# Vector Voronsovkaya type asymptotic expansions for sigmoid functions induced quasi-interpolation neural network operators revisited

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**Abstract**

Here we study further the quasi-interpolation arctangent-algebraic-Gudermannian-generalized symmetrical activation functions relied neural network operators of one hidden layer. Based on fractional calculus theory we derive fractional Voronovskaya type asymptotic expansions for the approximation of these operators to the unit operator, as we are studying the univariate case. We treat also analogously the multivariate case by using Fréchet derivatives. The functions under approximation are Banach space valued.

**2020 AMS Mathematics Subject Classification:** 26A33, 41A25, 41A45, 41A60.

**Keywords and Phrases:** Neural Network Fractional Approximation, Multivariate Neural Network Approximation, Voronovskaya Asymptotic Expansions, fractional derivative, arctangent-algebraic-Gudermannian-generalized symmetrical sigmoid activation functions.

## 1 Background

This is a continuation and generalization of [8].

## 1.1 About the arctangent activation function

We consider the

$$\arctan x=\int_0^x \frac{dz}{1+z^2},\quad x\in\mathbb{R}. \tag{1}$$

We will be using

$$h(x):=\frac{2}{\pi}\arctan\left(\frac{\pi}{2}x\right)=\frac{2}{\pi}\int_0^{\frac{\pi x}{2}}\frac{dz}{1+z^2},\ \ x\in\mathbb{R}, \tag{2}$$

which is a sigmoid type function and it is strictly increasing. We have that

$$h(0)=0,\ \ h(-x)=-h(x),\ \ h(+\infty)=1,\ \ h(-\infty)=-1,$$

and

$$h'(x)=\frac{4}{4+\pi^2x^2}>0,\ \text{ all } x\in\mathbb{R}. \tag{3}$$

We consider the activation function

$$\psi_1(x):=\frac{1}{4}\left(h(x+1)-h(x-1)\right),\ \ x\in\mathbb{R}, \tag{4}$$

and we notice that

$$\psi_1(-x)=\psi_1(x), \tag{5}$$

it is an even function.

Since $x+1>x-1$, then $h(x+1)>h(x-1)$, and $\psi_1(x)>0$, all $x\in\mathbb{R}$.

We see that

$$\psi_1(0)=\frac{1}{\pi}\arctan\frac{\pi}{2}\cong 18.31. \tag{6}$$

Let $x>0$, we have that

$$\psi_1'(x)=\frac{1}{4}\left(h'(x+1)-h'(x-1)\right)=$$

$$\frac{-4\pi^2x}{\left(4+\pi^2(x+1)^2\right)\left(4+\pi^2(x-1)^2\right)}<0. \tag{7}$$

That is

$$\psi_1'(x)<0,\text{ for } x>0. \tag{8}$$

That is $\psi_1$ is strictly decreasing on $[0,\infty)$ and clearly is strictly increasing on $(-\infty,0]$, and $\psi_1'(0)=0$.

Observe that

$$\begin{gathered}\lim_{x\to+\infty}\psi_1(x)=\tfrac{1}{4}\left(h(+\infty)-h(+\infty)\right)=0,\\ \text{and}\\ \lim_{x\to-\infty}\psi_1(x)=\tfrac{1}{4}\left(h(-\infty)-h(-\infty)\right)=0.\end{gathered} \tag{9}$$

That is the $x$-axis is the horizontal asymptote on $\psi_1$.

All in all, $\psi_1$ is a bell symmetric function with maximum $\psi_1(0)\cong 18.31$.

We need

**Theorem 1** *([10], p. 286) We have that*

$$\sum_{i=-\infty}^{\infty} \psi_1 (x - i) = 1, \ \ \forall \ x \in \mathbb{R}. \tag{10}$$

**Theorem 2** *([10], p. 287) It holds*

$$\int_{-\infty}^{\infty} \psi_1 (x) \, dx = 1. \tag{11}$$

So that $\psi_1 (x)$ is a density function on $\mathbb{R}$.

We mention

**Theorem 3** *([10], p. 288) Let $0 < \alpha < 1$, and $n \in \mathbb{N}$ with $n^{1-\alpha} > 2$. It holds*

$$\sum_{\begin{cases} k = -\infty \\ : |nx - k| \geq n^{1-\alpha} \end{cases}}^{\infty} \psi_1 (nx - k) < \frac{2}{\pi^2 (n^{1-\alpha} - 2)} =: c_1 (\alpha, n) \, . \tag{12}$$

Denote by $\lfloor \cdot \rfloor$ the integral part of the number and by $\lceil \cdot \rceil$ the ceiling of the number.

We need

**Theorem 4** *([10], p. 289) Let $x \in [a, b] \subset \mathbb{R}$ and $n \in \mathbb{N}$ so that $\lceil na \rceil \leq \lfloor nb \rfloor$. It holds*

$$\frac{1}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \psi_1 (nx - k)} < \frac{1}{\psi_1 (1)} \cong 0.0868 =: \alpha_1, \ \ \forall \ x \in [a, b] \, . \tag{13}$$

**Note 5** *([10], pp. 290-291)*

*i) We have that*

$$\lim_{n \to \infty} \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \psi_1 (nx - k) \neq 1, \tag{14}$$

*for at least some $x \in [a, b]$.*

*ii) For large enough $n \in \mathbb{N}$ we always obtain $\lceil na \rceil \leq \lfloor nb \rfloor$. Also $a \leq \frac{k}{n} \leq b$, iff $\lceil na \rceil \leq k \leq \lfloor nb \rfloor$.*

*In general, by Theorem 1, it holds*

$$\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \psi_1 (nx - k) \leq 1. \tag{15}$$

We introduce (see [15])

$$Z_1(x_1,...,x_N) := Z_1(x) := \prod_{i=1}^{N} \psi_1(x_i), \quad x = (x_1,...,x_N) \in \mathbb{R}^N, \ N \in \mathbb{N}. \tag{16}$$

Denote by $a = (a_1,...,a_N)$ and $b = (b_1,...,b_N)$.

It has the properties:

(i) $Z_1(x) > 0, \ \forall \ x \in \mathbb{R}^N$,

(ii)

$$\sum_{k=-\infty}^{\infty} Z_1(x-k) := \sum_{k_1=-\infty}^{\infty} \sum_{k_2=-\infty}^{\infty} ... \sum_{k_N=-\infty}^{\infty} Z_1(x_1-k_1,...,x_N-k_N) = 1, \tag{17}$$

where $k := (k_1,...,k_n) \in \mathbb{Z}^N$, $\forall \ x \in \mathbb{R}^N$,

hence

(iii)

$$\sum_{k=-\infty}^{\infty} Z_1(nx-k) = 1, \tag{18}$$

$\forall \ x \in \mathbb{R}^N$; $n \in \mathbb{N}$,

and

(iv)

$$\int_{\mathbb{R}^N} Z_1(x)\,dx = 1, \tag{19}$$

that is $Z_1$ is a multivariate density function.

(v) It is clear that

$$\sum_{\begin{cases} k=-\infty \\ \left\|\frac{k}{n}-x\right\|_{\infty} > \frac{1}{n^{\beta}} \end{cases}}^{\infty} Z_1(nx-k) < \frac{2}{\pi^2\left(n^{1-\beta}-2\right)} = c_1(\beta,n), \tag{20}$$

$0 < \beta < 1$, $n \in \mathbb{N} : n^{1-\beta} > 2$, $x \in \mathbb{R}^N$.

(vi) By Theorem 4 we get that

$$0 < \frac{1}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} Z_1(nx-k)} < \frac{1}{(\psi_1(1))^N} \cong (0.0868)^N =: \gamma_1(N), \tag{21}$$

$\forall \ x \in \left(\prod_{i=1}^{N}[a_i,b_i]\right), \ n \in \mathbb{N}$.

Furthermore it holds

$$\lim_{n\to\infty} \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} Z_1(nx-k) \neq 1, \tag{22}$$

for at least some $x \in \left( \prod_{i=1}^{N} [a_i, b_i] \right)$.

Above it is $\|x\|_\infty := \max \{|x_1|, ..., |x_N|\}$, $x \in \mathbb{R}^N$, also set $\infty := (\infty, ..., \infty)$, $-\infty = (-\infty, ... - \infty)$ upon the multivariate context.

## 1.2 About the algebraic activation function

Here see also [11].

We consider the generator algebraic function

$$\varphi(x) = \frac{x}{\sqrt[2m]{1 + x^{2m}}}, \quad m \in \mathbb{N}, x \in \mathbb{R}, \tag{23}$$

which is a sigmoidal type of function and is a strictly increasing function.

We see that $\varphi(-x) = -\varphi(x)$ with $\varphi(0) = 0$. We get that

$$\varphi'(x) = \frac{1}{(1 + x^{2m})^{\frac{2m+1}{2m}}} > 0, \ \forall \ x \in \mathbb{R}, \tag{24}$$

proving $\varphi$ as strictly increasing over $\mathbb{R}, \varphi'(x) = \varphi'(-x)$. We easily find that $\lim_{x \to +\infty} \varphi(x) = 1$, $\varphi(+\infty) = 1$, and $\lim_{x \to -\infty} \varphi(x) = -1$, $\varphi(-\infty) = -1$.

We consider the activation function

$$\psi_2(x) = \frac{1}{4}\left[\varphi(x+1) - \varphi(x-1)\right]. \tag{25}$$

Clearly it is $\psi_2(x) = \psi_2(-x)$, $\forall \ x \in \mathbb{R}$, so that $\psi_2$ is an even function and symmetric with respect to the $y$-axis. Clealry $\psi_2(x) > 0$, $\forall \ x \in \mathbb{R}$.

Also it is

$$\psi_2(0) = \frac{1}{2 \sqrt[2m]{2}}. \tag{26}$$

By [11], we have that $\psi_2'(x) < 0$ for $x > 0$. That is $\psi_2$ is strictly decreasing over $(0, +\infty)$.

Clearly, $\psi_2$ is strictly increasing over $(-\infty, 0)$ and $\psi_2'(0) = 0$.

Furthermore we obtain that

$$\lim_{x \to +\infty} \psi_2(x) = \frac{1}{4}\left[\varphi(+\infty) - \varphi(+\infty)\right] = 0, \tag{27}$$

and

$$\lim_{x \to -\infty} \psi_2(x) = \frac{1}{4}\left[\varphi(-\infty) - \varphi(-\infty)\right] = 0. \tag{28}$$

That is the $x$-axis is the horizontal asymptote of $\psi_2$.

Conclusion, $\psi_2$ is a bell shape symmetric function with maximum

$$\psi_2(0) = \frac{1}{2 \sqrt[2m]{2}}, \quad m \in \mathbb{N}. \tag{29}$$

We need

**Theorem 6** *([11]) We have that*

$$\sum_{i=-\infty}^{\infty} \psi_2 (x-i) = 1, \ \ \forall \ x \in \mathbb{R}. \tag{30}$$

**Theorem 7** *([11]) It holds*

$$\int_{-\infty}^{\infty} \psi_2 (x) \, dx = 1. \tag{31}$$

**Theorem 8** *([11]) Let $0 < \alpha < 1$, and $n \in \mathbb{N}$ with $n^{1-\alpha} > 2$. It holds*

$$\sum_{\begin{cases} k=-\infty \\ : |nx-k| \geq n^{1-\alpha} \end{cases}}^{\infty} \psi_2 (nx-k) < \frac{1}{4m \left(n^{1-\alpha}-2\right)^{2m}} =: c_2 (\alpha, n), \quad m \in \mathbb{N}. \tag{32}$$

We need

**Theorem 9** *([11]) Let $[a,b] \subset \mathbb{R}$ and $n \in \mathbb{N}$ so that $\lceil na \rceil \leq \lfloor nb \rfloor$. It holds*

$$\frac{1}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \psi_2 (nx-k)} < 2 \left( \sqrt[2m]{1+4^m} \right) =: \alpha_2, \tag{33}$$

$\forall \ x \in [a,b]$, $m \in \mathbb{N}$.

**Note 10** *1) By [11] we have that*

$$\lim_{n \to \infty} \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \psi_2 (nx-k) \neq 1, \tag{34}$$

*for at least some $x \in [a,b]$.*

*2) Let $[a,b] \subset \mathbb{R}$. For large $n \in \mathbb{N}$ we always have $\lceil na \rceil \leq \lfloor nb \rfloor$. Also $a \leq \frac{k}{n} \leq b$, iff $\lceil na \rceil \leq k \leq \lfloor nb \rfloor$.*

*In general it holds that*

$$\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \psi_2 (nx-k) \leq 1. \tag{35}$$

We introduce (see also [17])

$$Z_2 (x_1, ..., x_N) := Z_2 (x) := \prod_{i=1}^{N} \psi_2 (x_i), \quad x = (x_1, ..., x_N) \in \mathbb{R}^N, \ N \in \mathbb{N}. \tag{36}$$

It has the properties:

(i) $Z_2(x) > 0, \ \forall \ x \in \mathbb{R}^N$,

(ii)

$$\sum_{k=-\infty}^{\infty} Z_2(x-k) := \sum_{k_1=-\infty}^{\infty} \sum_{k_2=-\infty}^{\infty} \cdots \sum_{k_N=-\infty}^{\infty} Z_2(x_1-k_1, ..., x_N-k_N) = 1, \tag{37}$$

where $k := (k_1, ..., k_n) \in \mathbb{Z}^N$, $\forall \ x \in \mathbb{R}^N$,

hence

(iii)

$$\sum_{k=-\infty}^{\infty} Z_2(nx-k) = 1, \tag{38}$$

$\forall \ x \in \mathbb{R}^N$; $n \in \mathbb{N}$,

and

(iv)

$$\int_{\mathbb{R}^N} Z_2(x)\, dx = 1, \tag{39}$$

that is $Z_2$ is a multivariate density function.

(v) It is clear that

$$\sum_{\left\{\begin{array}{c} k=-\infty \\ \left\| \frac{k}{n} - x \right\|_{\infty} > \frac{1}{n^{\beta}} \end{array}\right.}^{\infty} Z_2(nx-k) < \frac{1}{4m\left(n^{1-\beta}-2\right)^{2m}} = c_2(\beta, n), \tag{40}$$

$0 < \beta < 1$, $n \in \mathbb{N} : n^{1-\beta} > 2$, $x \in \mathbb{R}^N$, $m \in \mathbb{N}$.

(vi) By Theorem 9 we get that

$$0 < \frac{1}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} Z_2(nx-k)} < \frac{1}{(\psi_2(1))^N} \cong \left[2\left(\sqrt[2m]{1+4^m}\right)\right]^N := \gamma_2(N), \tag{41}$$

$\forall \ x \in \left(\prod_{i=1}^N [a_i, b_i]\right), \ \ n \in \mathbb{N}$.

Furthermore it holds

$$\lim_{n\to\infty} \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} Z_2(nx-k) \neq 1, \tag{42}$$

for at least some $x \in \left(\prod_{i=1}^N [a_i, b_i]\right)$.

## 1.3 About the Gudermannian activation function

See also [29], [12].

Here we consider $gd(x)$ the Gudermannian function [29], which is a sigmoid function, as a generator function:

$$\sigma(x) = 2\arctan\left(\tanh\left(\frac{x}{2}\right)\right) = \int_0^x \frac{dt}{\cosh t} =: gd(x), \; x \in \mathbb{R}. \tag{43}$$

Let the normalized generator sigmoid function

$$f(x) := \frac{4}{\pi}\sigma(x) = \frac{4}{\pi}\int_0^x \frac{dt}{\cosh t} = \frac{8}{\pi}\int_0^x \frac{1}{e^t + e^{-t}}dt, \quad x \in \mathbb{R}. \tag{44}$$

Here

$$f'(x) = \frac{4}{\pi \cosh x} > 0, \quad \forall \; x \in \mathbb{R},$$

hence $f$ is strictly increasing on $\mathbb{R}$.

Notice that $\tanh(-x) = -\tanh x$ and $\arctan(-x) = -\arctan x$, $x \in \mathbb{R}$.

So, here the neural network activation function will be:

$$\psi_3(x) = \frac{1}{4}\left[f(x+1) - f(x-1)\right], \; x \in \mathbb{R}. \tag{45}$$

By [12], we get that

$$\psi_3(x) = \psi_3(-x), \quad \forall \; x \in \mathbb{R}, \tag{46}$$

i.e. it is even and symmetric with respect to the $y$-axis. Here we have $f(+\infty) = 1$, $f(-\infty) = -1$ and $f(0) = 0$. Clearly it is

$$f(-x) = -f(x), \;\; \forall \; x \in \mathbb{R}, \tag{47}$$

an odd function, symmetric with respect to the origin. Since $x+1 > x-1$, and $f(x+1) > f(x-1)$, we obtain $\psi_3(x) > 0$, $\forall \; x \in \mathbb{R}$.

By [12], we have that

$$\psi_3(0) = \frac{2}{\pi}gd(1) \cong 0.551. \tag{48}$$

By [12] $\psi_3$ is strictly decreasing on $(0, +\infty)$, and strictly increasing on $(-\infty, 0)$, and $\psi_3'(0) = 0$.

Also we have that

$$\lim_{x\to+\infty}\psi_3(x) = \lim_{x\to-\infty}\psi_3(x) = 0, \tag{49}$$

that is the $x$-axis is the horizontal asymptote for $\psi_3$.

Conclusion, $\psi_3$ is a bell shaped symmetric function with maximum $\psi_3(0) \cong 0.551$.

We need

**Theorem 11** *([12]) It holds that*

$$\sum_{i=-\infty}^{\infty} \psi_3(x-i) = 1, \ \ \forall\, x \in \mathbb{R}. \tag{50}$$

**Theorem 12** *([12]) We have that*

$$\int_{-\infty}^{\infty} \psi_3(x)\, dx = 1. \tag{51}$$

*So $\psi_3(x)$ is a density function.*

**Theorem 13** *([12]) Let $0 < \alpha < 1$, and $n \in \mathbb{N}$ with $n^{1-\alpha} > 2$. It holds*

$$\sum_{\begin{cases} k = -\infty \\ : |nx-k| \geq n^{1-\alpha} \end{cases}}^{\infty} \psi_3(nx-k) < \frac{1}{\pi e^{(n^{1-\alpha}-2)}} = \frac{4e^2}{\pi e^{n^{1-\alpha}}} =: c_3(\alpha, n). \tag{52}$$

**Theorem 14** *([12]) Let $[a,b] \subset \mathbb{R}$ and $n \in \mathbb{N}$, so that $\lceil na \rceil \leq \lfloor nb \rfloor$. It holds*

$$\frac{1}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \psi_3(nx-k)} < \frac{\pi}{gd(2)} \cong 2.412 =: \alpha_3, \tag{53}$$

$\forall\, x \in [a,b]$.

We make

**Remark 15** *([12])*

*(i) We have that*

$$\lim_{n\to\infty} \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \psi_3(nx-k) \neq 1, \tag{54}$$

*for at least some $x \in [a,b]$.*

*(ii) Let $[a,b] \subset \mathbb{R}$. For large $n$ we always have $\lceil na \rceil \leq \lfloor nb \rfloor$. Also $a \leq \frac{k}{n} \leq b$, iff $\lceil na \rceil \leq k \leq \lfloor nb \rfloor$.*

*In general it holds*

$$\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \psi_3(nx-k) \leq 1. \tag{55}$$

We introduce (see also [14])

$$Z_3(x_1, ..., x_N) := Z_3(x) := \prod_{i=1}^{N} \psi_3(x_i), \quad x = (x_1, ..., x_N) \in \mathbb{R}^N, \ N \in \mathbb{N}. \tag{56}$$

It has the properties:

(i) $Z_3(x) > 0, \ \forall\, x \in \mathbb{R}^N$,

(ii)

$$\sum_{k=-\infty}^{\infty} Z_3(x-k) := \sum_{k_1=-\infty}^{\infty} \sum_{k_2=-\infty}^{\infty} \dots \sum_{k_N=-\infty}^{\infty} Z_3(x_1-k_1, \dots, x_N-k_N) = 1, \tag{57}$$

where $k := (k_1, \dots, k_n) \in \mathbb{Z}^N$, $\forall\, x \in \mathbb{R}^N$,

hence

(iii)

$$\sum_{k=-\infty}^{\infty} Z_3(nx-k) = 1, \tag{58}$$

$\forall\, x \in \mathbb{R}^N$; $n \in \mathbb{N}$,

and

(iv)

$$\int_{\mathbb{R}^N} Z_3(x)\, dx = 1, \tag{59}$$

that is $Z_3$ is a multivariate density function.

(v) It is also clear that

$$\sum_{\begin{cases} k=-\infty \\ \left\| \frac{k}{n} - x \right\|_\infty > \frac{1}{n^\beta} \end{cases}}^{\infty} Z_3(nx-k) < \frac{4e^2}{\pi e^{n^{1-\beta}}} = c_3(\beta, n), \tag{60}$$

$0 < \beta < 1$, $n \in \mathbb{N} : n^{1-\beta} > 2$, $x \in \mathbb{R}^N$, $m \in \mathbb{N}$.

(vi) By Theorem 14 we get that

$$0 < \frac{1}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} Z_3(nx-k)} < \left( \frac{\pi}{gd(2)} \right)^N \cong (2.412)^N =: \gamma_3(N), \tag{61}$$

$\forall\, x \in \left( \prod_{i=1}^N [a_i, b_i] \right)$, $\ n \in \mathbb{N}$.

Furthermore it holds

$$\lim_{n \to \infty} \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} Z_3(nx-k) \neq 1, \tag{62}$$

for at least some $x \in \left( \prod_{i=1}^N [a_i, b_i] \right)$.

## 1.4 About the generalized symmetrical activation function

Here we consider the generalized symmetrical sigmoid function ([13], [22])

$$f_1(x) = \frac{x}{\left(1+|x|^{\mu}\right)^{\frac{1}{\mu}}}, \quad \mu > 0,\ x \in \mathbb{R}. \tag{63}$$

This has applications in immunology and protection from disease together with probability theory. It is also called a symmetrical protection curve.

The parameter $\mu$ is a shape parameter controling how fast the curve approaches the asymptotes for a given slope at the inflection point. When $\mu = 1$ $f_1$ is the absolute sigmoid function, and when $\mu = 2$, $f_1$ is the square root sigmoid function. When $\mu = 1.5$ the function approximates the arctangent function, when $\mu = 2.9$ it approximates the logistic function, and when $\mu = 3.4$ it approximates the error function. Parameter $\mu$ is estimated in the likelihood maximization ([22]). For more see [22].

Next we study the particular generator sigmoid function

$$f_2(x) = \frac{x}{\left(1+|x|^{\lambda}\right)^{\frac{1}{\lambda}}}, \quad \lambda \text{ is an odd number, } x \in \mathbb{R}. \tag{64}$$

We have that $f_2(0) = 0$, and

$$f_2(-x) = -f_2(x), \tag{65}$$

so $f_2$ is symmetric with respect to zero.

When $x \geq 0$, we get that ([13])

$$f_2'(x) = \frac{1}{\left(1+x^{\lambda}\right)^{\frac{\lambda+1}{\lambda}}} > 0, \tag{66}$$

that is $f_2$ is strictly increasing on $[0, +\infty)$ and $f_2$ is strictly increasing on $(-\infty, 0]$. Hence $f_2$ is strictly increasing on $\mathbb{R}$.

We also have $f_2(+\infty) = f_2(-\infty) = 1$.

Let us consider the activation function ([13]):

$$\psi_4(x) = \frac{1}{4}\left[f_2(x+1) - f_2(x-1)\right] =$$

$$\frac{1}{4}\left[\frac{(x+1)}{\left(1+|x+1|^{\lambda}\right)^{\frac{1}{\lambda}}} - \frac{(x-1)}{\left(1+|x-1|^{\lambda}\right)^{\frac{1}{\lambda}}}\right]. \tag{67}$$

Clearly it holds ([13])

$$\psi_4(x) = \psi_4(-x), \quad \forall\, x \in \mathbb{R}. \tag{68}$$

and

$$\psi_4(0) = \frac{1}{2\sqrt[\lambda]{2}}, \tag{69}$$

and $\psi_4(x) > 0$, $\forall\, x \in \mathbb{R}$.

Following [13], we have that $\psi_4$ is strictly decreasing over $[0, +\infty)$, and $\psi_4$ is strictly increasing on $(-\infty, 0]$, by $\psi_4$-symmetry with respect to $y$-axis, and $\psi_4'(0) = 0$.

Clearly it is

$$\lim_{x \to +\infty} \psi_4(x) = \lim_{x \to -\infty} \psi_4(x) = 0, \tag{70}$$

therefore the $x$-axis is the horizontal asymptote of $\psi_4(x)$.

The value

$$\psi_4(0) = \frac{1}{2\sqrt[\lambda]{2}},\ \lambda \text{ is an odd number,} \tag{71}$$

is the maximum of $\psi_4$, which is a bell shaped function.

We need

**Theorem 16** *([13]) It holds*

$$\sum_{i=-\infty}^{\infty} \psi_4(x - i) = 1, \quad \forall\, x \in \mathbb{R}. \tag{72}$$

**Theorem 17** *([13]) We have that*

$$\int_{-\infty}^{\infty} \psi_4(x)\, dx = 1. \tag{73}$$

So that $\psi_4(x)$ is a density function on $\mathbb{R}$.

We need

**Theorem 18** *([13]) Let $0 < \alpha < 1$, and $n \in \mathbb{N}$ with $n^{1-\alpha} > 2$. It holds*

$$\sum_{\begin{cases} j = -\infty \\ : |nx - j| \geq n^{1-\alpha} \end{cases}}^{\infty} \psi_4(nx - j) < \frac{1}{2\lambda\left(n^{1-\alpha} - 2\right)^{\lambda}} =: c_4(\alpha, n), \tag{74}$$

*where $\lambda \in \mathbb{N}$ is an odd number.*

We also need

**Theorem 19** *([13]) Let $[a, b] \subset \mathbb{R}$ and $n \in \mathbb{N}$ so that $\lceil na \rceil \leq \lfloor nb \rfloor$. Then*

$$\frac{1}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \psi_4(|nx - k|)} < 2\sqrt[\lambda]{1 + 2^{\lambda}} =: \alpha_4, \tag{75}$$

*where $\lambda$ is an odd number, $\forall\, x \in [a, b]$.*

We make

**Remark 20** *([13]) (1) We have that*

$$\lim_{n\to\infty}\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}\psi_4\left(nx-k\right)\neq 1,\quad \textit{for at least some } x\in\left[a,b\right]. \tag{76}$$

*(2) Let $[a,b]\subset\mathbb{R}$. For large enough $n$ we always obtain $\lceil na\rceil\leq\lfloor nb\rfloor$. Also $a\leq\frac{k}{n}\leq b$, iff $\lceil na\rceil\leq k\leq\lfloor nb\rfloor$.*

*In general it holds that*

$$\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}\psi_4\left(nx-k\right)\leq 1. \tag{77}$$

We introduce (see also [16])

$$Z_4\left(x_1,...,x_N\right):=Z_4\left(x\right):=\prod_{i=1}^{N}\psi_4\left(x_i\right),\quad x=\left(x_1,...,x_N\right)\in\mathbb{R}^N,\ N\in\mathbb{N}. \tag{78}$$

It has the properties:

(i) $Z_4\left(x\right)>0,\ \forall\ x\in\mathbb{R}^N$,

(ii)

$$\sum_{k=-\infty}^{\infty}Z_4\left(x-k\right):=\sum_{k_1=-\infty}^{\infty}\sum_{k_2=-\infty}^{\infty}...\sum_{k_N=-\infty}^{\infty}Z_4\left(x_1-k_1,...,x_N-k_N\right)=1, \tag{79}$$

where $k:=(k_1,...,k_n)\in\mathbb{Z}^N$, $\forall\ x\in\mathbb{R}^N$,

hence

(iii)

$$\sum_{k=-\infty}^{\infty}Z_4\left(nx-k\right)=1, \tag{80}$$

$\forall\ x\in\mathbb{R}^N$; $n\in\mathbb{N}$,

and

(iv)

$$\int_{\mathbb{R}^N}Z_4\left(x\right)dx=1, \tag{81}$$

that is $Z_4$ is a multivariate density function.

(v) It is clear that

$$\sum_{\begin{cases} k=-\infty \\ \left\|\frac{k}{n}-x\right\|_{\infty}>\frac{1}{n^{\beta}}\end{cases}}^{\infty}Z_4\left(nx-k\right)<\frac{1}{2\lambda\left(n^{1-\beta}-2\right)^{\lambda}}=c_4\left(\beta,n\right), \tag{82}$$

$0 < \beta < 1$, $n \in \mathbb{N} : n^{1-\beta} > 2$, $x \in \mathbb{R}^N$, $\lambda$ is odd.

(vi) By Theorem 19 we get that

$$0 < \frac{1}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} Z_4(nx-k)} < \left(2\sqrt[\lambda]{1+2^\lambda}\right)^N =: \gamma_4(N), \tag{83}$$

$\forall\ x \in \left(\prod_{i=1}^N [a_i, b_i]\right)$, $n \in \mathbb{N}$, $\lambda$ is odd.

Furthermore it holds

$$\lim_{n\to\infty} \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} Z_4(nx-k) \neq 1, \tag{84}$$

for at least some $x \in \left(\prod_{i=1}^N [a_i, b_i]\right)$.

The next integrals are of Bochner type ([25]).

We need

**Definition 21** *([10]) Let $[a,b] \subset \mathbb{R}$, $X$ be a Banach space, $\alpha > 0$; $m = \lceil \alpha \rceil \in \mathbb{N}$, ($\lceil \cdot \rceil$ is the ceiling of the number), $f : [a,b] \to X$. We assume that $f^{(m)} \in L_1([a,b], X)$. We call the Caputo-Bochner left fractional derivative of order $\alpha$:*

$$\left(D_{*a}^{\alpha} f\right)(x) := \frac{1}{\Gamma(m-\alpha)} \int_a^x (x-t)^{m-\alpha-1} f^{(m)}(t)\, dt, \quad \forall\ x \in [a,b]. \tag{85}$$

*If $\alpha \in \mathbb{N}$, we set $D_{*a}^{\alpha} f := f^{(m)}$ the ordinary $X$-valued derivative (defined similar to numerical one [28]), and also set $D_{*a}^{0} f := f$.*

By [10], $\left(D_{*a}^{\alpha} f\right)(x)$ exists almost everywhere in $x \in [a,b]$ and $D_{*a}^{\alpha} f \in L_1([a,b], X)$.

If $\left\| f^{(m)} \right\|_{L_\infty([a,b],X)} < \infty$, then by [10], $D_{*a}^{\alpha} f \in C([a,b], X)$, hence $\left\| D_{*a}^{\alpha} f \right\| \in C([a,b])$.

We mention

**Lemma 22** *([10]) Let $\alpha > 0$, $\alpha \notin \mathbb{N}$, $m = \lceil \alpha \rceil$, $f \in C^{m-1}([a,b], X)$ and $f^{(m)} \in L_\infty([a,b], X)$. Then $D_{*a}^{\alpha} f(a) = 0$.*

We mention

**Definition 23** *([10]) Let $[a,b] \subset \mathbb{R}$, $X$ be a Banach space, $\alpha > 0$, $m := \lceil \alpha \rceil$. We assume that $f^{(m)} \in L_1([a,b], X)$, where $f : [a,b] \to X$. We call the Caputo-Bochner right fractional derivative of order $\alpha$:*

$$\left(D_{b-}^{\alpha} f\right)(x) := \frac{(-1)^m}{\Gamma(m-\alpha)} \int_x^b (z-x)^{m-\alpha-1} f^{(m)}(z)\, dz, \quad \forall\ x \in [a,b]. \tag{86}$$

*We observe that $\left(D_{b-}^{m} f\right)(x) = (-1)^m f^{(m)}(x)$, for $m \in \mathbb{N}$, and $\left(D_{b-}^{0} f\right)(x) = f(x)$.*

By [10], $\left(D_{b-}^{\alpha}f\right)(x)$ exists almost everywhere on $[a,b]$ and $\left(D_{b-}^{\alpha}f\right) \in L_1([a,b],X)$.

If $\left\|f^{(m)}\right\|_{L_{\infty}([a,b],X)} < \infty$, and $\alpha \notin \mathbb{N}$, by [10], $D_{b-}^{\alpha}f \in C([a,b],X)$, hence $\left\|D_{b-}^{\alpha}f\right\| \in C([a,b])$.

We need

**Lemma 24** *([10]) Let $f \in C^{m-1}([a,b],X)$, $f^{(m)} \in L_{\infty}([a,b],X)$, $m = \lceil \alpha \rceil$, $\alpha > 0$, $\alpha \notin \mathbb{N}$. Then $D_{b-}^{\alpha}f(b) = 0$.*

We mention the left fractional vector Taylor formula

**Theorem 25** *([10]) Let $m \in \mathbb{N}$ and $f \in C^m([a,b],X)$, where $[a,b] \subset \mathbb{R}$ and $X$ is a Banach space, and let $\alpha > 0 : m = \lceil \alpha \rceil$. Then*

$$f(x) = \sum_{i=0}^{m-1} \frac{(x-a)^i}{i!} f^{(i)}(a) + \frac{1}{\Gamma(\alpha)} \int_a^x (x-z)^{\alpha-1} \left(D_{*a}^{\alpha}f\right)(z)\, dz, \tag{87}$$

$\forall\ x \in [a,b]$.

We also mention the right fractional vector Taylor formula

**Theorem 26** *([10]) Let $[a,b] \subset \mathbb{R}$, $X$ be a Banach space, $\alpha > 0$, $m = \lceil \alpha \rceil$, $f \in C^m([a,b],X)$. Then*

$$f(x) = \sum_{i=0}^{m-1} \frac{(x-b)^i}{i!} f^{(i)}(b) + \frac{1}{\Gamma(\alpha)} \int_x^b (z-x)^{\alpha-1} \left(D_{b-}^{\alpha}f\right)(z)\, dz, \tag{88}$$

$\forall\ x \in [a,b]$.

**Convention 27** *We assume that*

$$D_{*x_0}^{\alpha}f(x) = 0, \text{ for } x < x_0, \tag{89}$$

*and*

$$D_{x_0-}^{\alpha}f(x) = 0, \text{ for } x > x_0, \tag{90}$$

*for all $x, x_0 \in [a,b]$.*

We mention

**Proposition 28** *([10]) Let $f \in C^n([a,b],X)$, $n = \lceil \nu \rceil$, $\nu > 0$. Then $D_{*a}^{\nu}f(x)$ is continuous in $x \in [a,b]$.*

**Proposition 29** *([10]) Let $f \in C^m([a,b],X)$, $m = \lceil \alpha \rceil$, $\alpha > 0$. Then $D_{b-}^{\nu}f(x)$ is continuous in $x \in [a,b]$.*

We also mention

**Proposition 30** *([10]) Let $f \in C^{m-1}([a,b],X)$, $f^{(m)} \in L_\infty([a,b],X)$, $m = \lceil \alpha \rceil$, $\alpha > 0$ and*

$$D_{*x_0}^{\alpha} f(x) = \frac{1}{\Gamma(m-\alpha)} \int_{x_0}^{x} (x-t)^{m-\alpha-1} f^{(m)}(t)\,dt, \tag{91}$$

*for all $x, x_0 \in [a,b] : x \geq x_0$.*

*Then $D_{*x_0}^{\alpha} f(x)$ is continuous in $x_0$.*

**Proposition 31** *([10]) Let $f \in C^{m-1}([a,b],X)$, $f^{(m)} \in L_\infty([a,b],X)$, $m = \lceil \alpha \rceil$, $\alpha > 0$ and*

$$D_{x_0-}^{\alpha} f(x) = \frac{(-1)^m}{\Gamma(m-\alpha)} \int_{x}^{x_0} (\zeta-x)^{m-\alpha-1} f^{(m)}(\zeta)\,d\zeta, \tag{92}$$

*for all $x, x_0 \in [a,b] : x_0 \geq x$.*

*Then $D_{x_0-}^{\alpha} f(x)$ is continuous in $x_0$.*

**Corollary 32** *([10]) Let $f \in C^m([a,b],X)$, $m = \lceil \alpha \rceil$, $\alpha > 0$, $x, x_0 \in [a,b]$. Then $D_{*x_0}^{a} f(x)$, $D_{x_0-}^{a} f(x)$ are jointly continuous functions in $(x, x_0)$ from $[a,b]^2$ into $X$, $X$ is a Banach space.*

We make

**Remark 33** *([10], pp. 263-266) Let $\left(\mathbb{R}^N, \|\cdot\|_p\right)$, $N \in \mathbb{N}$; where $\|\cdot\|_p$ is the $L_p$-norm, $1 \leq p \leq \infty$. $\mathbb{R}^N$ is a Banach space, and $\left(\mathbb{R}^N\right)^{j_*}$ denotes the $j_*$-fold product space $\mathbb{R}^N \times ... \times \mathbb{R}^N$ endowed with the max-norm $\|x\|_{(\mathbb{R}^N)^{j_*}} := \max_{1 \leq \lambda \leq j_*} \|x_\lambda\|_p$, where $x := (x_1, ..., x_{j_*}) \in \left(\mathbb{R}^N\right)^{j_*}$.*

*Let $\left(X, \|\cdot\|_\gamma\right)$ be a general Banach space. Then the space $L_{j_*} := L_{j_*}\left(\left(\mathbb{R}^N\right)^{j_*}, X\right)$ of all $j_*$-multilinear continuous maps $g : \left(\mathbb{R}^N\right)^{j_*} \to X$, $j_* = 1, ..., m$, is a Banach space with norm*

$$\|g\| := \|g\|_{L_{j_*}} := \sup_{\left(\|x\|_{(\mathbb{R}^N)^{j_*}} = 1\right)} \|g(x)\|_\gamma = \sup \frac{\|g(x)\|_\gamma}{\|x_1\|_p \dots \|x_{j_*}\|_p}. \tag{93}$$

*Let $M$ be a non-empty convex and compact subset of $\mathbb{R}^N$ and $x_0 \in M$ is fixed.*

*Let $O$ be an open subset of $\mathbb{R}^N : M \subset O$. Let $f : O \to X$ be a continuous function, whose Fréchet derivatives (see [27]) $f^{(j_*)} : O \to L_{j_*} = L_{j_*}\left(\left(\mathbb{R}^N\right)^{j_*}; X\right)$ exist and are continuous for $1 \leq j_* \leq \overline{m}$, $\overline{m} \in \mathbb{N}$.*

*Call $(x - x_0)^{j_*} := (x - x_0, ..., x - x_0) \in \left(R^N\right)^{j_*}$, $x \in M$.*

*We will work with $f|_M$.*

*Then, by Taylor's formula ([18], [27], p. 124), we get*

$$f(x)=\sum_{j_*=0}^{\overline{m}-1}\frac{f^{(j_*)}(x_0)(x-x_0)^{j_*}}{j_*!}+R_{\overline{m}}(x,x_0),\quad \textit{all } x\in M, \tag{94}$$

*where the remainder is the Riemann integral*

$$R_{\overline{m}}(x,x_0):=\int_0^1\frac{(1-u)^{\overline{m}-1}}{(\overline{m}-1)!}f^{(\overline{m})}(x_0+u(x-x_0))(x-x_0)^{\overline{m}}\,du, \tag{95}$$

*here we set* $f^{(0)}(x_0)(x-x_0)^0=f(x_0)$.

*We obtain*

$$\left\|f^{(\overline{m})}(x_0+u(x-x_0))(x-x_0)^{\overline{m}}\right\|_{\gamma}\leq$$

$$\left\|f^{(\overline{m})}(x_0+u(x-x_0))\right\|\|x-x_0\|_p^{\overline{m}}\leq\left\|\left\|f^{(\overline{m})}\right\|\right\|_{\infty}\|x-x_0\|_p^{\overline{m}}, \tag{96}$$

*and*

$$\|R_{\overline{m}}(x,x_0)\|_{\gamma}\leq\frac{\left\|\left\|f^{(\overline{m})}\right\|\right\|_{\infty}}{m!}\|x-x_0\|_p^{\overline{m}}. \tag{97}$$

Let $\left(X,\|\cdot\|_{\gamma}\right)$ be a general Banach space.

We will study the following neural network operators.

**Definition 34** *Let* $f\in C([a,b],X)$, $n\in\mathbb{N}$. *We set*

$${}_jA_n(f,x):=\frac{\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}f\left(\frac{k}{n}\right)\psi_j(nx-k)}{\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}\psi_j(nx-k)},\ \ \forall\ x\in[a.b],\ j=1,2,3,4. \tag{98}$$

*These are univariate neural network operators.*

**Definition 35** *Let* $f\in C\left(\prod_{i=1}^{N}[a_i,b_i],X\right)$ *and* $n\in\mathbb{N}$ *such that* $\lceil na_i\rceil\leq\lfloor nb_i\rfloor$, $i=1,...,N$. *We will study the following multivariate linear neural network operators* ($x:=(x_1,...,x_N)\in\left(\prod_{i=1}^{N}[a_i,b_i]\right)$)

$${}_jH_n(f,x):={}_jH_n(f,x_1,...,x_N):=\frac{\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}f\left(\frac{k}{n}\right)Z_j(nx-k)}{\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}Z_j(nx-k)}= \tag{99}$$

$$\frac{\sum_{k_1=\lceil na_1\rceil}^{\lfloor nb_1\rfloor}\sum_{k_2=\lceil na_2\rceil}^{\lfloor nb_2\rfloor}...\sum_{k_N=\lceil na_N\rceil}^{\lfloor nb_N\rfloor}f\left(\frac{k_1}{n},...,\frac{k_N}{n}\right)\left(\prod_{i=1}^{N}\psi_j(nx_i-k_i)\right)}{\prod_{i=1}^{N}\left(\sum_{k_i=\lceil na_i\rceil}^{\lfloor nb_i\rfloor}\psi_j(nx_i-k_i)\right)},$$

*for* $j=1,2,3,4$.

*For large enough* $n$ *we always obtain* $\lceil na_i\rceil\leq\lfloor nb_i\rfloor$, $i=1,...,N$. *Also* $a_i\leq\frac{k_i}{n}\leq b_i$, *iff* $\lceil na_i\rceil\leq k_i\leq\lfloor nb_i\rfloor$, $i=1,...,N$.

In this article first we find fractional Voronskaya type asymptotic expansions for ${}_{j}A_{n}\left(f,x\right)$, $x\in\left[a,b\right]$, then we find multivariate Voronskaya type asymptotic expansions for ${}_{j}H_{n}\left(f,x\right)$, $x\in\left(\prod_{i=1}^{N}\left[a_{i},b_{i}\right]\right)$; $n\in\mathbb{N}$; $j=1,2,3,4$.

Our considered neural networks here are of one hidden layer.

For earlier motivational neural networks related work, see [1] - [10]. For neural networks in general, read [23], [24] and [26].

# 2 Main Results

We present our first univariate main result, as Voronovskaya type asymptotic expansion.

**Theorem 36** *Let* $\left(X,\left\Vert \cdot\right\Vert _{\gamma}\right)$ *be a Banach space,* $0<\beta<\frac{1}{2}$ *and* $0<\alpha\leq\frac{1-\beta}{\beta}$, $N\in\mathbb{N}:N=\left\lceil \alpha\right\rceil$, $f\in C^{N}\left(\left[a,b\right],X\right)$, $x\in\left[a,b\right]$, $n\in\mathbb{N}$ *large enough :* $n^{1-\beta}>2$. *Assume that* $\left\Vert \left\Vert D_{x-}^{\alpha}f\right\Vert _{\gamma}\right\Vert _{\infty,\left[a,x\right]}$, $\left\Vert \left\Vert D_{*x}^{\alpha}f\right\Vert _{\gamma}\right\Vert _{\infty,\left[x,b\right]}\leq M$, $M>0$. *Then*

$$
{}_{j}A_{n}\left(f,x\right)-f\left(x\right)=\sum_{j_{*}=1}^{N-1}\frac{f^{\left(j_{*}\right)}\left(x\right)}{j_{*}!}\ {}_{j}A_{n}\left(\left(\cdot-x\right)^{j_{*}}\right)\left(x\right)+o\left(\frac{1}{n^{\beta\left(\alpha-\varepsilon\right)}}\right),\tag{100}
$$

*where* $0<\varepsilon\leq\alpha$.

*If* $N=1$, *the sum in (100) collapses.*

*The last (100) implies that*

$$
n^{\beta\left(\alpha-\varepsilon\right)}\left[{}_{j}A_{n}\left(f,x\right)-f\left(x\right)-\sum_{j_{*}=1}^{N-1}\frac{f^{\left(j_{*}\right)}\left(x\right)}{j_{*}!}\ {}_{j}A_{n}\left(\left(\cdot-x\right)^{j_{*}}\right)\left(x\right)\right]\rightarrow0,\tag{101}
$$

*as* $n\rightarrow\infty$, $0<\varepsilon\leq\alpha$.

*When* $N=1$, *or* $f^{\left(j_{*}\right)}\left(x\right)=0$, $j_{*}=1,...,N-1$, *then we derive that*

$$
n^{\beta\left(\alpha-\varepsilon\right)}\left[{}_{j}A_{n}\left(f,x\right)-f\left(x\right)\right]\rightarrow0
$$

*as* $n\rightarrow\infty$, $0<\varepsilon\leq\alpha$. *Of great interest is the case of* $\alpha=\frac{1}{2}$.

*Above it is* $j=1,2,3,4$.

**Proof.** From Theorem 25 (87), we get by the left Caputo fractional vector Taylor formula that

$$
f\left(\frac{k}{n}\right)=\sum_{j_{*}=0}^{N-1}\frac{f^{\left(j_{*}\right)}\left(x\right)}{j_{*}!}\left(\frac{k}{n}-x\right)^{j_{*}}+\frac{1}{\Gamma\left(\alpha\right)}\int_{x}^{\frac{k}{n}}\left(\frac{k}{n}-J\right)^{\alpha-1}D_{*x}^{\alpha}f\left(J\right)dJ,\tag{102}
$$

for all $x\leq\frac{k}{n}\leq b$.

Also from Theorem 26 (88), using the right Caputo fractional vector Taylor formula we get

$$f\left(\frac{k}{n}\right)=\sum_{j_*=0}^{N-1}\frac{f^{(j_*)}(x)}{j_*!}\left(\frac{k}{n}-x\right)^{j_*}+\frac{1}{\Gamma(\alpha)}\int_{\frac{k}{n}}^{x}\left(J-\frac{k}{n}\right)^{\alpha-1}D_{x-}^{\alpha}f(J)\,dJ, \tag{103}$$

for all $a\leq\frac{k}{n}\leq x$.

We call

$$W_j(x):=\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}\psi_j(nx-k),\quad j=1,2,3,4. \tag{104}$$

Hence we have

$$\frac{f\left(\frac{k}{n}\right)\psi_j(nx-k)}{W_j(x)}=\sum_{j_*=0}^{N-1}\frac{f^{(j_*)}(x)}{j_*!}\frac{\psi_j(nx-k)}{W_j(x)}\left(\frac{k}{n}-x\right)^{j_*}+ \tag{105}$$

$$\frac{\psi_j(nx-k)}{W_j(x)\Gamma(\alpha)}\int_x^{\frac{k}{n}}\left(\frac{k}{n}-J\right)^{\alpha-1}D_{*x}^{\alpha}f(J)\,dJ,$$

all $x\leq\frac{k}{n}\leq b$, iff $\lceil nx\rceil\leq k\leq\lfloor nb\rfloor$, and

$$\frac{f\left(\frac{k}{n}\right)\psi_j(nx-k)}{W_j(x)}=\sum_{j_*=0}^{N-1}\frac{f^{(j_*)}(x)}{j_*!}\frac{\psi_j(nx-k)}{W_j(x)}\left(\frac{k}{n}-x\right)^{j_*}+ \tag{106}$$

$$\frac{\psi_j(nx-k)}{W_j(x)\Gamma(\alpha)}\int_{\frac{k}{n}}^{x}\left(J-\frac{k}{n}\right)^{\alpha-1}D_{x-}^{\alpha}f(J)\,dJ,$$

for all $a\leq\frac{k}{n}\leq x$, iff $\lceil na\rceil\leq k\leq\lfloor nx\rfloor$.

We have that $\lceil nx\rceil\leq\lfloor nx\rfloor+1$.

Therefore it holds

$$\sum_{k=\lfloor nx\rfloor+1}^{\lfloor nb\rfloor}\frac{f\left(\frac{k}{n}\right)\psi_j(nx-k)}{W_j(x)}=\sum_{j_*=0}^{N-1}\frac{f^{(j_*)}(x)}{j_*!}\sum_{k=\lfloor nx\rfloor+1}^{\lfloor nb\rfloor}\frac{\psi_j(nx-k)\left(\frac{k}{n}-x\right)^{j_*}}{W_j(x)}+ \tag{107}$$

$$\frac{1}{\Gamma(\alpha)}\left(\frac{\sum_{k=\lfloor nx\rfloor+1}^{\lfloor nb\rfloor}\psi_j(nx-k)}{W_j(x)}\int_x^{\frac{k}{n}}\left(\frac{k}{n}-J\right)^{\alpha-1}D_{*x}^{\alpha}f(J)\,dJ\right),$$

and

$$\sum_{k=\lceil na\rceil}^{\lfloor nx\rfloor}f\left(\frac{k}{n}\right)\frac{\psi_j(nx-k)}{W_j(x)}=\sum_{j_*=0}^{N-1}\frac{f^{(j_*)}(x)}{j_*!}\sum_{k=\lceil na\rceil}^{\lfloor nx\rfloor}\frac{\psi_j(nx-k)}{W_j(x)}\left(\frac{k}{n}-x\right)^{j_*}+ \tag{108}$$

$$\frac{1}{\Gamma(\alpha)}\left(\sum_{k=\lceil na\rceil}^{\lfloor nx\rfloor}\frac{\psi_j(nx-k)}{W_j(x)}\int_{\frac{k}{n}}^{x}\left(J-\frac{k}{n}\right)^{\alpha-1}D_{x-}^{\alpha}f(J)\,dJ\right).$$

Adding the last two equalities (107) and (108) we obtain

$$_jA_n(f,x)=\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}f\left(\frac{k}{n}\right)\frac{\psi_j(nx-k)}{W_j(x)}= \tag{109}$$

$$\sum_{j_*=0}^{N-1}\frac{f^{(j_*)}(x)}{j_*!}\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}\frac{\psi_j(nx-k)}{W_j(x)}\left(\frac{k}{n}-x\right)^{j_*}+$$

$$\frac{1}{\Gamma(\alpha)W_j(x)}\left\{\sum_{k=\lceil na\rceil}^{\lfloor nx\rfloor}\psi_j(nx-k)\int_{\frac{k}{n}}^{x}\left(J-\frac{k}{n}\right)^{\alpha-1}D_{x-}^{\alpha}f(J)\,dJ+\right.$$

$$\left.\sum_{k=\lfloor nx\rfloor+1}^{\lfloor nb\rfloor}\psi_j(nx-k)\int_{x}^{\frac{k}{n}}\left(\frac{k}{n}-J\right)^{\alpha-1}\left(D_{*x}^{\alpha}f(J)\right)dJ\right\}.$$

So we have derived ($j=1,2,3,4$)

$$\theta_j(x):=\ _jA_n(f,x)-f(x)-\sum_{j_*=1}^{N-1}\frac{f^{(j_*)}(x)}{j_*!}\ _jA_n\left((\cdot-x)^{j_*}\right)(x)=\theta_{jn}^{*}(x), \tag{110}$$

where

$$\theta_{jn}^{*}(x):=\frac{1}{\Gamma(\alpha)W_j(x)}\left\{\sum_{k=\lceil na\rceil}^{\lfloor nx\rfloor}\psi_j(nx-k)\int_{\frac{k}{n}}^{x}\left(J-\frac{k}{n}\right)^{\alpha-1}D_{x-}^{\alpha}f(J)\,dJ\right.$$

$$\left.+\sum_{k=\lfloor nx\rfloor+1}^{\lfloor nb\rfloor}\psi_j(nx-k)\int_{x}^{\frac{k}{n}}\left(\frac{k}{n}-J\right)^{\alpha-1}D_{*x}^{\alpha}f(J)\,dJ\right\}. \tag{111}$$

We set

$$_j\theta_{1n}^{*}(x):=\frac{1}{\Gamma(\alpha)}\left(\frac{\sum_{k=\lceil na\rceil}^{\lfloor nx\rfloor}\psi_j(nx-k)}{W_j(x)}\int_{\frac{k}{n}}^{x}\left(J-\frac{k}{n}\right)^{\alpha-1}D_{x-}^{\alpha}f(J)\,dJ\right), \tag{112}$$

and

$$_j\theta_{2n}^{*}:=\frac{1}{\Gamma(\alpha)}\left(\frac{\sum_{k=\lfloor nx\rfloor+1}^{\lfloor nb\rfloor}\psi_j(nx-k)}{W_j(x)}\int_{x}^{\frac{k}{n}}\left(\frac{k}{n}-J\right)^{\alpha-1}D_{*x}^{\alpha}f(J)\,dJ\right), \tag{113}$$

i.e.

$$\theta_{jn}^{*}(x) = {}_{j}\theta_{1n}^{*}(x) + {}_{j}\theta_{2n}^{*}(x), \quad j = 1,2,3,4. \tag{114}$$

We assume $b-a > \frac{1}{n^{\beta}}$, $0 < \beta < 1$, which is always the case for large enough $n \in \mathbb{N}$, that is when $n > \left\lceil (b-a)^{-\frac{1}{\beta}} \right\rceil$. It is always true that either $\left|\frac{k}{n} - x\right| \leq \frac{1}{n^{\beta}}$ or $\left|\frac{k}{n} - x\right| > \frac{1}{n^{\beta}}$.

For $k = \lceil na \rceil, ..., \lfloor nx \rfloor$, we consider

$$\gamma_{1k} := \left\| \int_{\frac{k}{n}}^{x} \left(J - \frac{k}{n}\right)^{\alpha-1} D_{x-}^{\alpha} f(J)\, dJ \right\|_{\gamma} \leq \tag{115}$$

$$\int_{\frac{k}{n}}^{x} \left(J - \frac{k}{n}\right)^{\alpha-1} \left\|D_{x-}^{\alpha} f(J)\right\|_{\gamma} dJ$$

$$\leq \left\| \left\|D_{x-}^{\alpha} f\right\|_{\gamma} \right\|_{\infty,[a,x]} \frac{\left(x - \frac{\kappa}{n}\right)^{\alpha}}{\alpha} \leq \left\| \left\|D_{x-}^{\alpha} f\right\|_{\gamma} \right\|_{\infty,[a,x]} \frac{(x-a)^{\alpha}}{\alpha}. \tag{116}$$

That is

$$\gamma_{1k} \leq \left\| \left\|D_{x-}^{\alpha} f\right\|_{\gamma} \right\|_{\infty,[a,x]} \frac{(x-a)^{\alpha}}{\alpha}, \tag{117}$$

for $k = \lceil na \rceil, ..., \lfloor nx \rfloor$.

Also we have in case of $\left|\frac{k}{n} - x\right| \leq \frac{1}{n^{\beta}}$ that

$$\gamma_{1k} \leq \int_{\frac{k}{n}}^{x} \left(J - \frac{k}{n}\right)^{\alpha-1} \left\|D_{x-}^{\alpha} f(J)\right\|_{\gamma} dJ \tag{118}$$

$$\leq \left\| \left\|D_{x-}^{\alpha} f\right\|_{\gamma} \right\|_{\infty,[a,x]} \frac{\left(x - \frac{\kappa}{n}\right)^{\alpha}}{\alpha} \leq \left\| \left\|D_{x-}^{\alpha} f\right\|_{\gamma} \right\|_{\infty,[a,x]} \frac{1}{n^{\alpha\beta}\alpha}.$$

So that, when $\left|x - \frac{k}{n}\right| \leq \frac{1}{n^{\beta}}$, we get

$$\gamma_{1k} \leq \left\| \left\|D_{x-}^{\alpha} f\right\|_{\gamma} \right\|_{\infty,[a,x]} \frac{1}{\alpha n^{\alpha\beta}}. \tag{119}$$

Therefore

$$\left\|{}_{j}\theta_{1n}^{*}(x)\right\|_{\gamma} \leq \frac{1}{\Gamma(\alpha)} \left( \frac{\sum_{k=\lceil na \rceil}^{\lfloor nx \rfloor} \psi_j(nx-k)}{W_j(x)} \gamma_{1k} \right) = \frac{1}{\Gamma(\alpha)} \cdot$$

$$\left\{ \frac{\sum_{\begin{cases} k = \lceil na \rceil \\ : \left|\frac{k}{n} - x\right| \leq \frac{1}{n^{\beta}} \end{cases}}^{\lfloor nx \rfloor} \psi_j(nx-k)}{W_j(x)} \gamma_{1k} + \frac{\sum_{\begin{cases} k = \lceil na \rceil \\ : \left|\frac{k}{n} - x\right| > \frac{1}{n^{\beta}} \end{cases}}^{\lfloor nx \rfloor} \psi_j(nx-k)}{W_j(x)} \gamma_{1k} \right\}$$

$$\leq \frac{1}{\Gamma(\alpha)} \left\{ \left( \frac{\sum_{\begin{cases} k = \lceil na \rceil \\ : \left| \frac{k}{n} - x \right| \leq \frac{1}{n^{\beta}} \end{cases}}^{\lfloor nx \rfloor} \psi_j(nx-k)}{W_j(x)} \right) \left\| \left\| D_{x-}^{\alpha} f \right\|_{\gamma} \right\|_{\infty,[a,x]} \frac{1}{\alpha n^{\alpha\beta}} + \right.$$

$$\left. \frac{1}{W_j(x)} \left( \sum_{\begin{cases} k = \lceil na \rceil \\ : \left| \frac{k}{n} - x \right| > \frac{1}{n^{\beta}} \end{cases}}^{\lfloor nx \rfloor} \psi_j(nx-k) \right) \left\| \left\| D_{x-}^{\alpha} f \right\|_{\gamma} \right\|_{\infty,[a,x]} \frac{(x-a)^{\alpha}}{\alpha} \right\} \quad (120)$$

(by (12), (13); (32), (33); (52), (53); (74), (75))

$$\leq \frac{\left\| \left\| D_{x-}^{\alpha} f \right\|_{\gamma} \right\|_{\infty,[a,x]}}{\Gamma(\alpha+1)} \left\{ \frac{1}{n^{\alpha\beta}} + \alpha_j c_j(\beta, n)(x-a)^{\alpha} \right\}.$$

Therefore we proved

$$\left\| {}_j\theta_{1n}^{*}(x) \right\|_{\gamma} \leq \frac{\left\| \left\| D_{x-}^{\alpha} f \right\|_{\gamma} \right\|_{\infty,[a,x]}}{\Gamma(\alpha+1)} \left\{ \frac{1}{n^{\alpha\beta}} + \alpha_j c_j(\beta, n)(x-a)^{\alpha} \right\}. \quad (121)$$

But for large enough $n \in \mathbb{N}$ we get

$$\left\| {}_j\theta_{1n}^{*}(x) \right\|_{\gamma} \leq \frac{2 \left\| \left\| D_{x-}^{\alpha} f \right\|_{\gamma} \right\|_{\infty,[a,x]}}{\Gamma(\alpha+1) n^{\alpha\beta}}. \quad (122)$$

Similarly, we have that

$$\gamma_{2k} := \left\| \int_x^{\frac{k}{n}} \left( \frac{k}{n} - J \right)^{\alpha-1} D_{*x}^{\alpha} f(J) \, dJ \right\|_{\gamma} \leq$$

$$\int_x^{\frac{k}{n}} \left( \frac{k}{n} - J \right)^{\alpha-1} \left\| D_{*x}^{\alpha} f(J) \right\|_{\gamma} dJ \leq$$

$$\left\| \left\| D_{*x}^{\alpha} f \right\|_{\gamma} \right\|_{\infty,[x,b]} \frac{\left( \frac{k}{n} - x \right)^{\alpha}}{\alpha} \leq \left\| \left\| D_{*x}^{\alpha} f \right\|_{\gamma} \right\|_{\infty,[x,b]} \frac{(b-x)^{\alpha}}{\alpha}. \quad (123)$$

That is

$$\gamma_{2k} \leq \left\| \left\| D_{*x}^{\alpha} f \right\|_{\gamma} \right\|_{\infty,[x,b]} \frac{(b-x)^{\alpha}}{\alpha}, \quad (124)$$

for $k = \lfloor nx \rfloor + 1, ..., \lfloor nb \rfloor$.

Also we have in case of $\left|\frac{k}{n}-x\right| \leq \frac{1}{n^{\beta}}$ that

$$\gamma_{2k} \leq \frac{\left\|\left\|D_{*x}^{\alpha}f\right\|_{\gamma}\right\|_{\infty,[x,b]}}{\alpha n^{\alpha\beta}}. \tag{125}$$

Consequently it holds

$$\left\|{}_{j}\theta_{2n}^{*}(x)\right\|_{\gamma} \leq \frac{1}{\Gamma(\alpha)}\left(\frac{\sum_{k=\lfloor nx\rfloor+1}^{\lfloor nb\rfloor}\psi_{j}(nx-k)}{W_{j}(x)}\gamma_{2k}\right) =$$

$$\frac{1}{\Gamma(\alpha)}\left\{\left(\frac{\sum_{\left\{\begin{array}{l}k=\lfloor nx\rfloor+1\\ :\left|\frac{k}{n}-x\right|\leq\frac{1}{n^{\beta}}\end{array}\right.}^{\lfloor nb\rfloor}\psi_{j}(nx-k)}{W_{j}(x)}\right)\frac{\left\|\left\|D_{*x}^{\alpha}f\right\|_{\gamma}\right\|_{\infty,[x,b]}}{\alpha n^{\alpha\beta}}+\right.$$

$$\left.\frac{1}{W_{j}(x)}\left(\sum_{\left\{\begin{array}{l}k=\lfloor nx\rfloor+1\\ :\left|\frac{k}{n}-x\right|>\frac{1}{n^{\beta}}\end{array}\right.}^{\lfloor nb\rfloor}\psi_{j}(nx-k)\right)\left\|\left\|D_{*x}^{\alpha}f\right\|_{\gamma}\right\|_{\infty,[x,b]}\frac{(b-x)^{\alpha}}{\alpha}\right\} \leq$$

$$\frac{\left\|\left\|D_{*x}^{\alpha}f\right\|_{\gamma}\right\|_{\infty,[x,b]}}{\Gamma(\alpha+1)}\left\{\frac{1}{n^{\alpha\beta}}+\alpha_{j}c_{j}(\beta,n)(b-x)^{\alpha}\right\}. \tag{126}$$

That is

$$\left\|{}_{j}\theta_{2n}^{*}(x)\right\|_{\gamma} \leq \frac{\left\|\left\|D_{*x}^{\alpha}f\right\|_{\gamma}\right\|_{\infty,[x,b]}}{\Gamma(\alpha+1)}\left\{\frac{1}{n^{\alpha\beta}}+\alpha_{j}c_{j}(\beta,n)(b-x)^{\alpha}\right\}. \tag{127}$$

But for large enough $n \in \mathbb{N}$ we get

$$\left\|{}_{j}\theta_{2n}^{*}(x)\right\|_{\gamma} \leq \frac{2\left\|\left\|D_{*x}^{\alpha}f\right\|_{\gamma}\right\|_{\infty,[x,b]}}{\Gamma(\alpha+1)\,n^{\alpha\beta}}. \tag{128}$$

Since $\left\|\left\|D_{x-}^{\alpha}f\right\|_{\gamma}\right\|_{\infty,[a,x]}$, $\left\|\left\|D_{*x}^{\alpha}f\right\|_{\gamma}\right\|_{\infty,[x,b]} \leq M$, $M>0$, we derive

$$\left\|\theta_{jn}^{*}(x)\right\|_{\gamma} \leq \left\|{}_{j}\theta_{1n}^{*}(x)\right\|_{\gamma}+\left\|{}_{j}\theta_{2n}^{*}(x)\right\|_{\gamma} \overset{\text{(by (122), (128))}}{\leq} \frac{4M}{\Gamma(\alpha+1)\,n^{\alpha\beta}}. \tag{129}$$

That is for large enough $n \in \mathbb{N}$ we get

$$\left\|\theta_{j}(x)\right\|_{\gamma} = \left\|\theta_{jn}^{*}(x)\right\|_{\gamma} \leq \left(\frac{4M}{\Gamma(\alpha+1)}\right)\left(\frac{1}{n^{\alpha\beta}}\right), \tag{130}$$

resulting to

$$\|\theta(x)\|_{\gamma} = O\left(\frac{1}{n^{\alpha\beta}}\right), \tag{131}$$

and

$$\|\theta(x)\|_{\gamma} = o(1). \tag{132}$$

And, letting $0 < \varepsilon \leq \alpha$, we derive

$$\frac{\|\theta(x)\|_{\gamma}}{\left(\frac{1}{n^{\beta(\alpha-\varepsilon)}}\right)} \leq \left(\frac{4M}{\Gamma(\alpha+1)}\right)\left(\frac{1}{n^{\beta\varepsilon}}\right) \to 0, \tag{133}$$

as $n \to \infty$.

I.e.

$$\|\theta(x)\|_{\gamma} = o\left(\frac{1}{n^{\beta(\alpha-\varepsilon)}}\right), \tag{134}$$

proving the claim. ■

It follows a multivariate Voronovskaya type asymptotic expansion.

**Theorem 37** *Let $\left(X, \|\cdot\|_{\gamma}\right)$ be a Banach space, $\overline{m} \in \mathbb{N}$ such that $\overline{m} \leq \frac{1-\beta}{\beta}$, where $0 < \beta < \frac{1}{2}$. Let $f \in C^{\overline{m}}\left(\prod_{i=1}^{N}[a_i, b_i], X\right)$ ($\overline{m}$-times continuously Fréchet differentiable functions), $x \in \prod_{i=1}^{N}[a_i, b_i]$, and $n \in \mathbb{N} : n^{1-\beta} > 2$; $j = 1, 2, 3, 4$. Then*

$$ {}_{j}H_n(f, x) - f(x) = $$

$$\sum_{j_*=1}^{\overline{m}-1} \frac{1}{j_*!}\ {}_{j}H_n\left(f^{(j_*)}(x)(\cdot - x)^{j_*}, x\right) + o\left(\frac{1}{n^{\beta(\overline{m}-\varepsilon)}}\right), \tag{135}$$

*where $0 < \varepsilon \leq \overline{m}$.*

*If $\overline{m} = 1$, the sum in (135) collapses.*

*The last (135) implies that*

$$n^{\beta(\overline{m}-\varepsilon)}\left[{}_{j}H_n(f, x) - f(x) - \sum_{j_*=1}^{\overline{m}-1} \frac{1}{j_*!}\ {}_{j}H_n\left(f^{(j_*)}(x)(\cdot - x)^{j_*}, x\right)\right] \to 0, \tag{136}$$

*as $n \to \infty$, $0 < \varepsilon \leq \overline{m}$.*

*When $\overline{m} = 1$, or $f^{(j_*)}(x) = 0$, $j_* = 1, ..., \overline{m} - 1$, then we derive that*

$$n^{\beta(\overline{m}-\varepsilon)}\left[{}_{j}H_n(f, x) - f(x)\right] \to 0, \tag{137}$$

*as $n \to \infty$, $0 < \varepsilon \leq \overline{m}$.*

*Above it is $j = 1, 2, 3, 4$.*

**Proof.** We have that

$$f\left(\frac{k}{n}\right)-f(x)=\sum_{j_*=1}^{\overline{m}-1}\frac{f^{(j_*)}(x)}{j_*!}\left(\frac{k}{n}-x\right)^{j_*}+R_{\overline{m}}\left(\frac{k}{n},x\right), \tag{138}$$

where

$$R_{\overline{m}}\left(\frac{k}{n},x\right):=\int_0^1\frac{(1-u)^{\overline{m}-1}}{(\overline{m}-1)!}f^{(\overline{m})}\left(x+u\left(\frac{k}{n}-x\right)\right)\left(\frac{k}{n}-x\right)^{\overline{m}}du, \tag{139}$$

here we set $f^{(0)}(x)\left(\frac{k}{n}-x\right)^0=f(x)$.

By (97) we get that

$$\left\|R_{\overline{m}}\left(\frac{k}{n},x\right)\right\|_\gamma\leq\frac{\left\|\left\|f^{(\overline{m})}\right\|\right\|_\infty}{\overline{m}!}\left\|\frac{k}{n}-x\right\|_\infty^{\overline{m}} \tag{140}$$

$$\leq\frac{\left\|\left\|f^{(\overline{m})}\right\|\right\|_\infty}{\overline{m}!}\|b-a\|_\infty^{\overline{m}}.$$

Call

$$V_j(x):=\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}Z_j(nx-k), \tag{141}$$

for $j=1,2,3,4$.

Hence, we have

$$\begin{gathered}{}_jU_n(x):=\frac{\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}Z_j(nx-k)R_{\overline{m}}\left(\frac{k}{n},x\right)}{V_j(x)}=\\ \frac{\sum_{\substack{k=\lceil na\rceil\\ :\left\|\frac{k}{n}-x\right\|_\infty\leq\frac{1}{n^\beta}}}^{\lfloor nb\rfloor}Z_j(nx-k)R_{\overline{m}}\left(\frac{k}{n},x\right)}{V_j(x)}+\\ \frac{\sum_{\substack{k=\lceil na\rceil\\ :\left\|\frac{k}{n}-x\right\|_\infty>\frac{1}{n^\beta}}}^{\lfloor nb\rfloor}Z_j(nx-k)R_{\overline{m}}\left(\frac{k}{n},x\right)}{V_j(x)}.\end{gathered} \tag{142}$$

Therefore, we obtain $(j=1,2,3,4)$

$$\left\|{}_jU_n(x)\right\|_j\leq\left(\sum_{\substack{k=\lceil na\rceil\\ :\left\|\frac{k}{n}-x\right\|_\infty\leq\frac{1}{n^\beta}}}^{\lfloor nb\rfloor}\frac{Z_j(nx-k)}{V_j(x)}\right)\frac{\left\|\left\|f^{(\overline{m})}\right\|\right\|_\infty}{\overline{m}!}\frac{1}{n^{\overline{m}\beta}}+$$

$$\left( \sum_{\left\{ \begin{array}{c} k=\lceil na \rceil \\ : \left\| \frac{k}{n} - x \right\|_{\infty} > \frac{1}{n^{\beta}} \end{array} \right.}^{\lfloor nb \rfloor} \frac{Z_j (nx-k)}{V_j (x)} \right) \frac{\left\| \left\| f^{(\overline{m})} \right\| \right\|_{\infty}}{\overline{m}!} \left\| b-a \right\|_{\infty}^{\overline{m}} \tag{143}$$

(by (20), (21); (40), (41); (60), (61); (82), (83))

$$\leq \frac{\left\| \left\| f^{(\overline{m})} \right\| \right\|_{\infty}}{\overline{m}!} \left[ \frac{1}{n^{\beta \overline{m}}} + \gamma_j (N) c_j (\beta, n) \left\| b-a \right\|_{\infty}^{\overline{m}} \right].$$

Consequently, we get that

$$\left\| {}_jU_n (x) \right\|_j \leq \frac{\left\| \left\| f^{(\overline{m})} \right\| \right\|_{\infty}}{\overline{m}!} \left[ \frac{1}{n^{\beta \overline{m}}} + \gamma_j (N) c_j (\beta, n) \left\| b-a \right\|_{\infty}^{\overline{m}} \right]. \tag{144}$$

For large enough $n \in \mathbb{N}$, we get

$$\left\| {}_jU_n (x) \right\|_j \leq \frac{2 \left\| \left\| f^{(\overline{m})} \right\| \right\|_{\infty}}{\overline{m}!} \left( \frac{1}{n^{\beta \overline{m}}} \right). \tag{145}$$

That is

$$\left\| {}_jU_n (x) \right\|_j = O \left( \frac{1}{n^{\beta \overline{m}}} \right), \tag{146}$$

and

$$\left\| {}_jU_n (x) \right\|_j = o (1). \tag{147}$$

And, letting $0 < \varepsilon \leq \overline{m}$, we derive

$$\frac{\left\| {}_jU_n (x) \right\|_j}{\left( \frac{1}{n^{\beta (\overline{m} - \varepsilon)}} \right)} \leq \left( \frac{2 \left\| \left\| f^{(\overline{m})} \right\| \right\|_{\infty}}{\overline{m}!} \right) \frac{1}{n^{\beta \varepsilon}} \to 0, \tag{148}$$

as $n \to \infty$.

I.e.

$$\left\| {}_jU_n (x) \right\|_j = o \left( \frac{1}{n^{\beta (\overline{m} - \varepsilon)}} \right). \tag{149}$$

By (138) we observe that

$$\frac{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} f \left( \frac{k}{n} \right) Z_j (nx-k)}{V_j (x)} - f (x) = \tag{150}$$

$$\sum_{j_*=1}^{\overline{m}-1} \left( \frac{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \left( f^{(j_*)} (x) \left( \frac{k}{n} - x \right)^{j_*} \right) Z_j (nx-k)}{j_*! V_j (x)} \right) +$$

$$\frac{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} Z_j (nx-k) R_{\overline{m}} \left( \frac{k}{n}, x \right)}{V_j (x)}.$$

The last says ($j = 1, 2, 3, 4$)

$$
{}_jH_n\left(f, x\right) - f\left(x\right) - \sum_{j_*=1}^{\overline{m}-1} \frac{1}{j_*!}\, {}_jH_n\left(f^{(j_*)}\left(x\right)\left(\cdot - x\right)^{j_*}, x\right) = \, {}_jU_n\left(x\right). \tag{151}
$$

The proof of the theorem is complete. ■

## References


[1] G.A. Anastassiou, *Rate of convergence of some neural network operators to the unit-univariate case*, J. Math. Anal. Appli. 212 (1997), 237-262.

[2] G.A. Anastassiou, *Inteligent Systems: Approximation by Artificial Neural Networks*, Intelligent Systems Reference Library, Vol. 19, Springer, Heidelberg, 2011.

[3] G.A. Anastassiou, *Univariate hyperbolic tangent neural network approximation*, Mathematics and Computer Modelling, 53(2011), 1111-1132.

[4] G.A. Anastassiou, *Multivariate hyperbolic tangent neural network approximation*, Computers and Mathematics 61(2011), 809-821.

[5] G.A. Anastassiou, *Multivariate sigmoidal neural network approximation*, Neural Networks 24(2011), 378-386.

[6] G.A. Anastassiou, *Univariate sigmoidal neural network approximation*, J. of Computational Analysis and Applications, Vol. 14, No. 4, 2012, 659-690.

[7] G.A. Anastassiou, *Fractional neural network approximation*, Computers and Mathematics with Applications, 64 (2012), 1655-1676.

[8] G.A. Anastassiou, *Voronovskaya type asymptotic expansions for error function based quasi-interpolation neural network operators*, Revista Colombiana De Matematicas, 49 (1), 2015, 171-192.

[9] G.A. Anastassiou, *Intelligent Systems II: Complete Approximation by Neural Network Operators*, Springer, Heidelberg, New York, 2016.

[10] G.A. Anastassiou, *Intelligent Computations: Abstract Fractional Calculus, Inequalities, Approximations*, Springer, Heidelberg, New York, 2018.

[11] G.A. Anastassiou, *Algebraic function based Banach space valued ordinary and fractional neural network approximations*, New Trends in Mathematical Sciences, 10 special issues (1) (2022), 100-125.

[12] G.A. Anastassiou, *Gudermannian function activated Banach space valued ordinary and fractional neural network approximation*, Advances in Nonlinear Variational Inequalities, 25 (2) (2022), 27-64.

[13] G.A. Anastassiou, *Generalized symmetrical sigmoid function activated Banach space valued ordinary and fractional neural network approximation*, Analele Universităţii Oradea, Fasc. Matematica, accepted for publication, 2022.

[14] G.A. Anastassiou, *Abstract multivariate Gudermannian function activated neural network approximations*, Panamerican Mathematical Journal, accepted, 2022.

[15] G.A. Anastassiou, *General multivariate arctangent function activated neural network approximations*, submitted, 2022.

[16] G.A. Anastassiou, *Generalized symmetrical sigmoid function activated neural network multivariate approximation*, submitted, 2022.

[17] G.A. Anastassiou, R. Kozma, *Abstract multivariate algebraic function activated neural network approximations*, submitted, 2022.

[18] H. Cartan, *Differential Calculus*, Hermann, Paris, 1971.

[19] Z. Chen and F. Cao, *The approximation operators with sigmoidal functions*, Computers and Mathematics with Applications, 58 (2009), 758-765.

[20] D. Costarelli, R. Spigler, *Approximation results for neural network operators activated by sigmoidal functions*, Neural Networks 44 (2013), 101-106.

[21] D. Costarelli, R. Spigler, *Multivariate neural network operators with sigmoidal activation functions*, Neural Networks 48 (2013), 72-77.

[22] A.J. Dunning, J. Kensler, L. Goudeville, F. Bailleux, *Some extensions in continuous methods for immunological correlates of protection*, BMC Medical Research Methodology 15 (107) (28 Dec. 2015), doi:10.1186/s12874-015-0096-9.

[23] I.S. Haykin, *Neural Networks: A Comprehensive Foundation* (2 ed.), Prentice Hall, New York, 1998.

[24] W. McCulloch and W. Pitts, *A logical calculus of the ideas immanent in nervous activity*, Bulletin of Mathematical Biophysics, 7 (1943), 115-133.

[25] J. Mikusinski, *The Bochner integral*, Academic Press, New York, 1978.

[26] T.M. Mitchell, *Machine Learning*, WCB-McGraw-Hill, New York, 1997.

[27] L.B. Rall, *Computational Solution of Nonlinear Operator Equations*, John Wiley & Sons, New York, 1969.

[28] G.E. Shilov, *Elementary Functional Analysis*, Dover, New York, 1996.

[29] E.W. Weisstein, *Gudermannian*, MathWorld.