M)}{2} - f\left(\frac{m+M}{2}\right)\right]$$
$$\times\left(\frac{1}{2}(M-m) + \left|\frac{1}{P_n}\sum_{j=1}^{n} p_j A_j - \frac{m+M}{2}\right|\right)$$
$$\leq 2\left[\frac{f(m)+f(M)}{2} - f\left(\frac{m+M}{2}\right)\right]$$

in the operator order.

We also have:

Theorem 3. *Assume that* $\mathbf{A} = (A_1, ..., A_n)$ *an* n*-tuple of selfadjoint operators with spectra in* $[m, M] \subset (0, \infty)$, *then for any* $\mathbf{p} \in \mathcal{P}_n^+$ *and* $x \in H$, $\|x\| = 1$ *we have*

$$1 \leq \frac{\left[\eta_x\left(\frac{1}{P_n}\sum_{j=1}^{n} p_j A_j\right)\right]^{P_n}}{\prod_{j=1}^{n}[\eta_x(A_j)]^{p_j}} \tag{3.2}$$
$$\leq \left(\frac{\sqrt{m^m M^M}}{\left(\frac{m+M}{2}\right)^{\frac{m+M}{2}}}\right)^{\left(1+\frac{2}{M-m}\left\langle\left|\frac{1}{P_n}\sum_{j=1}^{n} p_j A_j - \frac{m+M}{2}\right|x,x\right\rangle\right)}$$
$$\leq \left(\frac{\sqrt{m^m M^M}}{\left(\frac{m+M}{2}\right)^{\frac{m+M}{2}}}\right)^{2}.$$

Proof. If we write the inequality (3.1) for the operator convex function $f = \eta$, we derive

$$0 \leq \frac{1}{P_n}\sum_{j=1}^{n} p_j A_j \ln A_j - \left(\frac{1}{P_n}\sum_{j=1}^{n} p_j A_j\right)\ln\left(\frac{1}{P_n}\sum_{j=1}^{n} p_j A_j\right)$$
$$\leq \frac{2}{M-m}\left[\frac{m\ln m + M\ln M}{2} - \left(\frac{m+M}{2}\right)\ln\left(\frac{m+M}{2}\right)\right]$$
$$\times\left(\frac{1}{2}(M-m)1_H + \left|\frac{1}{P_n}\sum_{j=1}^{n} p_j A_j - \frac{m+M}{2}\right|\right)$$
$$\leq 2\left[\frac{m\ln m + M\ln M}{2} - \left(\frac{m+M}{2}\right)\ln\left(\frac{m+M}{2}\right)\right],$$

namely

$$\begin{aligned}
0 &\leq \frac{1}{P_n}\sum_{j=1}^{n} p_j A_j \ln A_j - \left(\frac{1}{P_n}\sum_{j=1}^{n} p_j A_j\right) \ln\left(\frac{1}{P_n}\sum_{j=1}^{n} p_j A_j\right) \\
&\leq \frac{2}{M-m} \ln\left(\frac{\sqrt{m^m M^M}}{\left(\frac{m+M}{2}\right)^{\frac{m+M}{2}}}\right) \\
&\times \left(\frac{1}{2}(M-m) 1_H + \left|\frac{1}{P_n}\sum_{j=1}^{n} p_j A_j - \frac{m+M}{2} 1_H\right|\right) \\
&\leq 2\ln\left(\frac{\sqrt{m^m M^M}}{\left(\frac{m+M}{2}\right)^{\frac{m+M}{2}}}\right).
\end{aligned}$$

If we take the inner product over $x \in H$, $\|x\| = 1$, we get

$$\begin{aligned}
0 &\leq \left\langle \frac{1}{P_n}\sum_{j=1}^{n} p_j A_j \ln A_j x, x\right\rangle - \left\langle \left(\frac{1}{P_n}\sum_{j=1}^{n} p_j A_j\right) \ln\left(\frac{1}{P_n}\sum_{j=1}^{n} p_j A_j\right) x, x\right\rangle \\
&\leq \frac{2}{M-m} \ln\left(\frac{\sqrt{m^m M^M}}{\left(\frac{m+M}{2}\right)^{\frac{m+M}{2}}}\right) \\
&\times \left(\frac{1}{2}(M-m) 1_H + \left\langle\left|\frac{1}{P_n}\sum_{j=1}^{n} p_j A_j - \frac{m+M}{2}\right| x, x\right\rangle\right) \\
&\leq 2\ln\left(\frac{\sqrt{m^m M^M}}{\left(\frac{m+M}{2}\right)^{\frac{m+M}{2}}}\right).
\end{aligned}$$

By taking the exponential we derive the desired result (3.2). □

Remark 3. *The case of two operators is as follows: if* $0 < m \leq A,\ B \leq M$ *and* $\alpha \in [0,1]$*, then*

$$\begin{aligned}
1 &\leq \frac{\eta_x((1-\alpha)A + \alpha B)}{[\eta_x(A)]^{1-\alpha}[\eta_x(B)]^{\alpha}} \\
&\leq \left(\frac{\sqrt{m^m M^M}}{\left(\frac{m+M}{2}\right)^{\frac{m+M}{2}}}\right)^{1+\frac{2}{M-m}\left\langle\left|(1-\alpha)A+\alpha B-\frac{m+M}{2}1_H\right|x,x\right\rangle} \\
&\leq \left(\frac{\sqrt{m^m M^M}}{\left(\frac{m+M}{2}\right)^{\frac{m+M}{2}}}\right)^{2}
\end{aligned} \tag{3.3}$$

for all $x \in H$, $\|x\| = 1$.

Corollary 5. *If $0 < m \leq A$, $B \leq M$ and $x \in H$, $\|x\| = 1$, then*

$$L(\eta_x(A), \eta_x(B)) \leq \int_0^1 \eta_x((1-t)A + tB)dt \leq \left(\frac{\sqrt{m^m M^M}}{\left(\frac{m+M}{2}\right)^{\frac{m+M}{2}}}\right)^2 L(\eta_x(A), \eta_x(B)), \tag{3.4}$$

where $L(\cdot, \cdot)$ is the logarithmic mean (1.1).

Proof. From (3.3) we have

$$[\eta_x(A)]^{1-t} [\eta_x(B)]^t \leq \eta_x((1-t)A + tB) \leq \left(\frac{\sqrt{m^m M^M}}{\left(\frac{m+M}{2}\right)^{\frac{m+M}{2}}}\right)^2 [\eta_x(A)]^{1-t} [\eta_x(B)]^t,$$

for all $t \in [0, 1]$.

If we take the integral over t, then we get

$$\int_0^1 [\eta_x(A)]^{1-t} [\eta_x(B)]^t dt \leq \int_0^1 \eta_x((1-t)A + tB)dt \leq \left(\frac{\sqrt{m^m M^M}}{\left(\frac{m+M}{2}\right)^{\frac{m+M}{2}}}\right)^2 \int_0^1 [\eta_x(A)]^{1-t} [\eta_x(B)]^t dt. \tag{3.5}$$

Since

$$\int_0^1 [\eta_x(A)]^{1-t} [\eta_x(B)]^t dt = L(\eta_x(A), \eta_x(B)),$$

hence by (3.5) we derive (3.4). □

In [5] we obtained, among others, the following reverse of Jensen's inequality:

Lemma 4. *Let $f : [m, M] \to \mathbb{R}$ be an operator convex function on $[m, M]$ and A_j selfadjoint operators with the spectrum $\mathrm{Sp}(A_j) \subset [m, M]$ for $j = 1, ..., k$. If $C_j \in \mathcal{B}(H)$ for $j = 1, ..., n$ satisfying the condition $\sum_{j=1}^n C_j^* C_j = 1_H$, then*

$$\begin{aligned} 0 &\leq \sum_{j=1}^n C_j^* f(A_j) C_j - f\left(\sum_{j=1}^n C_j^* A_j C_j\right) \\ &\leq \frac{f'_-(M) - f'_+(m)}{M - m} \left(M 1_H - \sum_{j=1}^n C_j^* A_j C_j\right) \left(\sum_{j=1}^n C_j^* A_j C_j - m 1_H\right) \\ &\leq \frac{1}{4}(M - m)\left[f'_-(M) - f'_+(m)\right] 1_H. \end{aligned} \tag{3.6}$$

By the use of this lemma we can state the following result as well:

Theorem 4. *Assume that $\mathbf{A}=(A_1,...,A_n)$ an n-tuple of selfadjoint operators with spectra in $[m,M]\subset(0,\infty)$, then for any $\mathbf{p}\in\mathcal{P}_n^+$ and $x\in H$, $\|x\|=1$ we have*

$$1\leq N_n(\mathbf{p};\mathbf{A},x) \leq \left(\frac{M}{m}\right)^{\frac{1}{M-m}\left(M-\frac{1}{P_n}\sum_{j=1}^n p_j\langle A_jx,x\rangle\right)\left(\frac{1}{P_n}\sum_{j=1}^n p_j\langle A_jx,x\rangle-m\right)} \leq \left(\frac{M}{m}\right)^{\frac{1}{4}(M-m)}. \tag{3.7}$$

Proof. If we take in (3.6) $C_j=\sqrt{\frac{p_j}{P_n}}I$, $j=1,...,n$, then we get

$$\begin{aligned}0&\leq\frac{1}{P_n}\sum_{j=1}^n p_jf(A_j)-f\left(\sum_{j=1}^n p_jA_j\right)\\&\leq\frac{f'_-(M)-f'_+(m)}{M-m}\left(M-\frac{1}{P_n}\sum_{j=1}^n p_jA_j\right)\left(\frac{1}{P_n}\sum_{j=1}^n p_jA_j-m\right)\\&\leq\frac{1}{4}(M-m)\left[f'_-(M)-f'_+(m)\right].\end{aligned}$$

If we take the inner product for $x\in H$, $\|x\|=1$, then we get

$$\begin{aligned}0&\leq\frac{1}{P_n}\sum_{j=1}^n p_j\langle f(A_j)x,x\rangle-\left\langle f\left(\sum_{j=1}^n p_jA_j\right)x,x\right\rangle\\&\leq\frac{f'_-(M)-f'_+(m)}{M-m}\\&\times\left(M-\frac{1}{P_n}\sum_{j=1}^n p_j\langle A_jx,x\rangle\right)\left(\frac{1}{P_n}\sum_{j=1}^n p_j\langle A_jx,x\rangle-m\right)\\&\leq\frac{1}{4}(M-m)\left[f'_-(M)-f'_+(m)\right].\end{aligned}$$

If we write this inequality for the operator convex function $f(t)=t\ln t$, $t>0$, then we get

$$\begin{aligned}0&\leq\frac{1}{P_n}\sum_{j=1}^n p_j\langle A_j\ln A_jx,x\rangle-\left\langle\left(\sum_{j=1}^n p_jA_j\right)\ln\left(\sum_{j=1}^n p_jA_j\right)x,x\right\rangle\\&\leq\frac{\ln M-\ln m}{M-m}\left(M-\frac{1}{P_n}\sum_{j=1}^n p_j\langle A_jx,x\rangle\right)\left(\frac{1}{P_n}\sum_{j=1}^n p_j\langle A_jx,x\rangle-m\right)\\&\leq\frac{1}{4}(M-m)(\ln M-\ln m).\end{aligned}\tag{3.8}$$

Now, if we take the exponential in (3.8), then we get

$$\begin{aligned}1 &\le \frac{\left[\eta_x\left(\frac{1}{P_n}\sum_{j=1}^n p_j A_j\right)\right]^{P_n}}{\prod_{j=1}^n \left[\eta_x\left(A_j\right)\right]^{p_j}}\\ &\le \exp\left[\frac{\ln M-\ln m}{M-m}\left(M-\frac{1}{P_n}\sum_{j=1}^n p_j\left\langle A_j x,x\right\rangle\right)\left(\frac{1}{P_n}\sum_{j=1}^n p_j\left\langle A_j x,x\right\rangle-m\right)\right]\\ &\le \exp\left[\frac{1}{4}\left(M-m\right)\left(\ln M-\ln m\right)\right],\end{aligned}$$

which is equivalent to (3.7). □

Finally, by the use of [5]

Lemma 5. *Let $f:[m,M]\to\mathbb{R}$ be an operator convex function on $[m,M]$ and A_j selfadjoint operators with the spectrum $\operatorname{Sp}(A_j)\subseteq[m,M]$ for $j=1,...,k$. If $C_j\in\mathcal{B}(H)$ for $j=1,...,k$ satisfying the condition $\sum_{j=1}^k C_j^*C_j=1_H$, then*

$$\begin{aligned}0 &\le \sum_{j=1}^k C_j^* f\left(A_j\right)C_j - f\left(\sum_{j=1}^k C_j^*A_jC_j\right)\\ &\le \frac{1}{2}\left\|f''\right\|_{[m,M],\infty}\left(M-\sum_{j=1}^k C_j^*A_jC_j\right)\left(\sum_{j=1}^k C_j^*A_jC_j-m\right)\\ &\le \frac{1}{8}\left\|f''\right\|_{[m,M],\infty}\left(M-m\right)^2,\end{aligned}\tag{3.9}$$

we can also state:

Theorem 5. *With the assumptions of Theorem 4, we have*

$$\begin{aligned}1 &\le D_n\left(\mathbf{p};\mathbf{A},x\right)\\ &\le \exp\left[\frac{1}{2m}\left(M-\frac{1}{P_n}\sum_{j=1}^n p_j\left\langle A_j x,x\right\rangle\right)\left(\frac{1}{P_n}\sum_{j=1}^n p_j\left\langle A_j x,x\right\rangle-m\right)\right]\\ &\le \exp\left[\frac{1}{8}m\left(\frac{M}{m}-1\right)^2\right].\end{aligned}\tag{3.10}$$

The case of two operators is as follows: if $0<m\le A,\ B\le M$ and $\alpha\in[0,1]$, then from (3.7) we obtain

$$\begin{aligned}1 &\le \frac{\Delta_x((1-\alpha)A+\alpha B)}{\left[\Delta_x\left(A\right)\right]^{1-\alpha}\left[\Delta_x\left(B\right)\right]^{\alpha}}\\ &\le \left(\frac{M}{m}\right)^{\frac{1}{M-m}(M-(1-\alpha)\langle Ax,x\rangle-\alpha\langle Bx,x\rangle)((1-\alpha)\langle Ax,x\rangle+\alpha\langle Bx,x\rangle-m)}\\ &\le \left(\frac{M}{m}\right)^{\frac{1}{4}(M-m)}\end{aligned}\tag{3.11}$$

for all $x \in H$, $\|x\| = 1$, while from (3.10) we derive

$$1 \leq \frac{\Delta_x((1-\alpha)A+\alpha B)}{[\Delta_x(A)]^{1-\alpha}[\Delta_x(B)]^{\alpha}} \tag{3.12}$$
$$\leq \exp\left[\frac{1}{2m}\left(M-(M-(1-\alpha)\langle Ax,x\rangle-\alpha\langle Bx,x\rangle)\right)\right.$$
$$\left.\times((1-\alpha)\langle Ax,x\rangle+\alpha\langle Bx,x\rangle-m)\right]$$
$$\leq \exp\left[\frac{1}{8}m\left(\frac{M}{m}-1\right)^2\right]$$

for all $x \in H$, $\|x\| = 1$.

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