A SUB-MULIPLICATIVE PROPERTY FOR THE NORMALIZED ENTROPIC DETERMINANT OF POSITIVE OPERATORS IN HILBERT SPACES

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Abstract. For positive invertible operators A on a Hilbert space H and a fixed unit vector $x \in H$, define the *normalized entropic determinant* by $\eta_x(A) := \exp\left[-\langle A \ln Ax, x\rangle\right]$. In this paper we show among others that, if A, $B > 0$ are such that $AB + BA \geq 0$, then

$$\eta_x(A)\eta_x(B) \geq \eta_x(A+B)$$

for $x \in H$, $\|x\| = 1$. If $0 < m \leq A \leq M$ and $0 < n \leq B \leq N$, then we have the reverse inequality

$$\frac{\eta_x(A)\eta_x(B)}{\eta_x(A+B)} \leq \exp\left(\frac{M+N}{m+n}\right)$$

for $x \in H$, $\|x\| = 1$. Moreover, if $2mn \geq \frac{1}{4}(M-m)(N-n)$, then

$$\frac{\eta_x(A)\eta_x(B)}{\eta_x(A+B)} \geq \exp\left[\frac{2mn - \frac{1}{4}(M-m)(N-n)}{M+N}\right] \geq 1$$

for $x \in H$, $\|x\| = 1$.

1. Introduction

Let $B(H)$ be the space of all bounded linear operators on a Hilbert space H, and I stands for the identity operator on H. An operator A in $B(H)$ is said to be positive (in symbol: $A \geq 0$) if $\langle Ax, x\rangle \geq 0$ for all $x \in H$. In particular, $A > 0$ means that A is positive and invertible. For a pair A, B of selfadjoint operators the order relation $A \geq B$ means as usual that $A - B$ is positive.

In 1998, Fujii et al. [1], [2], introduced the *normalized determinant* $\Delta_x(A)$ for positive invertible operators A on a Hilbert space H and a fixed unit vector $x \in H$, namely $\|x\| = 1$, defined by $\Delta_x(A) := \exp\langle \ln Ax, x\rangle$ and discussed it as a continuous geometric mean and observed some inequalities around the determinant from this point of view.

Some of the fundamental properties of normalized determinant are as follows, [1].

For each unit vector $x \in H$, see also [6], we have:

(i) *continuity*: the map $A \to \Delta_x(A)$ is norm continuous;
(ii) *bounds*: $\left\langle A^{-1}x, x\right\rangle^{-1} \leq \Delta_x(A) \leq \langle Ax, x\rangle$;
(iii) *continuous mean*: $\langle A^p x, x\rangle^{1/p} \downarrow \Delta_x(A)$ for $p \downarrow 0$ and $\langle A^p x, x\rangle^{1/p} \uparrow \Delta_x(A)$ for $p \uparrow 0$;
(iv) *power equality:* $\Delta_x(A^t) = \Delta_x(A)^t$ for all $t > 0$;
(v) *homogeneity*: $\Delta_x(tA) = t\Delta_x(A)$ and $\Delta_x(tI) = t$ for all $t > 0$;

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(vi) *monotonicity*: $0 < A \leq B$ implies $\Delta_x(A) \leq \Delta_x(B)$;
(vii) *multiplicativity*: $\Delta_x(AB) = \Delta_x(A)\Delta_x(B)$ for commuting A and B;
(viii) *Ky Fan type inequality*: $\Delta_x((1-\alpha)A + \alpha B) \geq \Delta_x(A)^{1-\alpha}\Delta_x(B)^{\alpha}$ for $0 < \alpha < 1$.

We define the *logarithmic mean* of two positive numbers a, b by

$$L(a,b) := \begin{cases} \frac{b-a}{\ln b - \ln a} \text{ if } b \neq a, \\ \\ a \text{ if } b = a. \end{cases}$$

In [1] the authors obtained the following additive reverse inequality for the operator A which satisfy the condition $0 < m \leq A \leq M$, where m, M are positive numbers,

$$0 \leq \langle Ax, x\rangle - \Delta_x(A) \leq L(m,M)\left[\ln L(m,M) + \frac{M\ln m - m\ln M}{M-m} - 1\right] \tag{1.1}$$

for all $x \in H$, $\|x\| = 1$.

The famous *Young inequality* for scalars says that if $a, b > 0$ and $\nu \in [0,1]$, then

$$a^{1-\nu}b^{\nu} \leq (1-\nu)a + \nu b \tag{1.2}$$

with equality if and only if $a = b$. The inequality (1.2) is also called *ν-weighted arithmetic-geometric mean inequality.*

We recall that *Specht's ratio* is defined by [8]

$$S(h) := \begin{cases} \frac{h^{\frac{1}{h-1}}}{e\ln\left(h^{\frac{1}{h-1}}\right)} \text{ if } h \in (0,1)\cup(1,\infty) \\ \\ 1 \text{ if } h = 1. \end{cases} \tag{1.3}$$

It is well known that $\lim_{h\to 1} S(h) = 1$, $S(h) = S\left(\frac{1}{h}\right) > 1$ for $h > 0$, $h \neq 1$. The function is decreasing on $(0,1)$ and increasing on $(1,\infty)$.

In [2], the authors obtained the following multiplicative reverse inequality as well

$$1 \leq \frac{\langle Ax, x\rangle}{\Delta_x(A)} \leq S\left(\frac{M}{m}\right) \tag{1.4}$$

for $0 < mI \leq A \leq MI$ and $x \in H$, $\|x\| = 1$.

Since $0 < M^{-1}I \leq A^{-1} \leq m^{-1}I$, then by (1.4) for A^{-1} we get

$$1 \leq \frac{\langle A^{-1}x, x\rangle}{\Delta_x(A^{-1})} \leq S\left(\frac{m^{-1}}{M^{-1}}\right) = S\left(\left(\frac{m}{M}\right)^{-1}\right) = S\left(\frac{M}{m}\right),$$

which is equivalent to

$$1 \leq \frac{\Delta_x(A)}{\langle A^{-1}x, x\rangle^{-1}} \leq S\left(\frac{M}{m}\right) \tag{1.5}$$

for $x \in H$, $\|x\| = 1$.

For the entropy function $\eta(t) = -t\ln t$, $t > 0$, the operator entropy has the following expression:

$$\eta(A) = -A\ln A$$

for positive A.

For $x \in H$, $\|x\| = 1$, we define the *normalized entropic determinant* $\eta_x(A)$ by

$$\eta_x(A) := \exp(-\langle A\ln Ax, x\rangle) = \exp\langle\eta(A)x, x\rangle. \tag{1.6}$$

Let $x \in H$, $\|x\| = 1$. Observe that the map $A \to \eta_x(A)$ is *norm continuous* and since

$$\begin{aligned}
&\exp\left(-\langle tA\ln(tA)\,x, x\rangle\right)\\
&= \exp\left(-\langle tA(\ln t + \ln A)\,x, x\rangle\right) = \exp\left(-\langle(tA\ln t + tA\ln A)\,x, x\rangle\right)\\
&= \exp\left(-\langle Ax, x\rangle\, t\ln t\right)\exp\left(-t\langle A\ln Ax, x\rangle\right)\\
&= \exp\ln\left(t^{-\langle Ax,x\rangle t}\right)\left[\exp\left(-\langle A\ln Ax, x\rangle\right)\right]^{-t},
\end{aligned}$$

hence

$$\eta_x(tA) = t^{-t\langle Ax,x\rangle}\left[\eta_x(A)\right]^{-t} \tag{1.7}$$

for $t > 0$ and $A > 0$.

Observe also that

$$\eta_x(I) = 1 \text{ and } \eta_x(tI) = t^{-t} \tag{1.8}$$

for $t > 0$.

Motivated by the above results, in this paper we show among others that, if A, $B > 0$ are such that $AB + BA \geq 0$, then

$$\eta_x(A)\eta_x(B) \geq \eta_x(A+B)$$

for $x \in H$, $\|x\| = 1$. If $0 < m \leq A \leq M$ and $0 < n \leq B \leq N$, then we have the reverse inequality

$$\frac{\eta_x(A)\eta_x(B)}{\eta_x(A+B)} \leq \exp\left(\frac{M+N}{m+n}\right)$$

for $x \in H$, $\|x\| = 1$. Moreover, if $2mn \geq \frac{1}{4}(M-m)(N-n)$, then

$$\frac{\eta_x(A)\eta_x(B)}{\eta_x(A+B)} \geq \exp\left[\frac{2mn - \frac{1}{4}(M-m)(N-n)}{M+N}\right] \geq 1$$

for $x \in H$, $\|x\| = 1$.

2. Main Results

We start with the following integral representation result:

Theorem 1. *For any A, $B > 0$ we have*

$$\begin{aligned}
&(A+B)\ln(A+B) - A\ln A - B\ln B \\
&= \int_0^\infty (A+B+\lambda)^{-1} K(A,B;\lambda)(A+B+\lambda)^{-1}\,d\lambda,
\end{aligned} \tag{2.1}$$

where

$$K(A,B;\lambda) := AB + BA + B(A+\lambda)^{-1}AB + A(B+\lambda)^{-1}BA. \tag{2.2}$$

Proof. Observe that for $t > 0$, $t \neq 1$, we have

$$\int_0^u \frac{d\lambda}{(\lambda+t)(\lambda+1)} = \frac{\ln t}{t-1} + \frac{1}{1-t}\ln\left(\frac{u+t}{u+1}\right)$$

for all $u > 0$.

By taking the limit over $u \to \infty$ in this equality, we derive

$$\frac{\ln t}{t-1} = \int_0^\infty \frac{d\lambda}{(\lambda+t)(\lambda+1)},$$

which gives the representation for the logarithm

$$\ln t=(t-1)\int_0^\infty \frac{d\lambda}{(\lambda+1)(\lambda+t)} \tag{2.3}$$

for all $t>0$.

If we use the continuous functional calculus for selfadjoint operators, we have

$$\ln T=\int_0^\infty \frac{1}{\lambda+1}(T-1)(\lambda+T)^{-1}\,d\lambda \tag{2.4}$$

for all operators $T>0$.

Observe that

$$\begin{aligned}\int_0^\infty \frac{1}{\lambda+1}(T-1)(\lambda+T)^{-1}\,d\lambda &= \int_0^\infty \frac{1}{\lambda+1}(T+\lambda-\lambda-1)(\lambda+T)^{-1}\,d\lambda\\ &=\int_0^\infty \left[(\lambda+1)^{-1}-(\lambda+T)^{-1}\right]d\lambda\end{aligned}$$

and then

$$\ln T=\int_0^\infty \left[(\lambda+1)^{-1}-(\lambda+T)^{-1}\right]d\lambda$$

giving the representation

$$T\ln T=\int_0^\infty \left[(\lambda+1)^{-1}T-T(\lambda+T)^{-1}\right]d\lambda \tag{2.5}$$

for all operators $T>0$.

For $A,\ B>0$ we have

$$\begin{aligned}&(A+B)\ln(A+B)-A\ln A-B\ln B\\ &=\int_0^\infty \left[(\lambda+1)^{-1}(A+B)-(A+B)(\lambda+(A+B))^{-1}\right]d\lambda\\ &-\int_0^\infty \left[(\lambda+1)^{-1}A-A(\lambda+A)^{-1}\right]d\lambda\\ &-\int_0^\infty \left[(\lambda+1)^{-1}B-B(\lambda+B)^{-1}\right]d\lambda\\ &=\int_0^\infty \left[A(\lambda+A)^{-1}+B(\lambda+B)^{-1}-(A+B)(\lambda+A+B)^{-1}\right]d\lambda.\end{aligned} \tag{2.6}$$

Now, observe that

$$\begin{aligned}&A(\lambda+A)^{-1}+B(\lambda+B)^{-1}-(A+B)(\lambda+A+B)^{-1}\\ &=(A+\lambda-\lambda)(\lambda+A)^{-1}+(B+\lambda-\lambda)(\lambda+B)^{-1}\\ &-(A+B+\lambda-\lambda)(\lambda+A+B)^{-1}\\ &=1-\lambda(\lambda+A)^{-1}+1-\lambda(\lambda+B)^{-1}-1+\lambda(\lambda+A+B)^{-1}\\ &=1+\lambda\left[(\lambda+A+B)^{-1}-(\lambda+A)^{-1}-(\lambda+B)^{-1}\right]\\ &=\lambda\left[(\lambda+A+B)^{-1}+\lambda^{-1}-(\lambda+A)^{-1}-(\lambda+B)^{-1}\right]\end{aligned}$$

Consider

$$L_\lambda:=(A+B+\lambda)^{-1}+\lambda^{-1}-(A+\lambda)^{-1}-(B+\lambda)^{-1}. \tag{2.7}$$

Then by (2.6) we obtain the representation

$$(A+B)\ln(A+B)-A\ln A-B\ln B=\int_0^\infty \lambda L_\lambda d\lambda \tag{2.8}$$

for all $A,\ B>0$.

If we multiply both sides of (2.7) by $A+B+\lambda$, then we get

$$\begin{aligned}
W_\lambda &:= (A+B+\lambda)\, L_\lambda\,(A+B+\lambda)\\
&= (A+B+\lambda)+\lambda^{-1}(A+B+\lambda)^2\\
&- (A+B+\lambda)(A+\lambda)^{-1}(A+B+\lambda)\\
&- (A+B+\lambda)(B+\lambda)^{-1}(A+B+\lambda)\\
&= (A+B+\lambda)+\lambda^{-1}(A+B+\lambda)^2\\
&- (A+B+\lambda)-B(A+\lambda)^{-1}(A+B+\lambda)\\
&- A(B+\lambda)^{-1}(A+B+\lambda)-(A+B+\lambda)\\
&= \lambda^{-1}(A+B+\lambda)^2-B(A+\lambda)^{-1}B-B\\
&- A(B+\lambda)^{-1}A-A-(A+B+\lambda)
\end{aligned}$$

$$\begin{aligned}
&= \lambda^{-1}\left(A^2+AB+\lambda A+BA+B^2+\lambda B+\lambda A+\lambda B+\lambda^2\right)\\
&- B(A+\lambda)^{-1}B-2B-A(B+\lambda)^{-1}A-2A-\lambda\\
&= \lambda^{-1}\left(A^2+AB+BA+B^2\right)+2B+2A+\lambda\\
&- B(A+\lambda)^{-1}B-A(B+\lambda)^{-1}A-2A-2B-\lambda\\
&= \lambda^{-1}\left(A^2+AB+BA+B^2\right)-B(A+\lambda)^{-1}B-A(B+\lambda)^{-1}A\\
&= \lambda^{-1}\left[A^2+AB+BA+B^2-\lambda B(A+\lambda)^{-1}B-\lambda A(B+\lambda)^{-1}A\right]\\
&= \lambda^{-1}\left[A^2+AB+BA+B^2-B\left(\lambda^{-1}A+1\right)^{-1}B-A\left(\lambda^{-1}B+1\right)^{-1}A\right].
\end{aligned}$$

Observe that

$$\begin{aligned}
&B^2-B\left(\lambda^{-1}A+1\right)^{-1}B\\
&= B\left(\lambda^{-1}A+1\right)^{-1}\left(\lambda^{-1}A+1\right)B-B\left(\lambda^{-1}A+1\right)^{-1}B\\
&= B\left(\lambda^{-1}A+1\right)^{-1}\left(\lambda^{-1}A+1-1\right)B\\
&= \lambda^{-1}B\left(\lambda^{-1}A+1\right)^{-1}AB=B(A+\lambda)^{-1}AB
\end{aligned}$$

and

$$\begin{aligned}
&A^2-A\left(\lambda^{-1}B+1\right)^{-1}A\\
&= A\left(\lambda^{-1}B+1\right)^{-1}\left(\lambda^{-1}B+1\right)A-A\left(\lambda^{-1}B+1\right)^{-1}A\\
&= A\left(\lambda^{-1}B+1\right)^{-1}\left(\lambda^{-1}B+1-1\right)A\\
&= \lambda^{-1}A\left(\lambda^{-1}B+1\right)^{-1}BA=A(B+\lambda)^{-1}BA.
\end{aligned}$$

Therefore

$$W_\lambda=\lambda^{-1}\left[AB+BA+B(A+\lambda)^{-1}AB+A(B+\lambda)^{-1}BA\right],$$

which gives that

$$L_\lambda := (A+B+\lambda)^{-1} W_\lambda (A+B+\lambda)^{-1}.$$

From (2.8) we then get the representation (2.1). □

The symmetrized product of two operators $A, B \in B(H)$ is defined by $S(A,B) = AB + BA$. In general, the symmetrized product of two operators A, B is not positive (see for instance [7]). Also Gustafson [5] showed that if $0 \le m \le A \le M$ and $0 \le n \le B \le N$, then we have the lower bound

$$S(A,B) \ge 2mn - \frac{1}{4}(M-m)(N-n) =: k, \tag{2.9}$$

which can take positive or negative values depending on the parameters m, M, n, N.

Corollary 1. *Let A, $B > 0$ and assume that $S(A,B) \ge k$ for some real constant k, then*

$$(A+B)\ln(A+B) - A\ln A - B\ln B \ge k(A+B)^{-1}. \tag{2.10}$$

If $k \ge 0$, then

$$(A+B)\ln(A+B) - A\ln A - B\ln B \ge k(A+B)^{-1} \ge 0. \tag{2.11}$$

Proof. Since for all A, $B > 0$,

$$(B+\lambda)^{-1} B > 0, \;\; (A+\lambda)^{-1} A > 0$$

for $\lambda \ge 0$, then

$$A(B+\lambda)^{-1} BA, \;\; B(A+\lambda)^{-1} AB \ge 0$$

that gives

$$A(B+\lambda)^{-1} BA + B(A+\lambda)^{-1} AB \ge 0,$$

which implies that

$$(A+B+\lambda)^{-1}\left[A(B+\lambda)^{-1} BA + B(A+\lambda)^{-1} AB\right](A+B+\lambda)^{-1} \ge 0$$

for$\lambda \ge 0$.

By the representation (2.1) we then get

$$\begin{aligned}
&(A+B)\ln(A+B) - A\ln A - B\ln B \\
&\ge \int_0^\infty (A+B+\lambda)^{-1} S(A,B)(A+B+\lambda)^{-1} d\lambda \\
&\ge k\int_0^\infty (A+B+\lambda)^{-2} d\lambda = k(A+B)^{-1}
\end{aligned}$$

since

$$t^{-1} = \int_0^\infty (\lambda+t)^{-2} d\lambda \text{ for } t > 0.$$

This proves the desired result □

Remark 1. *If $0 \le m \le A \le M$ and $0 \le n \le B \le N$, then*

$$\begin{aligned}
&(A+B)\ln(A+B) - A\ln A - B\ln B \\
&\ge \left[2mn - \frac{1}{4}(M-m)(N-n)\right](A+B)^{-1}.
\end{aligned} \tag{2.12}$$

If $2mn \geq \frac{1}{4}(M-m)(N-n)$, *then*

$$(A+B)\ln(A+B) \geq A\ln A + B\ln B.$$

We have the following identity for the entropic determinant:

Theorem 2. *For any* A, $B>0$ *and* $x \in H$, $\|x\|=1$, *we have*

$$\begin{aligned}(2.13)\qquad &\frac{\eta_x(A)\eta_x(B)}{\eta_x(A+B)}\\ &= \exp\left\langle \int_0^\infty \left\langle (A+B+\lambda)^{-1} K(A,B;\lambda)(A+B+\lambda)^{-1}x, x\right\rangle d\lambda \right\rangle.\end{aligned}$$

Proof. If we take the inner product over the vector $x \in H$, $\|x\|=1$ in (2.1), then we get

$$\begin{aligned}&\langle (A+B)\ln(A+B)x, x\rangle - \langle A\ln Ax, x\rangle - \langle B\ln Bx, x\rangle\\ &= \int_0^\infty \left\langle (A+B+\lambda)^{-1} K(A,B;\lambda)(A+B+\lambda)^{-1}x, x\right\rangle d\lambda.\end{aligned}$$

If we take the exponential in this equality, then we get

$$\begin{aligned}&\frac{\exp(-\langle A\ln Ax, x\rangle)\exp(-\langle B\ln Bx, x\rangle)}{\exp(-\langle (A+B)\ln(A+B)x, x\rangle)}\\ &= \exp\left(\int_0^\infty \left\langle (A+B+\lambda)^{-1} K(A,B;\lambda)(A+B+\lambda)^{-1}x, x\right\rangle d\lambda\right),\end{aligned}$$

which gives (2.13). □

Corollary 2. *Let* A, $B>0$ *and assume that* $S(A,B) \geq k$ *for some real constant* k, *then*

$$\frac{\eta_x(A)\eta_x(B)}{\eta_x(A+B)} \geq \exp\left(k(A+B)^{-1}\right) \tag{2.14}$$

for $x \in H$, $\|x\|=1$.

If $k \geq 0$, *then*

$$\frac{\eta_x(A)\eta_x(B)}{\eta_x(A+B)} \geq \exp\left(k(A+B)^{-1}\right) \geq 1, \tag{2.15}$$

for $x \in H$, $\|x\|=1$.

Remark 2. *Let* A, $B>0$. *We observe that, if* $S(A,B) \geq 0$, *then the following sub-multiplicative property holds,*

$$\eta_x(A)\eta_x(B) \geq \eta_x(A+B) \tag{2.16}$$

for $x \in H$, $\|x\|=1$.

Also, if $0<m \leq A \leq M$ *and* $0<n \leq B \leq N$, *then*

$$\frac{\eta_x(A)\eta_x(B)}{\eta_x(A+B)} \geq \exp\left(\left[2mn - \frac{1}{4}(M-m)(N-n)\right](A+B)^{-1}\right). \tag{2.17}$$

Moreover, if $2mn \geq \frac{1}{4}(M-m)(N-n)$, *then*

$$\frac{\eta_x(A)\eta_x(B)}{\eta_x(A+B)} \geq \exp\left[\frac{2mn - \frac{1}{4}(M-m)(N-n)}{M+N}\right] \geq 1 \tag{2.18}$$

for $x \in H$, $\|x\|=1$.

We have the following consequence of Theorem 1:

Corollary 3. *Assume that* A, $B > 0$ *with* $A + B \leq L$ *for some positive constant* L, *then*

$$(A+B)\ln(A+B) - A\ln A - B\ln B \leq L(A+B)^{-1}. \tag{2.19}$$

Moreover, if $0 < \ell \leq A + B$ *for some constant* $\ell > 0$, *then*

$$(A+B)\ln(A+B) - A\ln A - B\ln B \leq \frac{L}{\ell}. \tag{2.20}$$

Proof. Assume that$A$, $B \geq 0$. Observe that for $\lambda > 0$

$$(A+\lambda)^{-1}A = (A+\lambda)^{-1}(A+\lambda-\lambda) = 1 - \lambda(A+\lambda)^{-1},$$

which shows that

$$0 \leq (A+\lambda)^{-1}A \leq 1.$$

If we multiply this inequality both sides by B, then we get

$$0 \leq B(A+\lambda)^{-1}AB \leq B^2.$$

Similarly,

$$0 \leq A(B+\lambda)^{-1}BA \leq A^2.$$

Therefore

$$0 \leq B(A+\lambda)^{-1}AB + A(B+\lambda)^{-1}BA \leq A^2 + B^2$$

and

$$\begin{aligned} L(A,B;\lambda) &= AB + BA + B(A+\lambda)^{-1}AB + A(B+\lambda)^{-1}BA \\ &\leq AB + BA + A^2 + B^2 = (A+B)^2 \leq L, \end{aligned}$$

which implies that

$$\begin{aligned} &(A+B+\lambda)^{-1}L(A,B;\lambda)(A+B+\lambda)^{-1} \\ &\leq L(A+B+\lambda)^{-1}(A+B+\lambda)^{-1} \end{aligned}$$

for $\lambda > 0$.

By taking the integral and using the identity (2.1) we derive

$$\begin{aligned} &(A+B)\ln(A+B) - A\ln A - B\ln B \\ &\leq L\int_0^\infty (A+B+\lambda)^{-1}(A+B+\lambda)^{-1}\,dt = L(A+B)^{-1}, \end{aligned}$$

which proves the desired inequality (2.19). □

We also have the following result for the entropic determinants:

Theorem 3. *Let* A, $B > 0$ *with* $A + B \leq L$ *for some positive constant* L, *and* $x \in H$, $\|x\| = 1$, *then*

$$\frac{\eta_x(A)\eta_x(B)}{\eta_x(A+B)} \leq \exp\left[L\left\langle (A+B)^{-1}x, x\right\rangle\right]. \tag{2.21}$$

Moreover, if $0 < \ell \leq A + B$ *for some constant* $\ell > 0$, *then*

$$\frac{\eta_x(A)\eta_x(B)}{\eta_x(A+B)} \leq \exp\left(\frac{L}{\ell}\right). \tag{2.22}$$

The proof follows in a similar way to the one from Theorem 2 via Corollary 3.

Remark 3. *We observe that, if* $0 < m \leq A \leq M$ *and* $0 < n \leq B \leq N$*, then* $0 < m + n \leq A + B \leq M + N$ *and by (2.22) we obtain the simple upper bound*

$$\frac{\eta_x(A)\eta_x(B)}{\eta_x(A+B)} \leq \exp\left(\frac{M+N}{m+n}\right) \tag{2.23}$$

for all $x \in H$, $\|x\| = 1$.

3. Related Results

The following integral inequalities also hold:

Theorem 4. *Let* A, $B \geq 0$ *with* $AB + BA \geq 0$, *then*

$$\begin{aligned} \eta_x(A+B) &\leq \int_0^1 \eta_x((1-t)A + tB)\eta_x((1-t)B + tA)dt \\ &\leq \int_0^1 \eta_x^2((1-t)A + tB)dt \end{aligned} \tag{3.1}$$

and, if $A + B \leq L$, *then also*

$$\int_0^1 \eta_x((1-t)A + tB)\eta_x((1-t)B + tA)dt \leq \eta_x(A+B)\exp\left[L\left\langle (A+B)^{-1}x, x\right\rangle\right] \tag{3.2}$$

for all $x \in H$, $\|x\| = 1$.

Proof. We have

$$\begin{aligned} &((1-t)A + tB)((1-t)B + tA) \\ &= (1-t)^2 AB + t(1-t)B^2 + t(1-t)A^2 + t^2 BA \end{aligned}$$

and

$$\begin{aligned} &((1-t)B + tA)((1-t)A + tB) \\ &= (1-t)^2 BA + t(1-t)A^2 + (1-t)tB^2 + t^2 AB. \end{aligned}$$

Therefore, since $AB + BA \geq 0$, then

$$\begin{aligned} &((1-t)A + tB)((1-t)B + tA) \\ &+ ((1-t)B + tA)((1-t)A + tB) \\ &= (1-t)^2 AB + t(1-t)B^2 + t(1-t)A^2 + t^2 BA \\ &+ (1-t)^2 BA + t(1-t)A^2 + (1-t)tB^2 + t^2 AB \\ &= 2t(1-t)A^2 + 2t(1-t)B^2 + \left[(1-t)^2 + t^2\right](AB + BA) \\ &\geq 0 \end{aligned}$$

for all $t \in [0,1]$.

By utilising (2.16) for $(1-t)A + tB$ and $(1-t)B + tA$, $t \in [0,1]$, we get

$$\eta_x((1-t)A + tB)\eta_x((1-t)B + tA) \geq \eta_x(A+B)$$

for $x \in H$, $\|x\| = 1$.

If we integrate over $t\in[0,1]$, then we get

$$\begin{aligned}\eta_x(A+B) &\leq \int_0^1 \eta_x((1-t)\,A+tB)\eta_x((1-t)\,B+tA)dt \\ &\leq \left(\int_0^1 \eta_x^2((1-t)\,A+tB)dt\right)^{1/2}\left(\int_0^1 \eta_x^2((1-t)\,B+tA)dt\right)^{1/2} \\ &= \int_0^1 \eta_x^2((1-t)\,A+tB)dt,\end{aligned}$$

which proves (3.1).

From (2.21) we get

$$\eta_x((1-t)\,A+tB)\eta_x((1-t)\,B+tA)\leq \eta_x(A+B)\exp\left[L\left\langle (A+B)^{-1}x,x\right\rangle\right]$$

for all $t\in[0,1]$, which by integration gives (3.2). □

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