# REFINEMENTS AND REVERSES OF SOME INEQUALITIES FOR THE NORMALIZED ENTROPIC DETERMINANT OF POSITIVE OPERATORS IN HILBERT SPACES

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Abstract. For positive invertible operators $A$ on a Hilbert space $H$ and a fixed unit vector $x \in H$, define the normalized entropic determinant by $\eta_x(A) := \exp[-\langle A \ln Ax, x\rangle]$. In this paper we show among others that, if $A$ satisfies the condition $0 < m \le A \le M$, then

$$1 \le \exp\left\{\frac{1}{2M^2}\left(\langle A^2x, x\rangle - \langle Ax, x\rangle^2\right)\right\}$$
$$\le \frac{\langle Ax, x\rangle^{-\langle Ax, x\rangle}}{\eta_x(A)} \le \exp\left\{\frac{1}{2m^2}\left(\langle A^2x, x\rangle - \langle Ax, x\rangle^2\right)\right\}$$

and

$$1 \le \exp\left\{\frac{m}{2}\left(\langle A^2x, x\rangle - \langle Ax, x\rangle^2\right)\right\}$$
$$\le \frac{\eta_x(A)}{\left(\frac{\langle Ax, x\rangle}{\langle A^2x, x\rangle}\right)^{\langle Ax, x\rangle}} \le \exp\left\{\frac{M}{2}\left(\langle A^2x, x\rangle - \langle Ax, x\rangle^2\right)\right\}$$

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## 1. Introduction

Let $B(H)$ be the space of all bounded linear operators on a Hilbert space $H$, and $I$ stands for the identity operator on $H$. An operator $A$ in $B(H)$ is said to be positive (in symbol: $A \ge 0$) if $\langle Ax, x\rangle \ge 0$ for all $x \in H$. In particular, $A > 0$ means that $A$ is positive and invertible. For a pair $A$, $B$ of selfadjoint operators the order relation $A \ge B$ means as usual that $A - B$ is positive.

In 1998, Fujii et al. [6], [7], introduced the *normalized determinant* $\Delta_x(A)$ for positive invertible operators $A$ on a Hilbert space $H$ and a fixed unit vector $x \in H$, namely $\|x\| = 1$, defined by $\Delta_x(A) := \exp\langle \ln Ax, x\rangle$ and discussed it as a continuous geometric mean and observed some inequalities around the determinant from this point of view.

Some of the fundamental properties of normalized determinant are as follows, [6].

For each unit vector $x \in H$, see also [8], we have:

(i) *continuity*: the map $A \to \Delta_x(A)$ is norm continuous;
(ii) *bounds*: $\langle A^{-1}x, x\rangle^{-1} \le \Delta_x(A) \le \langle Ax, x\rangle$;
(iii) *continuous mean*: $\langle A^p x, x\rangle^{1/p} \downarrow \Delta_x(A)$ for $p \downarrow 0$ and $\langle A^p x, x\rangle^{1/p} \uparrow \Delta_x(A)$ for $p \uparrow 0$;

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(iv) *power equality:* $\Delta_x(A^t) = \Delta_x(A)^t$ for all $t > 0$;
(v) *homogeneity*: $\Delta_x(tA) = t\Delta_x(A)$ and $\Delta_x(tI) = t$ for all $t > 0$;
(vi) *monotonicity*: $0 < A \leq B$ implies $\Delta_x(A) \leq \Delta_x(B)$;
(vii) *multiplicativity*: $\Delta_x(AB) = \Delta_x(A)\Delta_x(B)$ for commuting $A$ and $B$;
(viii) *Ky Fan type inequality*: $\Delta_x((1-\alpha)A + \alpha B) \geq \Delta_x(A)^{1-\alpha}\Delta_x(B)^{\alpha}$ for $0 < \alpha < 1$.

We define the *logarithmic mean* of two positive numbers $a$, $b$ by

$$L(a,b) := \begin{cases} \frac{b-a}{\ln b - \ln a} \text{ if } b \neq a, \\ \\ a \text{ if } b = a. \end{cases}$$

In [6] the authors obtained the following additive reverse inequality for the operator $A$ which satisfy the condition $0 < m \leq A \leq M$, where $m, M$ are positive numbers,

$$0 \leq \langle Ax, x\rangle - \Delta_x(A) \leq L(m,M)\left[\ln L(m,M) + \frac{M\ln m - m\ln M}{M-m} - 1\right] \tag{1.1}$$

for all $x \in H$, $\|x\| = 1$.

The famous *Young inequality* for scalars says that if $a, b > 0$ and $\nu \in [0,1]$, then

$$a^{1-\nu}b^{\nu} \leq (1-\nu)a + \nu b \tag{1.2}$$

with equality if and only if $a = b$. The inequality (1.2) is also called *$\nu$-weighted arithmetic-geometric mean inequality.*

We recall that *Specht's ratio* is defined by [12]

$$S(h) := \begin{cases} \frac{h^{\frac{1}{h-1}}}{e\ln\left(h^{\frac{1}{h-1}}\right)} \text{ if } h \in (0,1)\cup(1,\infty) \\ \\ 1 \text{ if } h = 1. \end{cases} \tag{1.3}$$

It is well known that $\lim_{h\to 1} S(h) = 1$, $S(h) = S\left(\frac{1}{h}\right) > 1$ for $h > 0$, $h \neq 1$. The function is decreasing on $(0,1)$ and increasing on $(1,\infty)$.

In [7], the authors obtained the following multiplicative reverse inequality as well

$$1 \leq \frac{\langle Ax, x\rangle}{\Delta_x(A)} \leq S\left(\frac{M}{m}\right) \tag{1.4}$$

for $0 < mI \leq A \leq MI$ and $x \in H$, $\|x\| = 1$.

Since $0 < M^{-1}I \leq A^{-1} \leq m^{-1}I$, then by (1.4) for $A^{-1}$ we get

$$1 \leq \frac{\langle A^{-1}x, x\rangle}{\Delta_x(A^{-1})} \leq S\left(\frac{m^{-1}}{M^{-1}}\right) = S\left(\left(\frac{m}{M}\right)^{-1}\right) = S\left(\frac{M}{m}\right),$$

which is equivalent to

$$1 \leq \frac{\Delta_x(A)}{\langle A^{-1}x, x\rangle^{-1}} \leq S\left(\frac{M}{m}\right) \tag{1.5}$$

for $x \in H$, $\|x\| = 1$.

For the entropy function $\eta(t) = -t\ln t$, $t > 0$, the operator entropy has the following expression:

$$\eta(A) = -A\ln A$$

for positive $A$.

For $x\in H$, $\|x\|=1$, we define the *normalized entropic determinant* $\eta_x(A)$ by

$$\eta_x(A):=\exp\left(-\langle A\ln Ax,x\rangle\right)=\exp\langle\eta(A)x,x\rangle. \tag{1.6}$$

Let $x\in H$, $\|x\|=1$. Observe that the map $A\to\eta_x(A)$ is *norm continuous* and since

$$\begin{aligned}&\exp\left(-\langle tA\ln(tA)x,x\rangle\right)\\&=\exp\left(-\langle tA(\ln t+\ln A)x,x\rangle\right)=\exp\left(-\langle(tA\ln t+tA\ln A)x,x\rangle\right)\\&=\exp\left(-\langle Ax,x\rangle t\ln t\right)\exp\left(-t\langle A\ln Ax,x\rangle\right)\\&=\exp\ln\left(t^{-\langle Ax,x\rangle t}\right)\left[\exp\left(-\langle A\ln Ax,x\rangle\right)\right]^{-t},\end{aligned}$$

hence

$$\eta_x(tA)=t^{-t\langle Ax,x\rangle}\left[\eta_x(A)\right]^{-t} \tag{1.7}$$

for $t>0$ and $A>0$.

Observe also that

$$\eta_x(I)=1\text{ and }\eta_x(tI)=t^{-t} \tag{1.8}$$

for $t>0$.

In the recent paper [4] we obtained among others that, if $A$, $B>0$, then for all $x\in H,\|x\|=1$ and $t\in[0,1]$,

$$\eta_x((1-t)A+tB)\geq(\eta_x(A))^{1-t}(\eta_x(B))^{t}. \tag{1.9}$$

Also we have the bounds

$$\left(\frac{\langle A^2x,x\rangle}{\langle Ax,x\rangle}\right)^{-\langle Ax,x\rangle}\leq\eta_x(A)\leq\langle Ax,x\rangle^{-\langle Ax,x\rangle}, \tag{1.10}$$

where $A>0$ and $x\in H,\|x\|=1$.

Motivated by the above results, in this paper we show among others that, if $A$ satisfies the condition $0<m\leq A\leq M$, then

$$\begin{aligned}1&\leq\exp\left\{\frac{1}{2M^2}\left(\langle A^2x,x\rangle-\langle Ax,x\rangle^2\right)\right\}\\&\leq\frac{\langle Ax,x\rangle^{-\langle Ax,x\rangle}}{\eta_x(A)}\leq\exp\left\{\frac{1}{2m^2}\left(\langle A^2x,x\rangle-\langle Ax,x\rangle^2\right)\right\}\end{aligned}$$

and

$$\begin{aligned}1&\leq\exp\left\{\frac{m}{2}\left(\langle A^2x,x\rangle-\langle Ax,x\rangle^2\right)\right\}\\&\leq\frac{\eta_x(A)}{\left(\frac{\langle Ax,x\rangle}{\langle A^2x,x\rangle}\right)^{\langle Ax,x\rangle}}\leq\exp\left\{\frac{M}{2}\left(\langle A^2x,x\rangle-\langle Ax,x\rangle^2\right)\right\}\end{aligned}$$

for $x\in H$, $\|x\|=1$.

## 2. Inequalities for $p\in(-\infty,0)\cup(1,\infty)$

Assume that $A>0$. For a vector $y\neq 0$ we can extend the normalized entropic determinant as $\tilde{\eta}_y(A):=\exp\langle \ln Ay,y\rangle$ . We observe that

$$\begin{aligned}\tilde{\eta}_y(A)&:=\exp\langle -A\ln Ay,y\rangle=\exp\left(\|y\|^2\left\langle -A\ln A\frac{y}{\|y\|},\frac{y}{\|y\|}\right\rangle\right)\\&=\left[\exp\left(\left\langle -A\ln A\frac{y}{\|y\|},\frac{y}{\|y\|}\right\rangle\right)\right]^{\|y\|^2}=\left[\eta_{\frac{y}{\|y\|}}(A)\right]^{\|y\|^2}\end{aligned}$$

for any $y\neq 0$.

**Theorem 1.** *Assume that $A_j$ are operators such that $0<m\leq A_j\leq M$, $j\in\{1,...,n\}$. Define*

$$\phi_p(m,M):=\begin{cases}\frac{M^{1-p}}{p(p-1)} & \text{for } p\in(1,\infty),\\[2ex] \frac{m^{1-p}}{p(p-1)} & \text{for } p\in(-\infty,0)\end{cases}$$

*and*

$$\Phi_p(m,M):=\begin{cases}\frac{m^{1-p}}{p(p-1)} & \text{for } p\in(1,\infty),\\[2ex] \frac{M^{1-p}}{p(p-1)} & \text{for } p\in(-\infty,0).\end{cases}$$

*Then*

$$\begin{aligned}1&\leq\exp\left(\phi_p(m,M)\left[\sum_{j=1}^n\left\langle A_j^px_j,x_j\right\rangle-\left(\sum_{j=1}^n\langle A_jx_j,x_j\rangle\right)^p\right]\right)\\&\leq\frac{\left(\sum_{j=1}^n\langle A_jx_j,x_j\rangle\right)^{-\left(\sum_{j=1}^n\langle A_jx_j,x_j\rangle\right)}}{\prod_{j=1}^n\tilde{\eta}_{x_j}(A_j)}\\&\leq\exp\left(\Phi_p(m,M)\left[\sum_{j=1}^n\left\langle A_j^px_j,x_j\right\rangle-\left(\sum_{j=1}^n\langle A_jx_j,x_j\rangle\right)^p\right]\right),\end{aligned}\tag{2.1}$$

*for each $x_j\in H$, $j\in\{1,...,n\}$ with $\sum_{j=1}^n\|x_j\|^2=1$.*

*Proof.* Let $A_j$ be positive definite operators with $\operatorname{Sp}(A_j)\subseteq[m,M]\subset(0,\infty)$ $j\in\{1,...,n\}$. If $f$ is a twice differentiable function on $(m,M)$ and for $p\in(-\infty,0)\cup(1,\infty)$ we have for some $\phi<\Phi$ that

$$\phi\leq g(t):=\frac{t^{2-p}}{p(p-1)}f''(t)\leq\Phi \text{ for any } t\in(m,M),\tag{2.2}$$

then, see [3],

$$\phi\left[\sum_{j=1}^n\left\langle A_j^px_j,x_j\right\rangle-\left(\sum_{j=1}^n\langle A_jx_j,x_j\rangle\right)^p\right]\tag{2.3}$$

$$\leq \sum_{j=1}^{n} \langle f(A_j) x_j, x_j \rangle - f\left(\sum_{j=1}^{n} \langle A_j x_j, x_j \rangle\right)$$
$$\leq \Phi\left[\sum_{j=1}^{n} \langle A_j^p x_j, x_j \rangle - \left(\sum_{j=1}^{n} \langle A_j x_j, x_j \rangle\right)^p\right]$$

for each $x_j \in H$, $j \in \{1, ..., n\}$ with $\sum_{j=1}^{n} \|x_j\|^2 = 1$.

We consider the convex function $f(t) = t \ln t$, $t \in [m, M] \subset (0, \infty)$. Then

$$g(t) = \frac{t^{2-p}}{p(p-1)} \frac{1}{t} = \frac{1}{p(p-1)} t^{1-p}.$$

For $p \in (1, \infty)$, we have

$$\sup_{t\in[m,M]} g(t) = \frac{m^{1-p}}{p(p-1)} \text{ and } \inf_{t\in[m,M]} g(t) = \frac{M^{1-p}}{p(p-1)}$$

and for $p \in (-\infty, 0)$

$$\sup_{t\in[m,M]} g(t) = \sup_{t\in[m,M]} \frac{t^{1-p}}{p(p-1)} = \frac{M^{1-p}}{p(p-1)}$$

and

$$\inf_{t\in[m,M]} g(t) = \inf_{t\in[m,M]} \frac{t^{1-p}}{p(p-1)} = \frac{m^{1-p}}{p(p-1)}.$$

Therefore by (2.3) we get

$$0 \leq \phi_p(m, M)\left[\sum_{j=1}^{n} \langle A_j^p x_j, x_j \rangle - \left(\sum_{j=1}^{n} \langle A_j x_j, x_j \rangle\right)^p\right] \tag{2.4}$$
$$\leq \sum_{j=1}^{n} \langle A_j \ln A_j x_j, x_j \rangle - \left(\sum_{j=1}^{n} \langle A_j x_j, x_j \rangle\right) \ln\left(\sum_{j=1}^{n} \langle A_j x_j, x_j \rangle\right)$$
$$\leq \Phi_p(m, M)\left[\sum_{j=1}^{n} \langle A_j^p x_j, x_j \rangle - \left(\sum_{j=1}^{n} \langle A_j x_j, x_j \rangle\right)^p\right],$$

where $\phi_p(m, M)$ and $\Phi_p(m, M)$ are given above.

If we take the exponential in (2.4), then we obtain

$$1 \leq \exp\left(\phi_p(m, M)\left[\sum_{j=1}^{n} \langle A_j^p x_j, x_j \rangle - \left(\sum_{j=1}^{n} \langle A_j x_j, x_j \rangle\right)^p\right]\right) \tag{2.5}$$
$$\leq \frac{\exp\left[-\left(\sum_{j=1}^{n} \langle A_j x_j, x_j \rangle\right) \ln\left(\sum_{j=1}^{n} \langle A_j x_j, x_j \rangle\right)\right]}{\exp\left(\sum_{j=1}^{n} \langle -A_j \ln A_j x_j, x_j \rangle\right)}$$
$$\leq \exp\left(\Phi_p(m, M)\left[\sum_{j=1}^{n} \langle A_j^p x_j, x_j \rangle - \left(\sum_{j=1}^{n} \langle A_j x_j, x_j \rangle\right)^p\right]\right),$$

for each $x_j \in H$, $j \in \{1, ..., n\}$ with $\sum_{j=1}^{n} \|x_j\|^2 = 1$.

Since

$$\exp\left(\sum_{j=1}^{n}\langle -A_j \ln A_j x_j, x_j\rangle\right)=\prod_{j=1}^{n}\exp\langle -A_j \ln A_j x_j, x_j\rangle=\prod_{j=1}^{n}\tilde{\eta}_{x_j}(A_j),$$

hence by (2.5) we derive (2.1). □

**Remark 1.** *Assume that $A_j$ are operators such that $0<m\le A_j\le M$, $j\in\{1,...,n\}$. If we take $p=2$ in (2.1), then we get*

$$1\le\exp\left(\frac{1}{2M^2}\left[\sum_{j=1}^{n}\left\langle A_j^2 x_j, x_j\right\rangle-\left(\sum_{j=1}^{n}\langle A_j x_j, x_j\rangle\right)^2\right]\right) \tag{2.6}$$
$$\le\frac{\left(\sum_{j=1}^{n}\langle A_j x_j, x_j\rangle\right)^{-\left(\sum_{j=1}^{n}\langle A_j x_j, x_j\rangle\right)}}{\prod_{j=1}^{n}\tilde{\eta}_{x_j}(A_j)}$$
$$\le\exp\left(\frac{1}{2m^2}\left[\sum_{j=1}^{n}\left\langle A_j^2 x_j, x_j\right\rangle-\left(\sum_{j=1}^{n}\langle A_j x_j, x_j\rangle\right)^2\right]\right)$$

*for each $x_j\in H$, $j\in\{1,...,n\}$ with $\sum_{j=1}^{n}\|x_j\|^2=1$.*

*If we take $p=-1$ in (2.1), then we get*

$$1\le\exp\left(\frac{m}{2}\left[\sum_{j=1}^{n}\left\langle A_j^{-1} x_j, x_j\right\rangle-\left(\sum_{j=1}^{n}\langle A_j x_j, x_j\rangle\right)^{-1}\right]\right) \tag{2.7}$$
$$\le\frac{\left(\sum_{j=1}^{n}\langle A_j x_j, x_j\rangle\right)^{-\left(\sum_{j=1}^{n}\langle A_j x_j, x_j\rangle\right)}}{\prod_{j=1}^{n}\tilde{\eta}_{x_j}(A_j)}$$
$$\le\exp\left(\frac{M}{2}\left[\sum_{j=1}^{n}\left\langle A_j^{-1} x_j, x_j\right\rangle-\left(\sum_{j=1}^{n}\langle A_j x_j, x_j\rangle\right)^{-1}\right]\right)$$

*for each $x_j\in H$, $j\in\{1,...,n\}$ with $\sum_{j=1}^{n}\|x_j\|^2=1$.*

The case of normalized determinant is as follows:

**Corollary 1.** *Assume that $p_j \geq 0$ with $\sum_{j=1}^n p_j = 1$, then*

$$1 \leq \exp\left(\phi_p(m,M)\left[\sum_{j=1}^n p_j \left\langle A_j^p x, x\right\rangle - \left(\sum_{j=1}^n p_j \left\langle A_j x, x\right\rangle\right)^p\right]\right) \tag{2.8}$$
$$\leq \frac{\left\langle \sum_{j=1}^n p_j A_j x, x\right\rangle^{-\left\langle \sum_{j=1}^n p_j A_j x, x\right\rangle}}{\prod_{j=1}^n [\eta_x(A_j)]^{p_j}}$$
$$\leq \exp\left(\Phi_p(m,M)\left[\sum_{j=1}^n p_j \left\langle A_j^p x, x\right\rangle - \left(\sum_{j=1}^n p_j \left\langle A_j x, x\right\rangle\right)^p\right]\right),$$

*for $x \in H$, $\|x\| = 1$.*

The proof follows from (2.1) by taking $x_j = \sqrt{p_j}x$, $x \in H$, $\|x\| = 1$ and observing that

$$\prod_{j=1}^n \eta_{x_j}(A_j) = \prod_{j=1}^n \exp\left\langle -A_j \ln A_j \sqrt{p_j}x, \sqrt{p_j}x\right\rangle = \prod_{j=1}^n \exp\left[p_j \left\langle -A_j \ln A_j x, x\right\rangle\right]$$
$$= \prod_{j=1}^n \left[\exp\left\langle -A_j \ln A_j x, x\right\rangle\right]^{p_j} = \prod_{j=1}^n \left[\eta_x(A_j)\right]^{p_j}.$$

If we take $p = 2$ in (2.8), then we get

$$1 \leq \exp\left(\frac{1}{2M^2}\left[\sum_{j=1}^n p_j \left\langle A_j^2 x, x\right\rangle - \left(\sum_{j=1}^n \left\langle A_j x, x\right\rangle\right)^2\right]\right) \tag{2.9}$$
$$\leq \frac{\left\langle \sum_{j=1}^n p_j A_j x, x\right\rangle^{-\left\langle \sum_{j=1}^n p_j A_j x, x\right\rangle}}{\prod_{j=1}^n [\eta_x(A_j)]^{p_j}}$$
$$\leq \exp\left(\frac{1}{2m^2}\left[\sum_{j=1}^n p_j \left\langle A_j^2 x, x\right\rangle - \left(\sum_{j=1}^n \left\langle A_j x, x\right\rangle\right)^2\right]\right)$$

for each $x \in H$, $\|x\| = 1$.

If we take $p=-1$ in (2.8), then we get

$$1\leq \exp\left(\frac{m}{2}\left[\sum_{j=1}^{n}p_j\left\langle A_j^{-1}x,x\right\rangle-\left(\sum_{j=1}^{n}p_j\left\langle A_jx,x\right\rangle\right)^{-1}\right]\right) \tag{2.10}$$
$$\leq \frac{\left\langle \sum_{j=1}^{n}p_jA_jx,x\right\rangle^{-\left\langle \sum_{j=1}^{n}p_jA_jx,x\right\rangle}}{\prod_{j=1}^{n}\left[\eta_x(A_j)\right]^{p_j}}$$
$$\leq \exp\left(\frac{M}{2}\left[\sum_{j=1}^{n}p_j\left\langle A_j^{-1}x,x\right\rangle-\left(\sum_{j=1}^{n}p_j\left\langle A_jx,x\right\rangle\right)^{-1}\right]\right)$$

for $x\in H$, $\|x\|=1$.

The case of two operators is as follows. Assume that $0<m\leq A$, $B\leq M$, and $t\in[0,1]$. Then

$$1\leq \exp\left\{\phi_p(m,M)\right.$$
$$\left.\times\left[\left\langle\left[(1-t)A^p+tB^p\right]x,x\right\rangle-\left(\left\langle\left[(1-t)A+tB\right]x,x\right\rangle\right)^p\right]\right\} \tag{2.11}$$
$$\leq \frac{\left\langle\left[(1-t)A+tB\right]x,x\right\rangle^{-\left\langle\left[(1-t)A+tB\right]x,x\right\rangle}}{\left[\eta_x(A)\right]^{(1-t)}\left[\eta_x(B)\right]^t}$$
$$\leq \exp\left\{\Phi_p(m,M)\right.$$
$$\left.\times\left[\left\langle\left[(1-t)A^p+tB^p\right]x,x\right\rangle-\left(\left\langle\left[(1-t)A+tB\right]x,x\right\rangle\right)^p\right]\right\}$$

for $x\in H$, $\|x\|=1$.

If $B=A$, then we derive

$$1\leq \exp\left\{\phi_p(m,M)\left[\left\langle A^px,x\right\rangle-\left\langle Ax,x\right\rangle^p\right]\right\} \tag{2.12}$$
$$\leq \frac{\left\langle Ax,x\right\rangle^{-\left\langle Ax,x\right\rangle}}{\eta_x(A)}\leq \exp\left\{\Phi_p(m,M)\left[\left\langle A^px,x\right\rangle-\left\langle Ax,x\right\rangle^p\right]\right\}$$

for $x\in H$, $\|x\|=1$.

For $p=2$ we have

$$1\leq \exp\left\{\frac{1}{2M^2}\left[\left\langle A^2x,x\right\rangle-\left\langle Ax,x\right\rangle^2\right]\right\} \tag{2.13}$$
$$\leq \frac{\left\langle Ax,x\right\rangle^{-\left\langle Ax,x\right\rangle}}{\eta_x(A)}\leq \exp\left\{\frac{1}{2m^2}\left[\left\langle A^2x,x\right\rangle-\left\langle Ax,x\right\rangle^2\right]\right\}$$

for $x\in H$, $\|x\|=1$.

For $p=-1$ we obtain

$$1\leq \exp\left\{\frac{m}{2}\left[\left\langle A^{-1}x,x\right\rangle-\left\langle Ax,x\right\rangle^{-1}\right]\right\} \tag{2.14}$$
$$\leq \frac{\left\langle Ax,x\right\rangle^{-\left\langle Ax,x\right\rangle}}{\eta_x(A)}\leq \exp\left\{\frac{M}{2}\left[\left\langle A^{-1}x,x\right\rangle-\left\langle Ax,x\right\rangle^{-1}\right]\right\}$$

for $x\in H$, $\|x\|=1$.

We observe that the above inequalities (2.12)-(2.14) provide refinements and reverse of the fundamental bounds for the normalized determinant incorporated in the second part of (1.10) from the introduction.

It is well known that, see for instance [5, p. 28],

$$\left\langle A^2x, x\right\rangle - \left\langle Ax, x\right\rangle^2 \leq \frac{1}{4}(M-m)^2 \tag{2.15}$$

for $x \in H$, $\|x\| = 1$.

Then by (2.13) we get

$$\begin{aligned}\frac{\langle Ax, x\rangle^{-\langle Ax,x\rangle}}{\eta_x(A)} &\leq \exp\left\{\frac{1}{2m^2}\left[\left\langle A^2x, x\right\rangle - \left\langle Ax, x\right\rangle^2\right]\right\} \\ &\leq \exp\left\{\frac{1}{8}\left(\frac{M}{m}-1\right)^2\right\}\end{aligned} \tag{2.16}$$

for $x \in H$, $\|x\| = 1$.

We also use the well known inequality, see for instance [5, p. 28],

$$\left\langle A^{-1}x, x\right\rangle - \left\langle Ax, x\right\rangle^{-1} \leq \frac{\left(\sqrt{M}-\sqrt{m}\right)^2}{mM} \tag{2.17}$$

for $x \in H$, $\|x\| = 1$.

Then by (2.14) we obtain

$$\begin{aligned}\frac{\langle Ax, x\rangle^{-\langle Ax,x\rangle}}{\eta_x(A)} &\leq \exp\left\{\frac{M}{2}\left[\left\langle A^{-1}x, x\right\rangle - \left\langle Ax, x\right\rangle^{-1}\right]\right\} \\ &\leq \exp\left\{\frac{1}{2}\left(\sqrt{\frac{M}{m}}-1\right)^2\right\}\end{aligned} \tag{2.18}$$

for $x \in H$, $\|x\| = 1$.

## 3. Inequalities for $p \in (0,1)$

We also have:

**Theorem 2.** *Assume that $A_j$ are operators such that $0 < m \leq A_j \leq M$, $j \in \{1, ..., n\}$. Then for $p \in (0,1)$*

$$\begin{aligned}1 &\leq \exp\left(\frac{1}{p(1-p)M^p}\left[\left(\sum_{j=1}^n \langle A_j x_j, x_j\rangle\right)^p - \sum_{j=1}^n \left\langle A_j^p x_j, x_j\right\rangle\right]\right) \\ &\leq \frac{\left(\sum_{j=1}^n \langle A_j x_j, x_j\rangle\right)^{-\left(\sum_{j=1}^n \langle A_j x_j, x_j\rangle\right)}}{\prod_{j=1}^n \tilde{\eta}_{x_j}(A_j)} \\ &\leq \left(\frac{1}{p(1-p)m^p}\left[\left(\sum_{j=1}^n \langle A_j x_j, x_j\rangle\right)^p - \sum_{j=1}^n \left\langle A_j^p x_j, x_j\right\rangle\right]\right)\end{aligned} \tag{3.1}$$

*for each $x_j \in H$, $j \in \{1, ..., n\}$ with $\sum_{j=1}^n \|x_j\|^2 = 1$.*

*In particular,*

$$1 \le \exp\left(\frac{4}{M^{1/2}}\left[\left(\sum_{j=1}^{n}\langle A_j x_j, x_j\rangle\right)^{1/2} - \sum_{j=1}^{n}\left\langle A_j^{1/2} x_j, x_j\right\rangle\right]\right) \tag{3.2}$$
$$\le \frac{\left(\sum_{j=1}^{n}\langle A_j x_j, x_j\rangle\right)^{-\left(\sum_{j=1}^{n}\langle A_j x_j, x_j\rangle\right)}}{\prod_{j=1}^{n}\tilde{\eta}_{x_j}(A_j)}$$
$$\le \exp\left(\frac{4}{m^{1/2}}\left[\left(\sum_{j=1}^{n}\langle A_j x_j, x_j\rangle\right)^{1/2} - \sum_{j=1}^{n}\left\langle A_j^{1/2} x_j, x_j\right\rangle\right]\right)$$

*for each* $x_j \in H,\ j \in \{1, ..., n\}$ *with* $\sum_{j=1}^{n}\|x_j\|^2 = 1.$

*Proof.* If the following condition is satisfied

$$\delta \le h(t) := \frac{t^{2-p}}{p(1-p)} f''(t) \le \Delta \text{ for any } t \in (m, M) \tag{3.3}$$

and for some $\delta < \Delta$, where $p \in (0,1)$, then for $p \in (0,1)$, we also have [3]

$$\delta\left[\left(\sum_{j=1}^{n}\langle A_j x_j, x_j\rangle\right)^{p} - \sum_{j=1}^{n}\left\langle A_j^{p} x_j, x_j\right\rangle\right] \tag{3.4}$$
$$\le \sum_{j=1}^{n}\langle f(A_j) x_j, x_j\rangle - f\left(\sum_{j=1}^{n}\langle A_j x_j, x_j\rangle\right)$$
$$\le \Delta\left[\left(\sum_{j=1}^{n}\langle A_j x_j, x_j\rangle\right)^{p} - \sum_{j=1}^{n}\left\langle A_j^{p} x_j, x_j\right\rangle\right]$$

for each $x_j \in H,\ j \in \{1, ..., n\}$ with $\sum_{j=1}^{n}\|x_j\|^2 = 1.$

If we take $f(t) = t\ln t$, then

$$\begin{aligned} h(t) &= \frac{t^{2-p}}{p(1-p)}\frac{1}{t} \\ &= \frac{1}{p(1-p)t^{1-p}} \in \left[\frac{1}{p(1-p)M^{1-p}}, \frac{1}{p(1-p)m^{1-p}}\right] \end{aligned}$$

and by (3.4) we get

$$\begin{aligned}
0 &\leq \frac{1}{p(1-p)M^{1-p}}\left[\left(\sum_{j=1}^{n}\langle A_j x_j, x_j\rangle\right)^{p} - \sum_{j=1}^{n}\left\langle A_j^{p} x_j, x_j\right\rangle\right] \\
&\leq \sum_{j=1}^{n}\langle A_j \ln A_j x_j, x_j\rangle - \left(\sum_{j=1}^{n}\langle A_j x_j, x_j\rangle\right)\ln\left(\sum_{j=1}^{n}\langle A_j x_j, x_j\rangle\right) \\
&\leq \frac{1}{p(1-p)m^{1-p}}\left[\left(\sum_{j=1}^{n}\langle A_j x_j, x_j\rangle\right)^{p} - \sum_{j=1}^{n}\left\langle A_j^{p} x_j, x_j\right\rangle\right]
\end{aligned}$$

for each $x_j \in H$, $j \in \{1, ..., n\}$ with $\sum_{j=1}^{n}\|x_j\|^2 = 1$, which implies, by taking the exponential, the desired result (3.1). □

**Corollary 2.** *Assume that* $A_j$ *are operators such that* $0 < m \leq A_j \leq M$, $j \in \{1, ..., n\}$ *and* $p_j \geq 0$, $j \in \{1, ..., n\}$ *with* $\sum_{j=1}^{n} p_j = 1$*. Then for* $p \in (0, 1)$

$$\begin{aligned}
(3.5) \qquad 1 &\leq \exp\left(\frac{1}{p(1-p)M^{1-p}}\left[\left(\sum_{j=1}^{n} p_j\langle A_j x, x\rangle\right)^{p} - \sum_{j=1}^{n} p_j\left\langle A_j^{p} x, x\right\rangle\right]\right) \\
&\leq \frac{\left\langle\sum_{j=1}^{n} p_j A_j x, x\right\rangle^{-\left\langle\sum_{j=1}^{n} p_j A_j x, x\right\rangle}}{\prod_{j=1}^{n}[\eta_x(A_j)]^{p_j}} \\
&\leq \exp\left(\frac{1}{p(1-p)m^{1-p}}\left[\left(\sum_{j=1}^{n} p_j\langle A_j x, x\rangle\right)^{p} - \sum_{j=1}^{n} p_j\left\langle A_j^{p} x, x\right\rangle\right]\right)
\end{aligned}$$

*for each* $x \in H$ *with* $\|x\| = 1$*.*

*In particular,*

$$\begin{aligned}
(3.6) \qquad 1 &\leq \exp\left(\frac{4}{M^{1/2}}\left[\left(\sum_{j=1}^{n} p_j\langle A_j x, x\rangle\right)^{1/2} - \sum_{j=1}^{n} p_j\left\langle A_j^{1/2} x, x\right\rangle\right]\right) \\
&\leq \frac{\left\langle\sum_{j=1}^{n} p_j A_j x, x\right\rangle^{-\left\langle\sum_{j=1}^{n} p_j A_j x, x\right\rangle}}{\prod_{j=1}^{n}[\eta_x(A_j)]^{p_j}} \\
&\leq \exp\left(\frac{4}{m^{1/2}}\left[\left(\sum_{j=1}^{n} p_j\langle A_j x, x\rangle\right)^{1/2} - \sum_{j=1}^{n} p_j\left\langle A_j^{1/2} x, x\right\rangle\right]\right)
\end{aligned}$$

*for each* $x \in H$ *with* $\|x\| = 1$*.*

Similar particular inequalities may be stated, however we state only the case of one operators, namely, for the operator $A$ satisfying the condition $0 < m \le A \le M$,

$$1 \le \exp\left(\frac{1}{p(1-p)M^{1-p}}\left[\langle Ax,x\rangle^p - \langle A^p x,x\rangle\right]\right) \le \frac{\langle Ax,x\rangle^{-\langle Ax,x\rangle}}{\eta_x(A)} \le \exp\left(\frac{1}{p(1-p)m^{1-p}}\left[\langle Ax,x\rangle^p - \langle A^p x,x\rangle\right]\right) \tag{3.7}$$

for each $x \in H$ with $\|x\| = 1$, where $p \in (0,1)$.

For $p = 1/2$ we get

$$1 \le \exp\left(\frac{4}{M^{1/2}}\left[\langle Ax,x\rangle^{1/2} - \left\langle A^{1/2} x,x\right\rangle\right]\right) \le \frac{\langle Ax,x\rangle^{-\langle Ax,x\rangle}}{\eta_x(A)} \le \exp\left(\frac{4}{m^{1/2}}\left[\langle Ax,x\rangle^{1/2} - \left\langle A^{1/2} x,x\right\rangle\right]\right) \tag{3.8}$$

for each $x \in H$ with $\|x\| = 1$.

## 4. Related Results

We also have the following related results:

**Theorem 3.** *Assume that the operator $A$ satisfies the condition $0 < m \le A \le M$ and $p \in (-\infty, 0) \cup (1, \infty)$, then*

$$1 \le \exp\left\{\psi_p(m,M)\left[\frac{\left\langle A^{2-p}x,x\right\rangle \left\langle A^2 x,x\right\rangle^{p-1} - \langle Ax,x\rangle^p}{\left\langle A^2 x,x\right\rangle^{p-2}}\right]\right\} \le \frac{\eta_x(A)}{\left(\frac{\langle Ax,x\rangle}{\langle A^2x,x\rangle}\right)^{\langle Ax,x\rangle}} \le \exp\left\{\Psi_p(m,M)\left[\frac{\left\langle A^{2-p}x,x\right\rangle \left\langle A^2 x,x\right\rangle^{p-1} - \langle Ax,x\rangle^p}{\left\langle A^2 x,x\right\rangle^{p-2}}\right]\right\} \tag{4.1}$$

*for $x \in H$, $\|x\| = 1$, where*

$$\psi_p(m,M) := \begin{cases} \frac{m^{p-1}}{p(p-1)} & \text{for } p \in (1,\infty), \\ \frac{M^{p-1}}{p(p-1)} & \text{for } p \in (-\infty,0) \end{cases}$$

*and*

$$\Psi_p(m,M) := \begin{cases} \frac{M^{p-1}}{p(p-1)} & \text{for } p \in (1,\infty), \\ \frac{m^{p-1}}{p(p-1)} & \text{for } p \in (-\infty,0). \end{cases}$$

*Proof.* Observe that

$$
\begin{aligned}
&\eta_{\frac{Ax}{\|Ax\|}}\left(A^{-1}\right)\\
&=\exp\left(-\left\langle A^{-1}\left(\ln A^{-1}\right)\frac{Ax}{\|Ax\|},\frac{Ax}{\|Ax\|}\right\rangle\right)\\
&=\exp\left(\frac{1}{\|Ax\|^{2}}\left\langle A\ln Ax,x\right\rangle\right)=\exp\left(\frac{-1}{\|Ax\|^{2}}\left\langle -A\ln Ax,x\right\rangle\right)\\
&=\left[\eta_{x}(A)\right]^{-\frac{1}{\|Ax\|^{2}}}=\left[\eta_{x}(A)\right]^{-\frac{1}{\langle A^{2}x,x\rangle}}
\end{aligned}
$$

for $x\in H$, $\|x\|=1$, which gives that

$$
\eta_{x}(A)=\left[\eta_{\frac{Ax}{\|Ax\|}}\left(A^{-1}\right)\right]^{-\left\langle A^{2}x,x\right\rangle} \tag{4.2}
$$

for $x\in H$, $\|x\|=1$.

Since $0<M^{-1}\leq A^{-1}\leq m^{-1}$ written for $A^{-1}$ we get

$$
\begin{aligned}
\phi_{p}\left(M^{-1},m^{-1}\right)&:=\begin{cases}\frac{m^{p-1}}{p(p-1)} \text{ for } p\in(1,\infty)\,,\\ \\ \frac{M^{p-1}}{p(p-1)} \text{ for } p\in(-\infty,0)\end{cases}\\
&=\psi_{p}\left(m,M\right)
\end{aligned}
$$

and

$$
\begin{aligned}
\Phi_{p}\left(M^{-1},m^{-1}\right)&:=\begin{cases}\frac{M^{p-1}}{p(p-1)} \text{ for } p\in(1,\infty)\,,\\ \\ \frac{m^{p-1}}{p(p-1)} \text{ for } p\in(-\infty,0)\,.\end{cases}\\
&=\Psi_{p}\left(m,M\right),
\end{aligned}
$$

hence by (2.12) for $A^{-1}$ and $\frac{Ax}{\|Ax\|}$ we obtain

$$
\begin{aligned}
1&\leq\exp\left\{\phi_{p}\left(M^{-1},m^{-1}\right)\left[\frac{\langle A^{2-p}x,x\rangle}{\langle A^{2}x,x\rangle}-\frac{\langle Ax,x\rangle^{p}}{\langle A^{2}x,x\rangle^{p}}\right]\right\} \\
&\leq\frac{\left(\frac{\langle Ax,x\rangle}{\langle A^{2}x,x\rangle}\right)^{-\frac{\langle Ax,x\rangle}{\langle A^{2}x,x\rangle}}}{\eta_{\frac{Ax}{\|Ax\|}}\left(A^{-1}\right)}\\
&\leq\exp\left\{\Phi_{p}\left(M^{-1},m^{-1}\right)\left[\frac{\langle A^{2-p}x,x\rangle}{\langle A^{2}x,x\rangle}-\frac{\langle Ax,x\rangle^{p}}{\langle A^{2}x,x\rangle^{p}}\right]\right\}
\end{aligned} \tag{4.3}
$$

for $x\in H$, $\|x\|=1$.

If we take the power $-\langle A^2x,x\rangle \le 0$ in (4.3), then we get

$$1 \ge \exp\left\{-\psi_p(m,M)\left[\frac{\langle A^{2-p}x,x\rangle\langle A^2x,x\rangle^{p-1}-\langle Ax,x\rangle^p}{\langle A^2x,x\rangle^{p-2}}\right]\right\}$$
$$\ge \frac{\left(\frac{\langle Ax,x\rangle}{\langle A^2x,x\rangle}\right)^{\langle Ax,x\rangle}}{\left[\eta_{\frac{Ax}{\|Ax\|}}(A^{-1})\right]^{-\langle A^2x,x\rangle}}$$
$$\ge \exp\left\{-\Psi_p(m,M)\left[\frac{\langle A^{2-p}x,x\rangle\langle A^2x,x\rangle^{p-1}-\langle Ax,x\rangle^p}{\langle A^2x,x\rangle^{p-2}}\right]\right\}$$

and by (4.2) we get

$$1 \ge \exp\left\{-\psi_p(m,M)\left[\frac{\langle A^{2-p}x,x\rangle\langle A^2x,x\rangle^{p-1}-\langle Ax,x\rangle^p}{\langle A^2x,x\rangle^{p-2}}\right]\right\} \tag{4.4}$$
$$\ge \frac{\left(\frac{\langle Ax,x\rangle}{\langle A^2x,x\rangle}\right)^{\langle Ax,x\rangle}}{\eta_x(A)}$$
$$\ge \exp\left\{-\Psi_p(m,M)\left[\frac{\langle A^{2-p}x,x\rangle\langle A^2x,x\rangle^{p-1}-\langle Ax,x\rangle^p}{\langle A^2x,x\rangle^{p-2}}\right]\right\},$$

which is equivalent to (4.1) □

**Remark 2.** *If we take $p=2$ in (4.1), then we get*

$$1 \le \exp\left\{\frac{m}{2}\left(\langle A^2x,x\rangle-\langle Ax,x\rangle^2\right)\right\} \tag{4.5}$$
$$\le \frac{\eta_x(A)}{\left(\frac{\langle Ax,x\rangle}{\langle A^2x,x\rangle}\right)^{\langle Ax,x\rangle}} \le \exp\left\{\frac{M}{2}\left(\langle A^2x,x\rangle-\langle Ax,x\rangle^2\right)\right\}$$

*for $x\in H$, $\|x\|=1$, where $0<m\le A\le M$.*

*For $p=-1$ in (4.1) we obtain*

$$1 \le \exp\left\{\frac{1}{2M^2}\left[\frac{\langle A^3x,x\rangle\langle A^2x,x\rangle^{-2}-\langle Ax,x\rangle^{-2}}{\langle A^2x,x\rangle^{-3}}\right]\right\} \tag{4.6}$$
$$\le \frac{\eta_x(A)}{\left(\frac{\langle Ax,x\rangle}{\langle A^2x,x\rangle}\right)^{\langle Ax,x\rangle}}$$
$$\le \exp\left\{\frac{1}{2m^2}\left[\frac{\langle A^3x,x\rangle\langle A^2x,x\rangle^{-2}-\langle Ax,x\rangle^{-2}}{\langle A^2x,x\rangle^{-3}}\right]\right\}$$

*for $x\in H$, $\|x\|=1$, where $0<m\le A\le M$.*

*Since,*

$$\langle A^2x,x\rangle-\langle Ax,x\rangle^2 \le \frac{1}{4}(M-m)^2,$$

*hence by (4.5) we derive*

$$1 \le \exp\left\{\frac{m}{2}\left(\langle A^2x,x\rangle - \langle Ax,x\rangle^2\right)\right\} \le \frac{\eta_x(A)}{\left(\frac{\langle Ax,x\rangle}{\langle A^2x,x\rangle}\right)^{\langle Ax,x\rangle}} \le \exp\left\{\frac{M}{2}\left(\langle A^2x,x\rangle - \langle Ax,x\rangle^2\right)\right\} \le \exp\left(\frac{M}{8}(M-m)^2\right) \tag{4.7}$$

*for* $x \in H$, $\|x\| = 1$.

*The inequalities (4.6) and (4.7) provide refinements and reverses of the first inequality in* (**??**).

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