INEQUALITIES FOR NORMALIZED ENTROPIC DETERMINANT OF POSITIVE OPERATORS IN HILBERT SPACES

SILVESTRU SEVER DRAGOMIR[1,2]

Abstract. For positive invertible operators A on a Hilbert space H and a fixed unit vector $x \in H$, define the normalized determinant by $\Delta_x(A) := \exp[\langle \ln Ax, x\rangle]$ and the normalized entropic determinant by

$$\eta_x(A) := \exp[-\langle A \ln Ax, x\rangle].$$

In this paper we show among others that, if $A_j > 0$ and $p_j \geq 0$, $j \in \{1, ..., n\}$ with $\sum_{j=1}^{n} p_j = 1$, then

$$\prod_{j=1}^{n} [\eta_x(A_j)]^{p_j} \leq \left(\prod_{j=1}^{n} [\Delta_x(A_j)]^{p_j} \right)^{-\left\langle \sum_{j=1}^{n} p_j A_j x, x \right\rangle}$$

for all $x \in H$ with $\|x\| = 1$. In particular, we have

$$\eta_x(A) \leq [\Delta_x(A)]^{-\langle Ax, x\rangle}$$

for all $x \in H$ with $\|x\| = 1$.

1. Introduction

Let $B(H)$ be the space of all bounded linear operators on a Hilbert space H, and I stands for the identity operator on H. An operator A in $B(H)$ is said to be positive (in symbol: $A \geq 0$) if $\langle Ax, x\rangle \geq 0$ for all $x \in H$. In particular, $A > 0$ means that A is positive and invertible. For a pair A, B of selfadjoint operators the order relation $A \geq B$ means as usual that $A - B$ is positive.

In 1998, Fujii et al. [6], [7], introduced the *normalized determinant* $\Delta_x(A)$ for positive invertible operators A on a Hilbert space H and a fixed unit vector $x \in H$, namely $\|x\| = 1$, defined by $\Delta_x(A) := \exp\langle \ln Ax, x\rangle$ and discussed it as a continuous geometric mean and observed some inequalities around the determinant from this point of view.

Some of the fundamental properties of normalized determinant are as follows, [6].

For each unit vector $x \in H$, see also [9], we have:

(i) *continuity*: the map $A \to \Delta_x(A)$ is norm continuous;
(ii) *bounds*: $\left\langle A^{-1}x, x\right\rangle^{-1} \leq \Delta_x(A) \leq \langle Ax, x\rangle$;
(iii) *continuous mean*: $\langle A^p x, x\rangle^{1/p} \downarrow \Delta_x(A)$ for $p \downarrow 0$ and $\langle A^p x, x\rangle^{1/p} \uparrow \Delta_x(A)$ for $p \uparrow 0$;
(iv) *power equality:* $\Delta_x(A^t) = \Delta_x(A)^t$ for all $t > 0$;
(v) *homogeneity*: $\Delta_x(tA) = t\Delta_x(A)$ and $\Delta_x(tI) = t$ for all $t > 0$;

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(vi) *monotonicity*: $0 < A \leq B$ implies $\Delta_x(A) \leq \Delta_x(B)$;
(vii) *multiplicativity*: $\Delta_x(AB) = \Delta_x(A)\Delta_x(B)$ for commuting A and B;
(viii) *Ky Fan type inequality*: $\Delta_x((1-\alpha) A + \alpha B) \geq \Delta_x(A)^{1-\alpha}\Delta_x(B)^{\alpha}$ for $0 < \alpha < 1$.

We define the logarithmic mean of two positive numbers a, b by

$$L(a,b) := \begin{cases} \frac{b-a}{\ln b - \ln a} \text{ if } b \neq a, \\ \\ a \text{ if } b = a. \end{cases} \tag{1.1}$$

In [6] the authors obtained the following additive reverse inequality for the operator A which satisfy the condition $0 < mI \leq A \leq MI$, where m, M are positive numbers,

$$0 \leq \langle Ax, x\rangle - \Delta_x(A) \leq L(m,M)\left[\ln L(m,M) + \frac{M\ln m - m\ln M}{M-m} - 1\right] \tag{1.2}$$

for all $x \in H$, $\|x\| = 1$.

We recall that *Specht's ratio* is defined by [13]

$$S(h) := \begin{cases} \frac{h^{\frac{1}{h-1}}}{e\ln\left(h^{\frac{1}{h-1}}\right)} \text{ if } h \in (0,1)\cup(1,\infty), \\ \\ 1 \text{ if } h = 1. \end{cases} \tag{1.3}$$

It is well known that $\lim_{h\to 1} S(h) = 1$, $S(h) = S\left(\frac{1}{h}\right) > 1$ for $h > 0$, $h \neq 1$. The function is decreasing on $(0,1)$ and increasing on $(1,\infty)$.

In [7], the authors obtained the following multiplicative reverse inequality as well

$$1 \leq \frac{\langle Ax, x\rangle}{\Delta_x(A)} \leq S\left(\frac{M}{m}\right) \tag{1.4}$$

for $0 < mI \leq A \leq MI$ and $x \in H$, $\|x\| = 1$.

For the entropy function $\eta(t) = -t\ln t$, $t > 0$, the operator entropy has the following expression:

$$\eta(A) = -A\ln A$$

for positive A.

For $x \in H$, $\|x\| = 1$, we define the *normalized entropic determinant* $\eta_x(A)$ by

$$\eta_x(A) := \exp(-\langle A\ln Ax, x\rangle) = \exp\langle \eta(A)x, x\rangle. \tag{1.5}$$

Let $x \in H$, $\|x\| = 1$. Observe that the map $A \to \eta_x(A)$ is *norm continuous* and since

$$\begin{aligned} &\exp(-\langle tA\ln(tA)x, x\rangle) \\ &= \exp(-\langle tA(\ln t + \ln A)x, x\rangle) = \exp(-\langle (tA\ln t + tA\ln A)x, x\rangle) \\ &= \exp(-\langle Ax, x\rangle t\ln t)\exp(-t\langle A\ln Ax, x\rangle) \\ &= \exp\ln\left(t^{-\langle Ax,x\rangle t}\right)[\exp(-\langle A\ln Ax, x\rangle)]^{-t}, \end{aligned}$$

hence

$$\eta_x(tA) = t^{-t\langle Ax,x\rangle}[\eta_x(A)]^{-t} \tag{1.6}$$

for $t > 0$ and $A > 0$.

Observe also that

$$\eta_x(I) = 1 \text{ and } \eta_x(tI) = t^{-t} \tag{1.7}$$

for $t > 0$.

In [2] we showed that, if $A,\, B > 0$, then for all $x \in H$ with $\|x\| = 1$ and $t \in [0,1]$,

$$\eta_x((1-t)\,A + tB) \geq (\eta_x\,(A))^{1-t}\,(\eta_x\,(B))^t\,.$$

We also have the bounds

$$\left(\frac{\langle A^2 x, x\rangle}{\langle Ax, x\rangle}\right)^{-\langle Ax,x\rangle} \leq \eta_x(A) \leq \langle Ax, x\rangle^{-\langle Ax,x\rangle}\,,$$

where $A > 0$ and $x \in H$ with$\|x\| = 1$.

If $y \in H$, $y \neq 0$, then we can extend the definition of the normalized entropic determinant as follows

$$\breve{\eta}_y(A) := \exp\left(-\langle A \ln Ay, y\rangle\right) = \exp\left\langle \eta\,(A)\,y, y\right\rangle.$$

Also we can consider

$$\breve{\Delta}_y(A) := \exp\left\langle \ln Ay, y\right\rangle$$

for $y \in H$, $y \neq 0$.

We observe that

$$\breve{\eta}_y(A) = \left[\eta_{\frac{y}{\|y\|}}\,(A)\right]^{\|y\|^2} \text{ and } \breve{\Delta}_y(A) = \left[\Delta_{\frac{y}{\|y\|}}\,(A)\right]^{\|y\|^2}$$

for $y \in H$, $y \neq 0$.

Motivated by the above results, in this paper we show among others that, if $A_j > 0$ and $p_j \geq 0$, $j \in \{1, ..., n\}$ with $\sum_{j=1}^n p_j = 1$, then

$$\prod_{j=1}^n \left[\eta_x\,(A_j)\right]^{p_j} \leq \left(\prod_{j=1}^n \left[\Delta_x(A_j)\right]^{p_j}\right)^{-\left\langle \sum_{j=1}^n p_j A_j x, x\right\rangle}$$

for all $x \in H$ with $\|x\| = 1$. In particular, we have

$$\eta_x\,(A) \leq \left[\Delta_x(A)\right]^{-\langle Ax,x\rangle}$$

for all $x \in H$ with $\|x\| = 1$.

2. Main Results

We say that the functions $f,\, g : [a,b] \longrightarrow \mathbb{R}$ are *synchronous (asynchronous)* on the interval $[a,b]$ if they satisfy the following condition:

$$\left(f\,(t) - f\,(s)\right)\left(g\,(t) - g\,(s)\right) \geq (\leq)\, 0 \text{ for each } t,\ s \in [a,b]\,.$$

It is obvious that, if f, g are monotonic and have the same monotonicity on the interval $[a,b]$, then they are synchronous on $[a,b]$ while if they have opposite monotonicity, they are asynchronous.

Theorem 1. *Assume that A, $B > 0$ and x, $y \in H$ with $\|x\| = \|y\| = 1$, then*

$$\eta_x(A)\eta_y(B) \leq \left[\Delta_x(A)\right]^{-\langle By,y\rangle}\left[\Delta_y(B)\right]^{-\langle Ax,x\rangle}\,. \tag{2.1}$$

In particular, we have

$$\eta_x(A)\eta_x(B) \leq \left[\Delta_x(A)\right]^{-\langle Bx,x\rangle}\left[\Delta_x(B)\right]^{-\langle Ax,x\rangle}\,, \tag{2.2}$$

$$\eta_x(A)\eta_y(A) \le [\Delta_x(A)]^{-\langle Ay,y\rangle} [\Delta_y(A)]^{-\langle Ax,x\rangle} \tag{2.3}$$

and

$$\eta_x(A) \le [\Delta_x(A)]^{-\langle Ax,x\rangle}. \tag{2.4}$$

Proof. In [1] we obtained, between others, the following two operators and two vectors inequality: if A and B are selfadjoint operators and $\mathrm{Sp}(A), \mathrm{Sp}(B) \subseteq [m, M]$, then for any continuous synchronous (asynchronous) functions $f, g : [m, M] \longrightarrow \mathbb{R}$ we have the more general result

$$\begin{aligned} &\langle f(A) g(A) x, x\rangle + \langle f(B) g(B) y, y\rangle \\ &\ge (\le) \langle f(A) x, x\rangle \langle g(B) y, y\rangle + \langle f(B) y, y\rangle \langle g(A) x, x\rangle \end{aligned} \tag{2.5}$$

for each $x, y \in H$ with $\|x\| = \|y\| = 1$.

If we write (2.5) for the asynchronous functions $f(t) = -t$, $g(t) = \ln t$, $t > 0$, then we obtain

$$-\langle A \ln Ax, x\rangle - \langle B \ln By, y\rangle \le -\langle Ax, x\rangle \langle \ln By, y\rangle - \langle By, y\rangle \langle \ln Ax, x\rangle \tag{2.6}$$

for each $x, y \in H$ with $\|x\| = \|y\| = 1$.

If we take the exponential in (2.6), then we get

$$\begin{aligned} &\exp[-\langle A \ln Ax, x\rangle] \exp[-\langle B \ln By, y\rangle] \\ &\le \exp[-\langle Ax, x\rangle \langle \ln By, y\rangle] \exp[-\langle By, y\rangle \langle \ln Ax, x\rangle] \\ &= [\exp\langle \ln By, y\rangle]^{-\langle Ax,x\rangle} [\exp\langle \ln Ax, x\rangle]^{-\langle By,y\rangle} \end{aligned}$$

for each $x, y \in H$ with $\|x\| = \|y\| = 1$, which is equivalent to (2.1). □

We also have:

Theorem 2. *Assume that $A_j > 0$ and $x_j \in H$, $j \in \{1, ..., n\}$ with $\sum_{j=1}^n \|x_j\|^2 = 1$, then*

$$\prod_{j=1}^n \left[\eta_{\frac{x_j}{\|x_j\|}}(A)\right]^{\|x_j\|^2} \le \left(\prod_{j=1}^n \left[\Delta_{\frac{x_j}{\|x_j\|}}(A_j)\right]^{\|x_j\|^2}\right)^{-\sum_{j=1}^n \langle A_j x_j, x_j\rangle}. \tag{2.7}$$

Proof. In [1] we also obtained the following result: let A_j be selfadjoint operators with $\mathrm{Sp}(A_j) \subseteq [m, M]$ for $j \in \{1, ..., n\}$ and for some scalars $m < M$. If $f, g : [m, M] \longrightarrow \mathbb{R}$ are continuous and *synchronous (asynchronous) on* $[m, M]$, *then*

$$\sum_{j=1}^n \langle f(A_j) g(A_j) x_j, x_j\rangle \ge (\le) \sum_{j=1}^n \langle f(A_j) x_j, x_j\rangle \sum_{j=1}^n \langle g(A_j) x_j, x_j\rangle, \tag{2.8}$$

for each $x_j \in H$, $j \in \{1, ..., n\}$ with $\sum_{j=1}^n \|x_j\|^2 = 1$.

If we write the inequality (2.8) for the asynchronous functions $f(t) = -t$, $g(t) = \ln t$, $t > 0$, then we obtain

$$\sum_{j=1}^n -\langle A_j \ln A_j x_j, x_j\rangle \le -\sum_{j=1}^n \langle A_j x_j, x_j\rangle \sum_{j=1}^n \langle \ln A_j x_j, x_j\rangle, \tag{2.9}$$

for each $x_j \in H$, $j \in \{1, ..., n\}$ with $\sum_{j=1}^n \|x_j\|^2 = 1$.

By taking the exponential, we derive

$$\exp\left(\sum_{j=1}^{n} -\langle A_j \ln A_j x_j, x_j\rangle\right) \le \exp\left(-\sum_{j=1}^{n}\langle A_j x_j, x_j\rangle \sum_{j=1}^{n}\langle \ln A_j x_j, x_j\rangle\right).$$

Observe that

$$\begin{aligned}\exp\left(\sum_{j=1}^{n} -\langle A_j \ln A_j x_j, x_j\rangle\right) &= \prod_{j=1}^{n}\exp\left[-\langle A_j \ln A_j x_j, x_j\rangle\right]\\ &= \prod_{j=1}^{n}\breve{\eta}_{\frac{x_j}{\|x_j\|}}(A_j) = \prod_{j=1}^{n}\left[\eta_{\frac{x_j}{\|x_j\|}}(A)\right]^{\|x_j\|^2}\end{aligned}$$

and

$$\begin{aligned}&\exp\left(-\sum_{j=1}^{n}\langle A_j x_j, x_j\rangle \sum_{j=1}^{n}\langle \ln A_j x_j, x_j\rangle\right)\\ &= \left[\exp\left(\sum_{j=1}^{n}\langle \ln A_j x_j, x_j\rangle\right)\right]^{-\sum_{j=1}^{n}\langle A_j x_j, x_j\rangle}\\ &= \left[\prod_{j=1}^{n}\exp\left(\langle \ln A_j x_j, x_j\rangle\right)\right]^{-\sum_{j=1}^{n}\langle A_j x_j, x_j\rangle}\\ &= \left[\prod_{j=1}^{n}\breve{\Delta}_{\frac{x_j}{\|x_j\|}}(A_j)\right]^{-\sum_{j=1}^{n}\langle A_j x_j, x_j\rangle} = \left(\prod_{j=1}^{n}\left[\Delta_{\frac{x_j}{\|x_j\|}}(A_j)\right]^{\|x_j\|^2}\right)^{-\sum_{j=1}^{n}\langle A_j x_j, x_j\rangle}\end{aligned}$$

for each $x_j \in H$, $j \in \{1, ..., n\}$ with $\sum_{j=1}^{n}\|x_j\|^2 = 1$.

By utilizing (2.9) we derive (2.7). □

Corollary 1. *Let $A_j > 0$ and $p_j \ge 0$, $j \in \{1, ..., n\}$ with $\sum_{j=1}^{n} p_j = 1$. Then*

$$\prod_{j=1}^{n}\left[\eta_x(A_j)\right]^{p_j} \le \left(\prod_{j=1}^{n}\left[\Delta_x(A_j)\right]^{p_j}\right)^{-\left\langle \sum_{j=1}^{n} p_j A_j x, x\right\rangle} \tag{2.10}$$

for all $x \in H$ with $\|x\| = 1$.

Proof. If we choose in Theorem 2 $x_j = \sqrt{p_j}x$, $j \in \{1, ..., n\}$, where $p_j \ge 0$, $j \in \{1, ..., n\}$, with $\sum_{j=1}^{n} p_j = 1$ and $x \in H$, with $\|x\| = 1$ then a simple calculation shows that the inequality (2.7) becomes (2.10). The details are omitted. □

Remark 1. *The case of two operators is as follows*

$$\left[\eta_x(A)\right]^{1-t}\left[\eta_x(B)\right]^{t} \le \left(\left[\Delta_x(A)\right]^{1-t}\left[\Delta_x(B)\right]^{t}\right)^{-\langle (1-t)A + tBx, x\rangle}, \tag{2.11}$$

for A, $B > 0$, $t \in [0, 1]$ and $x \in H$ with $\|x\| = 1$.

For $t = 1/2$, we get

$$\eta_x(A)\,\eta_x(B) \le \left[\Delta_x(A)\,\Delta_x(B)\right]^{-\left\langle \frac{A+B}{2}x, x\right\rangle},$$

for A, $B > 0$ and $x \in H$ with $\|x\| = 1$.

If we take $B = A$, we recapture (2.18).

Theorem 3. *Assume that A, $B > 0$ and x, $y \in H$ with $\|x\| = \|y\| = 1$. If $q \in \mathbb{R} \setminus \{0\}$ and $p > 0$ with $p + q \neq 0$, then*

$$\begin{aligned} &\left[\eta_x\left(A^{p+q}\right)\right]^{\frac{1}{p+q}\langle B^q y,y\rangle}\left[\eta_y\left(B^{p+q}\right)\right]^{\frac{1}{p+q}\langle A^p x,x\rangle} \\ &\leq \left[\eta_y\left(B^q\right)\right]^{\frac{1}{q}\left\langle A^{p+q}x,x\right\rangle}\left[\eta_x\left(A^p\right)\right]^{\frac{1}{p}\left\langle B^{p+q}y,y\right\rangle}. \end{aligned} \tag{2.12}$$

If $p < 0$ and $p + q \neq 0$, then

$$\begin{aligned} &\left[\eta_x\left(A^{p+q}\right)\right]^{\frac{1}{p+q}\langle B^q y,y\rangle}\left[\eta_y\left(B^{p+q}\right)\right]^{\frac{1}{p+q}\langle A^p x,x\rangle} \\ &\geq \left[\eta_y\left(B^q\right)\right]^{\frac{1}{q}\left\langle A^{p+q}x,x\right\rangle}\left[\eta_x\left(A^p\right)\right]^{\frac{1}{p}\left\langle B^{p+q}y,y\right\rangle}. \end{aligned} \tag{2.13}$$

Proof. We consider only the case of synchronous functions. In this case we have then

$$f(t)\,g(t) + f(s)\,g(s) \geq f(t)\,g(s) + f(s)\,g(t)$$

for each t, $s \in [m, M]$.

Assume that h is positive If we multiply this inequality by $h(t)\,h(s) \geq 0$ we get

$$\begin{aligned} &h(t)\,f(t)\,g(t)\,h(s) + h(t)\,h(s)\,f(s)\,g(s) \\ &\geq h(t)\,f(t)\,h(s)\,g(s) + h(s)\,f(s)\,h(t)\,g(t) \end{aligned}$$

for each t, $s \in [m, M]$.

If we fix $s \in [m, M]$ and use the continuous functional calculus for operators with spectra in $[m, M]$ over t we get

$$\begin{aligned} &h(s)\,h(A)\,f(A)\,g(A) + h(s)\,f(s)\,g(s)\,h(A) \\ &\geq h(s)\,g(s)\,h(A)\,f(A) + h(s)\,f(s)\,h(A)\,g(A) \end{aligned}$$

and by taking the inner product over $x \in H$ with $\|x\| = 1$, we get

$$\begin{aligned} &h(s)\langle h(A)\,f(A)\,g(A)\,x,x\rangle + h(s)\,f(s)\,g(s)\langle h(A)\,x,x\rangle \\ &\geq h(s)\,g(s)\langle h(A)\,f(A)\,x,x\rangle + h(s)\,f(s)\langle h(A)\,g(A)\,x,x\rangle. \end{aligned}$$

If we apply again the functional calculus over B, then we get

$$\begin{aligned} &\langle h(A)\,f(A)\,g(A)\,x,x\rangle\,h(B) + \langle h(A)\,x,x\rangle\,h(B)\,f(B)\,g(B) \\ &\geq \langle h(A)\,f(A)\,x,x\rangle\,h(B)\,g(B) + \langle h(A)\,g(A)\,x,x\rangle\,h(B)\,f(B). \end{aligned}$$

If we take the inner product for $y \in H$ with $\|y\| = 1$, then we get

$$\begin{aligned} &\langle h(A)\,f(A)\,g(A)\,x,x\rangle\langle h(B)\,y,y\rangle + \langle h(A)\,x,x\rangle\langle h(B)\,f(B)\,g(B)\,y,y\rangle \\ &\geq \langle h(A)\,f(A)\,x,x\rangle\langle h(B)\,g(B)\,y,y\rangle + \langle h(A)\,g(A)\,x,x\rangle\langle h(B)\,f(B)\,y,y\rangle, \end{aligned} \tag{2.14}$$

which holds for x, $y \in H$ with $\|x\| = \|y\| = 1$.

If f and g are asynchronous, then the inequality (2.14) reverses.

For $p > 0$, $f(t) = t^p$ is increasing, $g(t) = -\ln t$ is decreasing and $h(t) = t^q$ is positive on $(0, \infty)$. By (2.14) we derive

$$\begin{aligned} &-\left\langle A^{p+q}\ln Ax,x\right\rangle\langle B^q y,y\rangle - \langle A^p x,x\rangle\left\langle B^{p+q}\ln By,y\right\rangle \\ &\leq -\left\langle A^{p+q}x,x\right\rangle\langle B^q\ln By,y\rangle - \langle A^p\ln Ax,x\rangle\left\langle B^{p+q}y,y\right\rangle, \end{aligned}$$

namely

$$\begin{aligned} & -\frac{1}{p+q}\left\langle A^{p+q}\ln A^{p+q}x,x\right\rangle\left\langle B^{q}y,y\right\rangle-\frac{1}{p+q}\left\langle A^{p}x,x\right\rangle\left\langle B^{p+q}\ln B^{p+q}y,y\right\rangle \\ & \leq -\frac{1}{q}\left\langle A^{p+q}x,x\right\rangle\left\langle B^{q}\ln B^{q}y,y\right\rangle-\frac{1}{p}\left\langle A^{p}\ln A^{p}x,x\right\rangle\left\langle B^{p+q}y,y\right\rangle, \end{aligned}$$

for $x,\ y\in H$ with $\|x\|=\|y\|=1$.

By taking the exponential, we get

$$\begin{aligned} & \left(\exp\left[-\left\langle A^{p+q}\ln A^{p+q}x,x\right\rangle\right]\right)^{\frac{1}{p+q}\left\langle B^{q}y,y\right\rangle}\left(\exp\left[-\left\langle B^{p+q}\ln B^{p+q}y,y\right\rangle\right]\right)^{\frac{1}{p+q}\left\langle A^{p}x,x\right\rangle} \\ & \leq\left(\exp\left[-\left\langle B^{q}\ln B^{q}y,y\right\rangle\right]\right)^{\frac{1}{q}\left\langle A^{p+q}x,x\right\rangle}\left(\exp\left[-\left\langle A^{p}\ln A^{p}x,x\right\rangle\right]\right)^{\frac{1}{p}\left\langle B^{p+q}y,y\right\rangle}, \end{aligned}$$

which is equivalent to (2.12). □

Corollary 2. *With the assumptions of Theorem 3, and $p>0$, we have the particular inequalities*

$$\begin{aligned} & \left[\eta_{x}\left(A^{p+q}\right)\right]^{\frac{1}{p+q}\left\langle B^{q}x,x\right\rangle}\left[\eta_{x}\left(B^{p+q}\right)\right]^{\frac{1}{p+q}\left\langle A^{p}x,x\right\rangle} \\ & \leq\left[\eta_{x}\left(B^{q}\right)\right]^{\frac{1}{q}\left\langle A^{p+q}x,x\right\rangle}\left[\eta_{x}\left(A^{p}\right)\right]^{\frac{1}{p}\left\langle B^{p+q}x,x\right\rangle}, \end{aligned} \tag{2.15}$$

$$\begin{aligned} & \left[\eta_{x}\left(A^{p+q}\right)\right]^{\frac{1}{p+q}\left\langle A^{q}y,y\right\rangle}\left[\eta_{y}\left(A^{p+q}\right)\right]^{\frac{1}{p+q}\left\langle A^{p}x,x\right\rangle} \\ & \leq\left[\eta_{y}\left(A^{q}\right)\right]^{\frac{1}{q}\left\langle A^{p+q}x,x\right\rangle}\left[\eta_{x}\left(A^{p}\right)\right]^{\frac{1}{p}\left\langle A^{p+q}y,y\right\rangle} \end{aligned} \tag{2.16}$$

and

$$\left[\eta_{x}\left(A^{p+q}\right)\right]^{\frac{1}{p+q}\left[\left\langle A^{p}x,x\right\rangle+\left\langle A^{q}x,x\right\rangle\right]}\leq\left[\eta_{x}\left(A^{q}\right)\right]^{\frac{1}{q}\left\langle A^{p+q}x,x\right\rangle}\left[\eta_{x}\left(A^{p}\right)\right]^{\frac{1}{p}\left\langle A^{p+q}x,x\right\rangle}. \tag{2.17}$$

If $p<0$, then we have the particular inequalities

$$\begin{aligned} & \left[\eta_{x}\left(A^{p+q}\right)\right]^{\frac{1}{p+q}\left\langle B^{q}x,x\right\rangle}\left[\eta_{x}\left(B^{p+q}\right)\right]^{\frac{1}{p+q}\left\langle A^{p}x,x\right\rangle} \\ & \geq\left[\eta_{x}\left(B^{q}\right)\right]^{\frac{1}{q}\left\langle A^{p+q}x,x\right\rangle}\left[\eta_{x}\left(A^{p}\right)\right]^{\frac{1}{p}\left\langle B^{p+q}x,x\right\rangle}, \end{aligned} \tag{2.18}$$

$$\begin{aligned} & \left[\eta_{x}\left(A^{p+q}\right)\right]^{\frac{1}{p+q}\left\langle A^{q}y,y\right\rangle}\left[\eta_{y}\left(A^{p+q}\right)\right]^{\frac{1}{p+q}\left\langle A^{p}x,x\right\rangle} \\ & \geq\left[\eta_{y}\left(A^{q}\right)\right]^{\frac{1}{q}\left\langle A^{p+q}x,x\right\rangle}\left[\eta_{x}\left(A^{p}\right)\right]^{\frac{1}{p}\left\langle A^{p+q}y,y\right\rangle} \end{aligned} \tag{2.19}$$

and

$$\left[\eta_{x}\left(A^{p+q}\right)\right]^{\frac{1}{p+q}\left[\left\langle A^{p}x,x\right\rangle+\left\langle A^{q}x,x\right\rangle\right]}\geq\left[\eta_{x}\left(A^{q}\right)\right]^{\frac{1}{q}\left\langle A^{p+q}x,x\right\rangle}\left[\eta_{x}\left(A^{p}\right)\right]^{\frac{1}{p}\left\langle A^{p+q}x,x\right\rangle}. \tag{2.20}$$

If we take $q=p>0$ in (2.17), then we get

$$\left[\eta_{x}\left(A^{2p}\right)\right]^{\frac{1}{2p}\left\langle A^{p}x,x\right\rangle}\leq\left[\eta_{x}\left(A^{p}\right)\right]^{\frac{1}{p}\left\langle A^{2p}x,x\right\rangle} \tag{2.21}$$

for $A>0$ and $x\in H$ with $\|x\|=1$.

Remark 2. *If we take $p=1-t$, $q=t\in(0,1)$ in (2.12), then we get*

$$\left[\eta_{x}\left(A\right)\right]^{\left\langle B^{t}y,y\right\rangle}\left[\eta_{y}\left(B\right)\right]^{\left\langle A^{1-t}x,x\right\rangle}\leq\left[\eta_{y}\left(B^{t}\right)\right]^{\frac{1}{t}\left\langle Ax,x\right\rangle}\left[\eta_{x}\left(A^{1-t}\right)\right]^{\frac{1}{1-t}\left\langle By,y\right\rangle}, \tag{2.22}$$

for all $t\in(0,1)$, $A,\ B>0$ and $x,\ y\in H$ with $\|x\|=\|y\|=1$.

In particular, for $B=A$ and $y=x$ we derive

$$\left[\eta_{x}\left(A\right)\right]^{\left\langle\left(A^{t}+A^{1-t}\right)x,x\right\rangle}\leq\left[\eta_{x}\left(A^{t}\right)\right]^{\frac{1}{t}\left\langle Ax,x\right\rangle}\left[\eta_{x}\left(A^{1-t}\right)\right]^{\frac{1}{1-t}\left\langle Ax,x\right\rangle}, \tag{2.23}$$

for $x\in H$ with $\|x\|=1$.

3. Related Results

We also have:

Theorem 4. *Assume that* $A_j > 0$ *and* $x_j \in H$, $j \in \{1, ..., n\}$ *with* $\sum_{j=1}^{n} \|x_j\|^2 = 1$, *then for all* $p \geq 1$,

$$\frac{\left(\prod_{j=1}^{n} \left[\eta_{\frac{x_j}{\|x_j\|}}\left(A_j^p\right)\right]^{\|x_j\|^2}\right)^{1/p}}{\left[\prod_{j=1}^{n} \left(\Delta_{\frac{x_j}{\|x_j\|}}\left(A_j\right)\right)^{\|x_j\|^2}\right]^{\sum_{j=1}^{n}\left\langle A_j^p x_j, x_j\right\rangle}} \tag{3.1}$$
$$\geq \left(\frac{\sum_{j=1}^{n} \left\langle A_j x_j, x_j\right\rangle}{\prod_{j=1}^{n} \left(\Delta_{\frac{x_j}{\|x_j\|}}\left(A_j\right)\right)^{\|x_j\|^2}}\right)^{\sum_{j=1}^{n}\left\langle A_j^p x_j, x_j\right\rangle - \left(\sum_{j=1}^{n}\left\langle A_j x_j, x_j\right\rangle\right)^p} \geq 1$$

Proof. Let A_j be selfadjoint operators with $\operatorname{Sp}(A_j) \subseteq [m, M]$ for $j \in \{1, ..., n\}$ and for some scalars $m < M$. In [1] we also obtained the following result: if f, g are a*synchronous on* $[m, M]$, then

$$\sum_{j=1}^{n} \left\langle f\left(A_j\right) x_j, x_j\right\rangle \sum_{j=1}^{n} \left\langle g\left(A_j\right) x_j, x_j\right\rangle - \sum_{j=1}^{n} \left\langle f\left(A_j\right) g\left(A_j\right) x_j, x_j\right\rangle \tag{3.2}$$
$$\geq \left[\sum_{j=1}^{n} \left\langle f\left(A_j\right) x_j, x_j\right\rangle - f\left(\sum_{j=1}^{n} \left\langle A_j x_j, x_j\right\rangle\right)\right]$$
$$\times \left[\sum_{j=1}^{n} \left\langle g\left(A_j\right) x_j, x_j\right\rangle - g\left(\sum_{j=1}^{n} \left\langle A_j x_j, x_j\right\rangle\right)\right]$$

for each $x_j \in H$, $j \in \{1, ..., n\}$ with $\sum_{j=1}^{n} \|x_j\|^2 = 1$. Moreover, if either both of them are convex or both of them are concave on $[m, M]$, then the right hand side of (3.2) is nonnegative as well.

Assume that A_j are positive definite and $p \geq 1$. Then by writing the the inequality (3.2) for the functions $f(t) = t^p$ and $g(t) = -\ln t$, $t > 0$

$$\sum_{j=1}^{n} \left\langle A_j^p \log A x_j, x_j\right\rangle - \sum_{j=1}^{n} \left\langle A_j^p x_j, x_j\right\rangle \sum_{j=1}^{n} \left\langle \log A_j x_j, x_j\right\rangle \tag{3.3}$$

$$
\begin{aligned}
&\geq \left[\sum_{j=1}^{n}\left\langle A_j^p x_j, x_j\right\rangle - \left(\sum_{j=1}^{n}\left\langle A_j x_j, x_j\right\rangle\right)^p\right] \\
&\times \left[\log\left(\sum_{j=1}^{n}\left\langle A_j x_j, x_j\right\rangle\right) - \sum_{j=1}^{n}\log\left\langle A_j x_j, x_j\right\rangle\right] \\
&\geq 0
\end{aligned}
$$

for each $x_j \in H$, $j \in \{1, ..., n\}$ with $\sum_{j=1}^{n}\|x_j\|^2 = 1$.

By taking the exponential in (3.3), we get

$$
\begin{aligned}
&\exp\left[\sum_{j=1}^{n}\left\langle A_j^p \log A x_j, x_j\right\rangle - \sum_{j=1}^{n}\left\langle A_j^p x_j, x_j\right\rangle \sum_{j=1}^{n}\left\langle \log A_j x_j, x_j\right\rangle\right] \\
&\geq \left(\exp\left[\log\left(\sum_{j=1}^{n}\left\langle A_j x_j, x_j\right\rangle\right) - \sum_{j=1}^{n}\log\left\langle A_j x_j, x_j\right\rangle\right]\right)^{\sum_{j=1}^{n}\left\langle A_j^p x_j, x_j\right\rangle - \left(\sum_{j=1}^{n}\left\langle A_j x_j, x_j\right\rangle\right)^p} \\
&\geq 1.
\end{aligned}
$$

Observe that

$$
\begin{aligned}
&\exp\left[\sum_{j=1}^{n}\left\langle A_j^p \log A x_j, x_j\right\rangle - \sum_{j=1}^{n}\left\langle A_j^p x_j, x_j\right\rangle \sum_{j=1}^{n}\left\langle \log A_j x_j, x_j\right\rangle\right] \\
&= \exp\left[\frac{1}{p}\sum_{j=1}^{n}\left\langle A_j^p \log A_j^p x_j, x_j\right\rangle - \sum_{j=1}^{n}\left\langle A_j^p x_j, x_j\right\rangle \sum_{j=1}^{n}\left\langle \log A_j x_j, x_j\right\rangle\right] \\
&= \frac{\exp\left(\frac{1}{p}\sum_{j=1}^{n}\left\langle A_j^p \log A_j^p x_j, x_j\right\rangle\right)}{\exp\left[\sum_{j=1}^{n}\left\langle A_j^p x_j, x_j\right\rangle \sum_{j=1}^{n}\left\langle \log A_j x_j, x_j\right\rangle\right]} \\
&= \frac{\left(\prod_{j=1}^{n}\exp\left\langle A_j^p \log A_j^p x_j, x_j\right\rangle\right)^{1/p}}{\left[\exp\left(\prod_{j=1}^{n}\left\langle \log A_j x_j, x_j\right\rangle\right)\right]^{\sum_{j=1}^{n}\left\langle A_j^p x_j, x_j\right\rangle}}
\end{aligned}
$$

$$
= \frac{\left(\prod_{j=1}^{n}\breve{\eta}_{\frac{x_j}{\|x_j\|}}\left(A_j^p\right)\right)^{-1/p}}{\left[\prod_{j=1}^{n}\breve{\Delta}_{\frac{x_j}{\|x_j\|}}\left(A_j\right)\right]^{\sum_{j=1}^{n}\left\langle A_j^p x_j, x_j\right\rangle}} = \frac{\left(\prod_{j=1}^{n}\left[\eta_{\frac{x_j}{\|x_j\|}}\left(A_j^p\right)\right]^{\|x_j\|^2}\right)^{-1/p}}{\left[\prod_{j=1}^{n}\left(\Delta_{\frac{x_j}{\|x_j\|}}\left(A_j\right)\right)^{\|x_j\|^2}\right]^{\sum_{j=1}^{n}\left\langle A_j^p x_j, x_j\right\rangle}}
$$

and

$$
\exp\left[\log\left(\sum_{j=1}^{n}\left\langle A_j x_j, x_j\right\rangle\right) - \sum_{j=1}^{n}\log\left\langle A_j x_j, x_j\right\rangle\right]
$$

$$
= \frac{\exp\log\left(\sum_{j=1}^{n}\left\langle A_j x_j, x_j\right\rangle\right)}{\exp\left(\sum_{j=1}^{n}\log\left\langle A_j x_j, x_j\right\rangle\right)} = \frac{\sum_{j=1}^{n}\left\langle A_j x_j, x_j\right\rangle}{\prod_{j=1}^{n}\breve{\Delta}_{\frac{x_j}{\|x_j\|}}(A_j)}
$$

$$
= \frac{\sum_{j=1}^{n}\left\langle A_j x_j, x_j\right\rangle}{\prod_{j=1}^{n}\left(\Delta_{\frac{x_j}{\|x_j\|}}(A_j)\right)^{\|x_j\|^2}}
$$

for each $x_j \in H$, $j \in \{1, ..., n\}$ with $\sum_{j=1}^{n}\|x_j\|^2 = 1$ and the inequality (3.1) is thus proved. □

Corollary 3. *Let $A_j > 0$ and $p_j \geq 0$, $j \in \{1, ..., n\}$ with $\sum_{j=1}^{n} p_j = 1$. Then*

$$
\frac{\left(\prod_{j=1}^{n}\left[\eta_x\left(A_j^p\right)\right]^{p_j}\right)^{-1/p}}{\left[\prod_{j=1}^{n}\left[\Delta_x(A_j)\right]^{p_j}\right]^{\left\langle\sum_{j=1}^{n} p_j A_j^p x, x\right\rangle}} \tag{3.4}
$$

$$
\geq \left(\frac{\left\langle\sum_{j=1}^{n} p_j A_j x, x\right\rangle}{\prod_{j=1}^{n}\left[\Delta_x(A_j)\right]^{p_j}}\right)^{\left\langle\sum_{j=1}^{n} p_j A_j^p x, x\right\rangle - \left\langle\sum_{j=1}^{n} p_j A_j x, x\right\rangle^p} \geq 1
$$

for all $x \in H$ with $\|x\| = 1$.

Remark 3. *For $p = 1$ we get*

$$
\frac{\left(\prod_{j=1}^{n}\left[\eta_x\left(A_j\right)\right]^{p_j}\right)^{-1}}{\left[\prod_{j=1}^{n}\left[\Delta_x(A_j)\right]^{p_j}\right]^{\left\langle\sum_{j=1}^{n} p_j A_j x, x\right\rangle}} \geq 1, \tag{3.5}
$$

while for $n = 1$ we obtain

$$
\frac{\left[\eta_x\left(A^p\right)\right]^{-1/p}}{\left[\Delta_x(A)\right]^{\langle A^p x, x\rangle}} \geq \left(\frac{\langle Ax, x\rangle}{\Delta_x(A)}\right)^{\langle A^p x, x\rangle - \langle Ax, x\rangle^p} \geq 1 \tag{3.6}
$$

provided that $A > 0$, $x \in H$ with $\|x\| = 1$ and $p \geq 1$.

For $p = 1$ in (3.6) we recapture (2.4), while for $p = 2$ in (3.6) we derive

$$
\frac{\left[\eta_x\left(A^2\right)\right]^{-1/2}}{\left[\Delta_x(A)\right]^{\langle A^2 x, x\rangle}} \geq \left(\frac{\langle Ax, x\rangle}{\Delta_x(A)}\right)^{\langle A^2 x, x\rangle - \langle Ax, x\rangle^2} \geq 1 \tag{3.7}
$$

for all $x \in H$ with $\|x\| = 1$.

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[1]Mathematics, College of Engineering & Science, Victoria University, PO Box 14428, Melbourne City, MC 8001, Australia.

E-mail address: sever.dragomir@vu.edu.au

URL: http://rgmia.org/dragomir

[2]DST-NRF Centre of Excellence in the Mathematical, and Statistical Sciences, School of Computer Science, & Applied Mathematics, University of the Witwatersrand,, Private Bag 3, Johannesburg 2050, South Africa