Fuzzy Fractional more sigmoid function activated neural network approximations revisited

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Abstract

Here we study the univariate fuzzy fractional quantitative approximation of fuzzy real valued functions on a compact interval by quasi-interpolation arctangent-algebraic-Gudermannian-generalized symmetrical activation function relied fuzzy neural network operators. These approximations are derived by establishing fuzzy Jackson type inequalities involving the fuzzy moduli of continuity of the right and left Caputo fuzzy fractional derivatives of the involved function. The approximations are fuzzy pointwise and fuzzy uniform. The related feed-forward fuzzy neural networks are with one hidden layer. We study also the fuzzy integer derivative and just fuzzy continuous cases. Our fuzzy fractional approximation result using higher order fuzzy differentiation converges better than in the fuzzy just continuous case.

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1 Introduction

The author in [1] and [2], see chapters 2-5, was the first to derive quantitative neural network approximations to continuous functions with rates by very specifically defined neural network operators of Cardaliaguet-Euvrard and "Squash-

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ing" types, by employing the modulus of continuity of the engaged function or its high order derivative, and producing very tight Jackson type inequalities. He studied there both the univariate and multivariate cases. The defining these operators "bell-shaped" and "squashing" function are assumed to be of compact support.

The author inspired by [28], continued his studies on neural networks approximation by introducing and using the proper quasi-interpolation operators of sigmoidal and hyperbolic tangent type which resulted into [10], [13] - [20], by treating both the univariate and multivariate cases.

Continuation of the author's works ([17], [18] and [19], Chapter 20) is this article where fuzzy neural network approximation based on arctangent-algebraic-Gudermannian-generalized symmetrical activation functions is taken at the fractional and ordinary levels resulting into higher rates of approximation. We involve the fuzzy ordinary derivatives and the right and left Caputo fuzzy fractional derivatives of the fuzzy function under approximation and we establish tight fuzzy Jackson type inequalities. An extensive background is given on fuzzyness, fractional calculus and neural networks, all needed to present our work.

Our fuzzy feed-forward neural networks (FFNNs) are with one hidden layer. About neural networks in general study [35], [38], [39].

2 Fuzzy Fractional Mathematical Analysis Basics

(see also [19], pp. 432-444)

We need the following basic background

Definition 1 *(see [43]) Let* $\mu : \mathbb{R} \to [0,1]$ *with the following properties:*

(i) is normal, i.e., $\exists\ x_0 \in \mathbb{R}$; $\mu(x_0) = 1$.

(ii) $\mu(\lambda x + (1-\lambda) y) \geq \min\{\mu(x), \mu(y)\}$, $\forall\ x, y \in \mathbb{R}$, $\forall\ \lambda \in [0,1]$ *(μ is called a convex fuzzy subset).*

(iii) μ *is upper semicontinuous on* $\mathbb{R}$*, i.e.* $\forall\ x_0 \in \mathbb{R}$ *and* $\forall\ \varepsilon > 0$, $\exists$ *neighborhood* $V(x_0) : \mu(x) \leq \mu(x_0) + \varepsilon$, $\forall\ x \in V(x_0)$.

(iv) The set $\overline{supp(\mu)}$ *is compact in* $\mathbb{R}$ *(where supp*$(\mu) := \{x \in \mathbb{R} : \mu(x) > 0\}$*).*

We call μ *a fuzzy real number. Denote the set of all* μ *with* $\mathbb{R}_{\mathcal{F}}$.

E.g. $\chi_{\{x_0\}} \in \mathbb{R}_{\mathcal{F}}$*, for any* $x_0 \in \mathbb{R}$*, where* $\chi_{\{x_0\}}$ *is the characteristic function at* x_0.

For $0 < r \leq 1$ *and* $\mu \in \mathbb{R}_{\mathcal{F}}$ *define*

$$[\mu]^r := \{x \in \mathbb{R} : \mu(x) \geq r\}$$

and

$$[\mu]^0 := \overline{\{x \in \mathbb{R} : \mu(x) \geq 0\}}.$$

Then it is well known that for each $r \in [0,1]$, $[\mu]^r$ *is a closed and bounded interval on* $\mathbb{R}$ *([34]).*

For $u, v \in \mathbb{R}_{\mathcal{F}}$ and $\lambda \in \mathbb{R}$, we define uniquely the sum $u \oplus v$ and the product $\lambda \odot u$ by

$$[u \oplus v]^r = [u]^r + [v]^r, \quad [\lambda \odot u]^r = \lambda [u]^r, \;\; \forall \, r \in [0,1],$$

where

$[u]^r + [v]^r$ means the usual addition of two intervals (as substes of $\mathbb{R}$) and

$\lambda [u]^r$ means the usual product between a scalar and a subset of $\mathbb{R}$ (see, e.g. [43]).

Notice $1 \odot u = u$ and it holds

$$u \oplus v = v \oplus u, \; \lambda \odot u = u \odot \lambda.$$

If $0 \leq r_1 \leq r_2 \leq 1$ then

$$[u]^{r_2} \subseteq [u]^{r_1}.$$

Actually $[u]^r = \left[u_-^{(r)}, u_+^{(r)}\right]$, where $u_-^{(r)} \leq u_+^{(r)}$, $u_-^{(r)}$, $u_+^{(r)} \in \mathbb{R}$, $\forall \, r \in [0,1]$.

For $\lambda > 0$ one has $\lambda u_\pm^{(r)} = (\lambda \odot u)_\pm^{(r)}$, respectively.

Define $D : \mathbb{R}_{\mathcal{F}} \times \mathbb{R}_{\mathcal{F}} \to \mathbb{R}_{\mathcal{F}}$ by

$$D(u,v) := \sup_{r \in [0,1]} \max \left\{ \left| u_-^{(r)} - v_-^{(r)} \right|, \left| u_+^{(r)} - v_+^{(r)} \right| \right\},$$

where

$$[v]^r = \left[v_-^{(r)}, v_+^{(r)}\right]; \;\; u, v \in \mathbb{R}_{\mathcal{F}}.$$

We have that D is a metric on $\mathbb{R}_{\mathcal{F}}$.

Then $(\mathbb{R}_{\mathcal{F}}, D)$ is a complete metric space, see [43], [44].

Here $\overset{*}{\sum}$ stands for fuzzy summation and $\widetilde{o} := \chi_{\{0\}} \in \mathbb{R}_{\mathcal{F}}$ is the neural element with respect to $\oplus$, i.e.,

$$u \oplus \widetilde{0} = \widetilde{0} \oplus u = u, \;\; \forall \, u \in \mathbb{R}_{\mathcal{F}}.$$

Denote

$$D^*(f,g) = \sup_{x \in X \subseteq \mathbb{R}} D(f,g),$$

where $f, g : X \to \mathbb{R}_{\mathcal{F}}$.

We mention

Definition 2 *Let $f : X \subseteq \mathbb{R} \to \mathbb{R}_{\mathcal{F}}$, X interval, we define the (first) fuzzy modulus of continuity of f by*

$$\omega_1^{(\mathcal{F})}(f,\delta)_X = \sup_{x,y \in X,\ |x-y| \leq \delta} D(f(x), f(y)), \quad \delta > 0.$$

When $g : X \subseteq \mathbb{R} \to \mathbb{R}$, we define

$$\omega_1(g,\delta) = \omega_1(g,\delta)_X = \sup_{x,y \in X,\ |x-y| \leq \delta} |g(x) - g(y)|.$$

We define by $C_{\mathcal{F}}^{U}(\mathbb{R})$ the space of fuzzy uniformly continuous functions from $\mathbb{R} \to \mathbb{R}_{\mathcal{F}}$, also $C_{\mathcal{F}}(\mathbb{R})$ is the space of fuzzy continuous functions on $\mathbb{R}$, and $C_b(\mathbb{R}, \mathbb{R}_{\mathcal{F}})$ is the fuzzy continuous and bounded functions.

We mention

Proposition 3 *([5]) Let $f \in C_{\mathcal{F}}^{U}(X)$. Then $\omega_1^{(\mathcal{F})}(f,\delta)_X < \infty$, for any $\delta > 0$.*

By [9], p. 129 we have that $C_{\mathcal{F}}^{U}([a,b]) = C_{\mathcal{F}}([a,b])$, fuzzy continuous functions on $[a,b] \subset \mathbb{R}$.

Proposition 4 *([5]) It holds*

$$\lim_{\delta \to 0} \omega_1^{(\mathcal{F})}(f,\delta)_X = \omega_1^{(\mathcal{F})}(f,0)_X = 0,$$

iff $f \in C_{\mathcal{F}}^{U}(X)$, where X is a compact interval.

Proposition 5 *([5]) Here $[f]^r = \left[f_-^{(r)}, f_+^{(r)}\right]$, $r \in [0,1]$. Let $f \in C_{\mathcal{F}}(\mathbb{R})$. Then $f_{\pm}^{(r)}$ are equicontinuous with respect to $r \in [0,1]$ over $\mathbb{R}$, respectively in $\pm$.*

Note 6 *It is clear by Propositions 4, 5, that if $f \in C_{\mathcal{F}}^{U}(\mathbb{R})$, then $f_{\pm}^{(r)} \in C_U(\mathbb{R})$ (uniformly continuous on $\mathbb{R}$). Also if $f \in C_b(\mathbb{R}, \mathbb{R}_{\mathcal{F}})$ implies $f_{\pm}^{(r)} \in C_b(\mathbb{R})$ (continuous and bounded functions on $\mathbb{R}$).*

Proposition 7 *Let $f : \mathbb{R} \to \mathbb{R}_{\mathcal{F}}$. Assume that $\omega_1^{\mathcal{F}}(f,\delta)_X$, $\omega_1\left(f_-^{(r)}, \delta\right)_X$, $\omega_1\left(f_+^{(r)}, \delta\right)_X$ are finite for any $\delta > 0$, $r \in [0,1]$, where X any interval of $\mathbb{R}$.*

Then

$$\omega_1^{(\mathcal{F})}(f,\delta)_X = \sup_{r \in [0,1]} \max\left\{\omega_1\left(f_-^{(r)}, \delta\right)_X, \omega_1\left(f_+^{(r)}, \delta\right)_X\right\}.$$

Proof. Similar to Proposition 14.15, p. 246 of [9]. ∎

We need

Remark 8 *([3]). Here* $r \in [0,1]$*,* $x_i^{(r)}, y_i^{(r)} \in \mathbb{R}$*,* $i = 1, ..., m \in \mathbb{N}$*. Suppose that*

$$\sup_{r \in [0,1]} \max \left(x_i^{(r)}, y_i^{(r)} \right) \in \mathbb{R}, \text{ for } i = 1, ..., m.$$

Then one sees easily that

$$\sup_{r \in [0,1]} \max \left(\sum_{i=1}^{m} x_i^{(r)}, \sum_{i=1}^{m} y_i^{(r)} \right) \leq \sum_{i=1}^{m} \sup_{r \in [0,1]} \max \left(x_i^{(r)}, y_i^{(r)} \right). \tag{1}$$

We need

Definition 9 *Let* $x, y \in \mathbb{R}_{\mathcal{F}}$*. If there exists* $z \in \mathbb{R}_{\mathcal{F}} : x = y \oplus z$*, then we call* z *the* H*-difference on* x *and* y*, denoted* $x - y$*.*

Definition 10 *([42]) Let* $T := [x_0, x_0 + \beta] \subset \mathbb{R}$*, with* $\beta > 0$*. A function* $f : T \to \mathbb{R}_{\mathcal{F}}$ *is* H*-differentiable at* $x \in T$ *if there exists an* $f'(x) \in \mathbb{R}_{\mathcal{F}}$ *such that the limits (with respect to* D*)*

$$\lim_{h \to 0+} \frac{f(x+h) - f(x)}{h}, \quad \lim_{h \to 0+} \frac{f(x) - f(x-h)}{h} \tag{2}$$

exist and are equal to $f'(x)$*.*

We call f' *the* H*-derivative or fuzzy derivative of* f *at* x*.*

Above is assumed that the H-differences $f(x+h) - f(x)$, $f(x) - f(x-h)$ exists in $\mathbb{R}_{\mathcal{F}}$ in a neighborhood of x.

Higher order H-fuzzy derivatives are defined the obvious way, like in the real case.

We denote by $C_{\mathcal{F}}^N(\mathbb{R})$, $N \geq 1$, the space of all N-times continuously H-fuzzy differentiable functions from $\mathbb{R}$ into $\mathbb{R}_{\mathcal{F}}$, similarly is defined $C_{\mathcal{F}}^N([a,b])$, $[a,b] \subset \mathbb{R}$.

We mention

Theorem 11 *([36]) Let* $f : \mathbb{R} \to \mathbb{R}_{\mathcal{F}}$ *be* H*-fuzzy differentiable. Let* $t \in \mathbb{R}$*,* $0 \leq r \leq 1$*. Clearly*

$$[f(t)]^r = \left[f(t)_-^{(r)}, f(t)_+^{(r)} \right] \subseteq \mathbb{R}.$$

Then $(f(t))_{\pm}^{(r)}$ *are differentiable and*

$$[f'(t)]^r = \left[\left(f(t)_-^{(r)} \right)', \left(f(t)_+^{(r)} \right)' \right].$$

I.e.

$$(f')_{\pm}^{(r)} = \left(f_{\pm}^{(r)} \right)', \quad \forall \, r \in [0,1].$$

Remark 12 *([4]) Let $f \in C_{\mathcal{F}}^{N}(\mathbb{R})$, $N \geq 1$. Then by Theorem 11 we obtain*

$$\left[f^{(i)}(t)\right]^{r}=\left[\left(f(t)_{-}^{(r)}\right)^{(i)},\left(f(t)_{+}^{(r)}\right)^{(i)}\right],$$

for $i=0,1,2,\ldots,N$, and in particular we have that

$$\left(f^{(i)}\right)_{\pm}^{(r)}=\left(f_{\pm}^{(r)}\right)^{(i)},$$

for any $r \in [0,1]$, all $i=0,1,2,\ldots,N$.

Note 13 *([4]) Let $f \in C_{\mathcal{F}}^{N}(\mathbb{R})$, $N \geq 1$. Then by Theorem 11 we have $f_{\pm}^{(r)} \in C^{N}(\mathbb{R})$, for any $r \in [0,1]$.*

Items 11-13 are valid also on $[a,b]$.

By [9], p. 131, if $f \in C_{\mathcal{F}}([a,b])$, then f is a fuzzy bounded function.

We need also a particular case of the Fuzzy Henstock integral ($\delta(x)=\frac{\delta}{2}$), see [43].

Definition 14 *([33], p. 644) Let $f:[a,b] \to \mathbb{R}_{\mathcal{F}}$. We say that f is Fuzzy-Riemann integrable to $I \in \mathbb{R}_{\mathcal{F}}$ if for any $\varepsilon>0$, there exists $\delta>0$ such that for any division $P=\{[u,v];\xi\}$ of $[a,b]$ with the norms $\Delta(P)<\delta$, we have*

$$D\left(\sum_{P}^{*}(v-u)\odot f(\xi), I\right)<\varepsilon.$$

We write

$$I:=(FR)\int_{a}^{b} f(x)\,dx. \tag{3}$$

We mention

Theorem 15 *([34]) Let $f:[a,b] \to \mathbb{R}_{\mathcal{F}}$ be fuzzy continuous. Then*

$$(FR)\int_{a}^{b} f(x)\,dx$$

exists and belongs to $\mathbb{R}_{\mathcal{F}}$, furthermore it holds

$$\left[(FR)\int_{a}^{b} f(x)\,dx\right]^{r}=\left[\int_{a}^{b}(f)_{-}^{(r)}(x)\,dx, \int_{a}^{b}(f)_{+}^{(r)}(x)\,dx\right],$$

$\forall\, r \in [0,1]$.

For the definition of general fuzzy integral we follow [37] next.

Definition 16 *Let* (Ω, Σ, μ) *be a complete* σ*-finite measure space. We call* $F : \Omega \rightarrow R_{\mathcal{F}}$ *measurable iff* $\forall$ *closed* $B \subseteq \mathbb{R}$ *the function* $F^{-1}(B) : \Omega \rightarrow [0,1]$ *defined by*

$$F^{-1}(B)(w) := \sup_{x \in B} F(w)(x), \text{ all } w \in \Omega$$

is measurable, see [37].

Theorem 17 *([37]) For* $F : \Omega \rightarrow \mathbb{R}_{\mathcal{F}}$,

$$F(w) = \left\{ \left(F_{-}^{(r)}(w), F_{+}^{(r)}(w) \right) | 0 \leq r \leq 1 \right\},$$

the following are equivalent

(1) F *is measurable,*

(2) $\forall\ r \in [0,1]$, $F_{-}^{(r)}$, $F_{+}^{(r)}$ *are measurable.*

Following [37], given that for each $r \in [0,1]$, $F_{-}^{(r)}$, $F_{+}^{(r)}$ are integrable we have that the parametrized representation

$$\left\{ \left(\int_A F_{-}^{(r)} d\mu, \int_A F_{+}^{(r)} d\mu \right) | 0 \leq r \leq 1 \right\} \tag{4}$$

is a fuzzy real number for each $A \in \Sigma$.

The last fact leads to

Definition 18 *([37]) A measurable function* $F : \Omega \rightarrow \mathbb{R}_{\mathcal{F}}$,

$$F(w) = \left\{ \left(F_{-}^{(r)}(w), F_{+}^{(r)}(w) \right) | 0 \leq r \leq 1 \right\}$$

is integrable if for each $r \in [0,1]$, $F_{\pm}^{(r)}$ *are integrable, or equivalently, if* $F_{\pm}^{(0)}$ *are integrable.*

In this case, the fuzzy integral of F over $A \in \Sigma$ is defined by

$$\int_A F d\mu := \left\{ \left(\int_A F_{-}^{(r)} d\mu, \int_A F_{+}^{(r)} d\mu \right) | 0 \leq r \leq 1 \right\}.$$

By [37], F is integrable iff $w \rightarrow \|F(w)\|_{\mathcal{F}}$ is real-valued integrable.

Here denote

$$\|u\|_{\mathcal{F}} := D\left(u, \widetilde{0}\right), \ \ \forall\ u \in \mathbb{R}_{\mathcal{F}}.$$

We need also

Theorem 19 *([37]) Let* $F, G : \Omega \rightarrow \mathbb{R}_{\mathcal{F}}$ *be integrable. Then*

(1) Let $a, b \in \mathbb{R}$, *then* $aF + bG$ *is integrable and for each* $A \in \Sigma$,

$$\int_A (aF + bG) d\mu = a \int_A F d\mu + b \int_A G d\mu;$$

(2) $D(F,G)$ is a real- valued integrable function and for each $A \in \Sigma$,

$$D\left(\int_A F d\mu, \int_A G d\mu\right) \leq \int_A D(F,G)\, d\mu.$$

In particular,

$$\left\| \int_A F d\mu \right\|_{\mathcal{F}} \leq \int_A \|F\|_{\mathcal{F}}\, d\mu.$$

Above μ could be the Lebesgue measure, with all the basic properties valid here too.

Basically here we have

$$\left[\int_A F d\mu\right]^r = \left[\int_A F_-^{(r)} d\mu, \int_A F_+^{(r)} d\mu\right], \tag{5}$$

i.e.

$$\left(\int_A F d\mu\right)_{\pm}^{(r)} = \int_A F_{\pm}^{(r)} d\mu, \quad \forall\ r \in [0,1].$$

We need

Definition 20 *Let $\nu \geq 0$, $n = \lceil \nu \rceil$ ($\lceil \cdot \rceil$ is the ceiling of the number), $f \in AC^n([a,b])$ (space of functions f with $f^{(n-1)} \in AC([a,b])$, absolutely continuous functions). We call left Caputo fractional derivative (see [29], pp. 49-52, [32], [40]) the function*

$$D_{*a}^{\nu} f(x) = \frac{1}{\Gamma(n-\nu)} \int_a^x (x-t)^{n-\nu-1} f^{(n)}(t)\, dt, \tag{6}$$

$\forall\ x \in [a,b]$, where Γ is the gamma function $\Gamma(\nu) = \int_0^\infty e^{-t} t^{\nu-1} dt$, $\nu > 0$.

Notice $D_{*a}^{\nu} f \in L_1([a,b])$ and $D_{*a}^{\nu} f$ exists a.e. on $[a,b]$.

We set $D_{*a}^{0} f(x) = f(x)$, $\forall\ x \in [a,b]$.

Lemma 21 *([8]) Let $\nu > 0$, $\nu \notin \mathbb{N}$, $n = \lceil \nu \rceil$, $f \in C^{n-1}([a,b])$ and $f^{(n)} \in L_\infty([a,b])$. Then $D_{*a}^{\nu} f(a) = 0$.*

Definition 22 *(see also [6], [31], [32]) Let $f \in AC^m([a,b])$, $m = \lceil \beta \rceil$, $\beta > 0$. The right Caputo fractional derivative of order $\beta > 0$ is given by*

$$D_{b-}^{\beta} f(x) = \frac{(-1)^m}{\Gamma(m-\beta)} \int_x^b (\zeta - x)^{m-\beta-1} f^{(m)}(\zeta)\, d\zeta, \tag{7}$$

$\forall\ x \in [a,b]$. We set $D_{b-}^{0} f(x) = f(x)$. Notice that $D_{b-}^{\beta} f \in L_1([a,b])$ and $D_{b-}^{\beta} f$ exists a.e. on $[a,b]$.

Lemma 23 *([8]) Let $f \in C^{m-1}([a,b])$, $f^{(m)} \in L_\infty([a,b])$, $m = \lceil \beta \rceil$, $\beta > 0$, $\beta \notin \mathbb{N}$. Then $D_{b-}^{\beta} f(b) = 0$.*

Convention 24 *We assume that*

$$D_{*x_0}^{\beta} f(x) = 0, \quad \text{for } x < x_0, \tag{8}$$

and

$$D_{x_0-}^{\beta} f(x) = 0, \quad \text{for } x > x_0, \tag{9}$$

for all $x, x_0 \in [a, b]$.

We mention

Proposition 25 *([8]) Let* $f \in C^n([a,b])$, $n = \lceil \nu \rceil$, $\nu > 0$. *Then* $D_{*a}^{\nu} f(x)$ *is continuous in* $x \in [a, b]$.

Also we have

Proposition 26 *([8]) Let* $f \in C^m([a,b])$, $m = \lceil \beta \rceil$, $\beta > 0$. *Then* $D_{b-}^{\beta} f(x)$ *is continuous in* $x \in [a, b]$.

We further mention

Proposition 27 *([8]) Let* $f \in C^{m-1}([a,b])$, $f^{(m)} \in L_{\infty}([a,b])$, $m = \lceil \beta \rceil$, $\beta > 0$ *and*

$$D_{*x_0}^{\beta} f(x) = \frac{1}{\Gamma(m-\beta)} \int_{x_0}^{x} (x-t)^{m-\beta-1} f^{(m)}(t) \, dt, \tag{10}$$

for all $x, x_0 \in [a, b] : x \geq x_0$.

Then $D_{*x_0}^{\beta} f(x)$ *is continuous in* x_0.

Proposition 28 *([8]) Let* $f \in C^{m-1}([a,b])$, $f^{(m)} \in L_{\infty}([a,b])$, $m = \lceil \beta \rceil$, $\beta > 0$ *and*

$$D_{x_0-}^{\beta} f(x) = \frac{(-1)^m}{\Gamma(m-\beta)} \int_{x}^{x_0} (\zeta - x)^{m-\beta-1} f^{(m)}(\zeta) \, d\zeta, \tag{11}$$

for all $x, x_0 \in [a, b] : x \leq x_0$.

Then $D_{x_0-}^{\beta} f(x)$ *is continuous in* x_0.

We need

Proposition 29 *([8]) Let* $g \in C([a,b])$, $0 < c < 1$, $x, x_0 \in [a, b]$. *Define*

$$L(x, x_0) = \int_{x_0}^{x} (x-t)^{c-1} g(t) \, dt, \quad \text{for } x \geq x_0, \tag{12}$$

and $L(x, x_0) = 0$, *for* $x < x_0$.

Then L *is jointly continuous in* (x, x_0) *on* $[a, b]^2$.

We mention

Proposition 30 *([8]) Let $g \in C([a,b])$, $0 < c < 1$, $x, x_0 \in [a,b]$. Define*

$$K(x, x_0) = \int_{x_0}^{x} (\zeta - x)^{c-1} g(\zeta)\, d\zeta, \quad \text{for } x \leq x_0, \tag{13}$$

and $K(x, x_0) = 0$, for $x > x_0$.

Then $K(x, x_0)$ is jointly continuous from $[a,b]^2$ into $\mathbb{R}$.

Based on Propositions 29, 30 we derive

Corollary 31 *([8]) Let $f \in C^m([a,b])$, $m = \lceil \beta \rceil$, $\beta > 0$, $\beta \notin \mathbb{N}$, $x, x_0 \in [a,b]$. Then $D^{\beta}_{*x_0} f(x)$, $D^{\beta}_{x_0-} f(x)$ are jointly continuous functions in (x, x_0) from $[a,b]^2$ into $\mathbb{R}$.*

We need

Theorem 32 *([8]) Let $f : [a,b]^2 \to \mathbb{R}$ be jointly continous. Consider*

$$G(x) = \omega_1 (f(\cdot, x), \delta)_{[x,b]}, \tag{14}$$

$\delta > 0$, $x \in [a,b]$.

Then G is continuous in $x \in [a,b]$.

Also it holds

Theorem 33 *([8]) Let $f : [a,b]^2 \to \mathbb{R}$ be jointly continous. Then*

$$H(x) = \omega_1 (f(\cdot, x), \delta)_{[a,x]}, \tag{15}$$

$x \in [a,b]$, is continuous in $x \in [a,b]$, $\delta > 0$.

So that for $f \in C^m([a,b])$, $m = \lceil \beta \rceil$, $\beta > 0$, $\beta \notin \mathbb{N}$, $x, x_0 \in [a,b]$, we have that $\omega_1 \left(D^{\beta}_{*x} f, h \right)_{[x,b]}$, $\omega_1 \left(D^{\beta}_{x-} f, h \right)_{[a,x]}$ are continuous functions in $x \in [a,b]$, $h > 0$ is fixed.

We make

Remark 34 *([8]) Let $f \in C^{n-1}([a,b])$, $f^{(n)} \in L_\infty([a,b])$, $n = \lceil \nu \rceil$, $\nu > 0$, $\nu \notin \mathbb{N}$. Then we have*

$$|D^{\nu}_{*a} f(x)| \leq \frac{\left\| f^{(n)} \right\|_\infty}{\Gamma(n - \nu + 1)} (x-a)^{n-\nu}, \quad \forall\, x \in [a,b]. \tag{16}$$

Thus we observe

$$\omega_1 (D^{\nu}_{*a} f, \delta) = \sup_{\substack{x,y \in [a,b] \\ |x-y| \leq \delta}} |D^{\nu}_{*a} f(x) - D^{\nu}_{*a} f(y)| \tag{17}$$

$$\leq \sup_{\substack{x,y\in[a,b]\\|x-y|\leq\delta}} \left(\frac{\left\|f^{(n)}\right\|_{\infty}}{\Gamma(n-\nu+1)}(x-a)^{n-\nu} + \frac{\left\|f^{(n)}\right\|_{\infty}}{\Gamma(n-\nu+1)}(y-a)^{n-\nu} \right)$$

$$\leq \frac{2\left\|f^{(n)}\right\|_{\infty}}{\Gamma(n-\nu+1)}(b-a)^{n-\nu}. \tag{18}$$

Consequently

$$\omega_1\left(D_{*a}^{\nu}f,\delta\right) \leq \frac{2\left\|f^{(n)}\right\|_{\infty}}{\Gamma(n-\nu+1)}(b-a)^{n-\nu}. \tag{19}$$

Similarly, let $f \in C^{m-1}([a,b])$, $f^{(m)} \in L_{\infty}([a,b])$, $m = \lceil\beta\rceil$, $\beta > 0$, $\beta \notin \mathbb{N}$, *then*

$$\omega_1\left(D_{b-}^{\beta}f,\delta\right) \leq \frac{2\left\|f^{(m)}\right\|_{\infty}}{\Gamma(m-\beta+1)}(b-a)^{m-\beta}. \tag{20}$$

So for $f \in C^{m-1}([a,b])$, $f^{(m)} \in L_{\infty}([a,b])$, $m = \lceil\beta\rceil$, $\beta > 0$, $\beta \notin \mathbb{N}$, *we find*

$$\sup_{x_0\in[a,b]} \omega_1\left(D_{*x_0}^{\beta}f,\delta\right)_{[x_0,b]} \leq \frac{2\left\|f^{(m)}\right\|_{\infty}}{\Gamma(m-\beta+1)}(b-a)^{m-\beta}, \tag{21}$$

and

$$\sup_{x_0\in[a,b]} \omega_1\left(D_{x_0-}^{\beta}f,\delta\right)_{[a,x_0]} \leq \frac{2\left\|f^{(m)}\right\|_{\infty}}{\Gamma(m-\beta+1)}(b-a)^{m-\beta}. \tag{22}$$

By Proposition 15.114, p. 388 of [7], we get here that $D_{*x_0}^{\beta}f \in C([x_0,b])$, *and by [12] we obtain that* $D_{x_0-}^{\beta}f \in C([a,x_0])$.

We need

Definition 35 *([11]) Let* $f \in C_{\mathcal{F}}([a,b])$ *(fuzzy continuous on* $[a,b] \subset \mathbb{R}$*),* $\nu > 0$.

We define the Fuzzy Fractional left Riemann-Liouville operator as

$$J_a^{\nu}f(x) := \frac{1}{\Gamma(\nu)} \odot \int_a^x (x-t)^{\nu-1} \odot f(t)\,dt, \quad x \in [a,b], \tag{23}$$

$$J_a^0 f := f.$$

Also, we define the Fuzzy Fractional right Riemann-Liouville operator as

$$I_{b-}^{\nu}f(x) := \frac{1}{\Gamma(\nu)} \odot \int_x^b (t-x)^{\nu-1} \odot f(t)\,dt, \quad x \in [a,b], \tag{24}$$

$$I_{b-}^0 f := f.$$

We mention

Definition 36 *([11]) Let $f : [a,b] \to \mathbb{R}_{\mathcal{F}}$ is called fuzzy absolutely continuous iff $\forall\ \epsilon > 0$, $\exists\ \delta > 0$ for every finite, pairwise disjoint, family*

$$(c_k, d_k)_{k=1}^{n} \subseteq (a,b) \quad \textit{with} \quad \sum_{k=1}^{n} (d_k - c_k) < \delta$$

we get

$$\sum_{k=1}^{n} D\left(f\left(d_k\right), f\left(c_k\right)\right) < \epsilon. \tag{25}$$

We denote the related space of functions by $AC_{\mathcal{F}}([a,b])$.

If $f \in AC_{\mathcal{F}}([a,b])$, then $f \in C_{\mathcal{F}}([a,b])$.

It holds

Proposition 37 *([11]) $f \in AC_{\mathcal{F}}([a,b]) \Leftrightarrow f_{\pm}^{(r)} \in AEC([a,b])$, $\forall\ r \in [0,1]$ (absolutely equicontinuous).*

We need

Definition 38 *([11]) We define the Fuzzy Fractional left Caputo derivative, $x \in [a,b]$.*

Let $f \in C_{\mathcal{F}}^{n}([a,b])$, $n = \lceil \nu \rceil$, $\nu > 0$ ($\lceil \cdot \rceil$ denotes the ceiling). We define

$$D_{*a}^{\nu\mathcal{F}} f(x) := \frac{1}{\Gamma(n-\nu)} \odot \int_a^x (x-t)^{n-\nu-1} \odot f^{(n)}(t)\, dt \tag{26}$$

$$= \left\{ \left(\frac{1}{\Gamma(n-\nu)} \int_a^x (x-t)^{n-\nu-1} \left(f^{(n)}\right)_{-}^{(r)}(t)\, dt, \right.\right.$$

$$\left.\left. \frac{1}{\Gamma(n-\nu)} \int_a^x (x-t)^{n-\nu-1} \left(f^{(n)}\right)_{+}^{(r)}(t)\, dt \right) | 0 \leq r \leq 1 \right\} =$$

$$= \left\{ \left(\frac{1}{\Gamma(n-\nu)} \int_a^x (x-t)^{n-\nu-1} \left(f_{-}^{(r)}\right)^{(n)}(t)\, dt, \right.\right.$$

$$\left.\left. \frac{1}{\Gamma(n-\nu)} \int_a^x (x-t)^{n-\nu-1} \left(f_{+}^{(r)}\right)^{(n)}(t)\, dt \right) | 0 \leq r \leq 1 \right\}. \tag{27}$$

So, we get

$$\left[D_{*a}^{\nu\mathcal{F}} f(x)\right]^{r} = \left[\left(\frac{1}{\Gamma(n-\nu)} \int_a^x (x-t)^{n-\nu-1} \left(f_{-}^{(r)}\right)^{(n)}(t)\, dt, \right.\right.$$

$$\left.\left. \frac{1}{\Gamma(n-\nu)} \int_a^x (x-t)^{n-\nu-1} \left(f_{+}^{(r)}\right)^{(n)}(t)\, dt \right) \right], \quad 0 \leq r \leq 1. \tag{28}$$

That is

$$\left(D_{*a}^{\nu\mathcal{F}} f(x)\right)_{\pm}^{(r)} = \frac{1}{\Gamma(n-\nu)} \int_a^x (x-t)^{n-\nu-1} \left(f_{\pm}^{(r)}\right)^{(n)}(t)\, dt = \left(D_{*a}^{\nu}\left(f_{\pm}^{(r)}\right)\right)(x),$$

see [7], [29].

I.e. we get that

$$\left(D_{*a}^{\nu\mathcal{F}} f(x)\right)_{\pm}^{(r)} = \left(D_{*a}^{\nu}\left(f_{\pm}^{(r)}\right)\right)(x), \tag{29}$$

$\forall\, x \in [a,b]$, *in short*

$$\left(D_{*a}^{\nu\mathcal{F}} f\right)_{\pm}^{(r)} = D_{*a}^{\nu}\left(f_{\pm}^{(r)}\right), \quad \forall\, r \in [0,1]. \tag{30}$$

We need

Lemma 39 *([11]) $D_{*a}^{\nu\mathcal{F}} f(x)$ is fuzzy continuous in $x \in [a,b]$.*

We need

Definition 40 *([11]) We define the Fuzzy Fractional right Caputo derivative, $x \in [a,b]$.*

Let $f \in C_{\mathcal{F}}^{n}([a,b])$, $n = \lceil \nu \rceil$, $\nu > 0$. We define

$$\begin{aligned}
D_{b-}^{\nu\mathcal{F}} f(x) &:= \frac{(-1)^n}{\Gamma(n-\nu)} \odot \int_x^b (t-x)^{n-\nu-1} \odot f^{(n)}(t)\, dt \\
&= \left\{\left(\frac{(-1)^n}{\Gamma(n-\nu)} \int_x^b (t-x)^{n-\nu-1} \left(f^{(n)}\right)_{-}^{(r)}(t)\, dt,\right.\right. \\
&\left.\left.\frac{(-1)^n}{\Gamma(n-\nu)} \int_x^b (t-x)^{n-\nu-1} \left(f^{(n)}\right)_{+}^{(r)}(t)\, dt\right) \middle| 0 \le r \le 1\right\} \qquad (31) \\
&= \left\{\left(\frac{(-1)^n}{\Gamma(n-\nu)} \int_x^b (t-x)^{n-\nu-1} \left(f_{-}^{(r)}\right)^{(n)}(t)\, dt,\right.\right. \\
&\left.\left.\frac{(-1)^n}{\Gamma(n-\nu)} \int_x^b (t-x)^{n-\nu-1} \left(f_{+}^{(r)}\right)^{(n)}(t)\, dt\right) \middle| 0 \le r \le 1\right\}.
\end{aligned}$$

We get

$$\begin{aligned}
\left[D_{b-}^{\nu\mathcal{F}} f(x)\right]^r &= \left[\left(\frac{(-1)^n}{\Gamma(n-\nu)} \int_x^b (t-x)^{n-\nu-1} \left(f_{-}^{(r)}\right)^{(n)}(t)\, dt,\right.\right. \\
&\left.\left.\frac{(-1)^n}{\Gamma(n-\nu)} \int_x^b (t-x)^{n-\nu-1} \left(f_{+}^{(r)}\right)^{(n)}(t)\, dt\right)\right], \quad 0 \le r \le 1.
\end{aligned}$$

That is

$$\left(D_{b-}^{\nu\mathcal{F}} f(x)\right)_{\pm}^{(r)} = \frac{(-1)^n}{\Gamma(n-\nu)} \int_x^b (t-x)^{n-\nu-1} \left(f_{\pm}^{(r)}\right)^{(n)}(t)\, dt = \left(D_{b-}^{\nu}\left(f_{\pm}^{(r)}\right)\right)(x),$$

see [6].

I.e. we get that

$$\left(D_{b-}^{\nu\mathcal{F}} f(x)\right)_{\pm}^{(r)} = \left(D_{b-}^{\nu}\left(f_{\pm}^{(r)}\right)\right)(x), \tag{32}$$

$\forall\ x \in [a,b]$, *in short*

$$\left(D_{b-}^{\nu\mathcal{F}} f\right)_{\pm}^{(r)} = D_{b-}^{\nu}\left(f_{\pm}^{(r)}\right), \quad \forall\ r \in [0,1]. \tag{33}$$

Clearly,

$$D_{b-}^{\nu}\left(f_{-}^{(r)}\right) \leq D_{b-}^{\nu}\left(f_{+}^{(r)}\right), \quad \forall\ r \in [0,1].$$

We need

Lemma 41 *([11])* $D_{b-}^{\nu\mathcal{F}} f(x)$ *is fuzzy continuous in* $x \in [a,b]$.

3 Real neural network approximation

3.1 About the arctangent activation function neural networks

We consider the

$$\arctan x = \int_0^x \frac{dz}{1+z^2}, \quad x \in \mathbb{R}. \tag{34}$$

We will be using

$$h(x) := \frac{2}{\pi}\arctan\left(\frac{\pi}{2}x\right) = \frac{2}{\pi}\int_0^{\frac{\pi x}{2}} \frac{dz}{1+z^2}, \quad x \in \mathbb{R}, \tag{35}$$

which is a sigmoid type function and it is strictly increasing. We have that

$$h(0) = 0, \quad h(-x) = -h(x), \quad h(+\infty) = 1, \quad h(-\infty) = -1,$$

and

$$h'(x) = \frac{4}{4+\pi^2 x^2} > 0, \quad \text{all } x \in \mathbb{R}. \tag{36}$$

We consider the activation function

$$\psi_1(x) := \frac{1}{4}\left(h(x+1) - h(x-1)\right), \quad x \in \mathbb{R}, \tag{37}$$

and we notice that

$$\psi_1(-x) = \psi_1(x), \tag{38}$$

it is an even function.

Since $x+1 > x-1$, then $h(x+1) > h(x-1)$, and $\psi_1(x) > 0$, all $x \in \mathbb{R}$.

We see that

$$\psi_1(0) = \frac{1}{\pi} \arctan \frac{\pi}{2} \cong 18.31. \tag{39}$$

Let $x > 0$, we have that

$$\psi_1'(x) = \frac{1}{4}\left(h'(x+1) - h'(x-1)\right) =$$

$$\frac{-4\pi^2 x}{\left(4 + \pi^2 (x+1)^2\right)\left(4 + \pi^2 (x-1)^2\right)} < 0. \tag{40}$$

That is

$$\psi_1'(x) < 0, \text{ for } x > 0. \tag{41}$$

That is ψ_1 is strictly decreasing on $[0, \infty)$ and clearly is strictly increasing on $(-\infty, 0]$, and $\psi_1'(0) = 0$.

Observe that

$$\begin{gathered} \lim_{x \to +\infty} \psi_1(x) = \tfrac{1}{4}\left(h(+\infty) - h(+\infty)\right) = 0, \\ \text{and} \\ \lim_{x \to -\infty} \psi_1(x) = \tfrac{1}{4}\left(h(-\infty) - h(-\infty)\right) = 0. \end{gathered} \tag{42}$$

That is the x-axis is the horizontal asymptote on ψ_1.

All in all, ψ_1 is a bell symmetric function with maximum $\psi_1(0) \cong 18.31$.

We need

Theorem 42 *([20], p. 286) We have that*

$$\sum_{i=-\infty}^{\infty} \psi_1(x - i) = 1, \quad \forall\, x \in \mathbb{R}. \tag{43}$$

Theorem 43 *([20], p. 287) It holds*

$$\int_{-\infty}^{\infty} \psi_1(x)\, dx = 1. \tag{44}$$

So that $\psi_1(x)$ is a density function on $\mathbb{R}$.

We mention

Theorem 44 *([20], p. 288) Let $0 < \alpha < 1$, and $n \in \mathbb{N}$ with $n^{1-\alpha} > 2$. It holds*

$$\sum_{\begin{cases} k = -\infty \\ : |nx - k| \geq n^{1-\alpha} \end{cases}}^{\infty} \psi_1(nx - k) < \frac{2}{\pi^2\left(n^{1-\alpha} - 2\right)} =: c_1(\alpha, n). \tag{45}$$

Denote by $\lfloor \cdot \rfloor$ the integral part of the number and by $\lceil \cdot \rceil$ the ceiling of the number.

We further mention

Theorem 45 *([20]) Let $x \in [a,b] \subset \mathbb{R}$ and $n \in \mathbb{N}$ so that $\lceil na \rceil \leq \lfloor nb \rfloor$. It holds*

$$\frac{1}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \psi_1 (nx-k)} < \frac{1}{\psi_1 (1)} \cong 0.0868 =: \alpha_1, \quad \forall\, x \in [a,b]. \tag{46}$$

We make

Remark 46 *([20]) (i) We have that*

$$\lim_{n \to \infty} \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \psi_1 (nx-k) \neq 1, \tag{47}$$

for at least some $x \in [a,b]$.

(ii) For large enough n we always obtain $\lceil na \rceil \leq \lfloor nb \rfloor$. Also $a \leq \frac{k}{n} \leq b$, iff $\lceil na \rceil \leq k \leq \lfloor nb \rfloor$. In general it holds (by (43)) that

$$\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \psi_1 (nx-k) \leq 1. \tag{48}$$

We give

Definition 47 *([20]) Let $f \in C([a,b])$ and $n \in \mathbb{N} : \lceil na \rceil \leq \lfloor nb \rfloor$. We define the real positive linear neural network operator*

$$_1A_n (f,x) := \frac{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} f\left(\frac{k}{n}\right) \psi_1 (nx-k)}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \psi_1 (nx-k)}, \quad x \in [a,b]. \tag{49}$$

Clearly here $_1A_n (f,x) \in C([a,b])$. In [20] we studied the pointwise and uniform convergence of $_1A_n (f,x)$ to $f(x)$ with rates.

We mention

Theorem 48 *([20]) Let $f \in C([a,b])$, $0 < \alpha < 1$, $n \in \mathbb{N} : n^{1-\alpha} > 2$, $x \in [a,b]$. Then*

i)

$$\left| {}_1A_n (f,x) - f(x) \right| \leq 0.0868 \left[\omega_1 \left(f, \frac{1}{n^\alpha} \right) + \frac{4 \|f\|_\infty}{\pi^2 (n^{1-\alpha} - 2)} \right] =: \rho_1 (f), \tag{50}$$

and

ii)

$$\|A_n (f) - f\|_\infty \leq \rho_1 (f). \tag{51}$$

We notice that $\lim_{n \to \infty} A_n (f) = f$, pointwise and uniformly.

We mention

Theorem 49 *([20]) Let $f \in C^N([a,b])$, $n, N \in \mathbb{N}$, $0 < \alpha < 1$, $x \in [a,b]$ and $n^{1-\alpha} > 2$. Then*

i)

$$\left|{}_1A_n(f,x) - f(x)\right| \leq 0.0868 \left\{ \sum_{j=1}^{N} \frac{\left|f^{(j)}(x)\right|}{j!} \left[\frac{1}{n^{\alpha j}} + \frac{2(b-a)^j}{\pi^2 (n^{1-\alpha} - 2)} \right] + \right. \tag{52}$$

$$\left. \left[\omega_1 \left(f^{(N)}, \frac{1}{n^{\alpha}} \right) \frac{1}{n^{\alpha N} N!} + \frac{4 \left\| f^{(N)} \right\|_{\infty} (b-a)^N}{N! \pi^2 (n^{1-\alpha} - 2)} \right] \right\} =: \gamma_{11}(f,x),$$

ii) assume further $f^{(j)}(x_0) = 0$, $j = 1, ..., N$, for some $x_0 \in [a,b]$, it holds

$$\left|{}_1A_n(f,x_0) - f(x_0)\right| \leq 0.0868 \cdot$$

$$\left\{ \omega_1 \left(f^{(N)}, \frac{1}{n^{\alpha}} \right) \frac{1}{n^{\alpha N} N!} + \frac{4 \left\| f^{(N)} \right\|_{\infty} (b-a)^N}{N! \pi^2 (n^{1-\alpha} - 2)} \right\} =: \gamma_{12}(f), \tag{53}$$

and

iii)

$$\left\|{}_1A_n(f) - f\right\|_{\infty} \leq 0.0868 \left\{ \sum_{j=1}^{N} \frac{\left\|f^{(j)}\right\|_{\infty}}{j!} \left[\frac{1}{n^{\alpha j}} + \frac{2(b-a)^j}{\pi^2 (n^{1-\alpha} - 2)} \right] + \right.$$

$$\left. \left[\omega_1 \left(f^{(N)}, \frac{1}{n^{\alpha}} \right) \frac{1}{n^{\alpha N} N!} + \frac{4 \left\| f^{(N)} \right\|_{\infty} (b-a)^N}{N! \pi^2 (n^{1-\alpha} - 2)} \right] \right\} =: \gamma_{13}(f). \tag{54}$$

Again we obtain $\lim_{n \to \infty} {}_1A_n(f) = f$, pointwise and uniformly.

We also mention the real valued fractional approximation result by neural networks.

Theorem 50 *([20]) Let $\alpha > 0$, $N = \lceil \alpha \rceil$, $\alpha \notin \mathbb{N}$, $f \in C^N([a,b])$, $0 < \beta < 1$, $x \in [a,b]$, $n \in \mathbb{N} : n^{1-\beta} > 2$. Then*

i)

$$\left| {}_1A_n(f,x) - \sum_{j=1}^{N-1} \frac{f^{(j)}(x)}{j!} \, {}_1A_n\left((\cdot - x)^j \right)(x) - f(x) \right| \leq$$

$$\frac{(0.0868)}{\Gamma(\alpha+1)} \left\{ \frac{\left(\omega_1 \left(D_{x-}^{\alpha} f, \frac{1}{n^{\beta}} \right)_{[a,x]} + \omega_1 \left(D_{*x}^{\alpha} f, \frac{1}{n^{\beta}} \right)_{[x,b]} \right)}{n^{\alpha\beta}} + \right.$$

$$\frac{2}{\pi^2\left(n^{1-\beta}-2\right)}\left(\left\|D_{x-}^{\alpha}f\right\|_{\infty,[a,x]}(x-a)^{\alpha}+\left\|D_{*x}^{\alpha}f\right\|_{\infty,[x,b]}(b-x)^{\alpha}\right)\Bigg\} =: \delta_{11}(f,x), \tag{55}$$

ii) if $f^{(j)}(x)=0$, for $j=1,...,N-1$, we have

$$|{}_1A_n(f,x)-f(x)| \leq \frac{(0.0868)}{\Gamma(\alpha+1)}$$

$$\left\{\frac{\left(\omega_1\left(D_{x-}^{\alpha}f,\frac{1}{n^{\beta}}\right)_{[a,x]}+\omega_1\left(D_{*x}^{\alpha}f,\frac{1}{n^{\beta}}\right)_{[x,b]}\right)}{n^{\alpha\beta}}+\right.$$

$$\frac{2}{\pi^2\left(n^{1-\beta}-2\right)}\left(\left\|D_{x-}^{\alpha}f\right\|_{\infty,[a,x]}(x-a)^{\alpha}+\left\|D_{*x}^{\alpha}f\right\|_{\infty,[x,b]}(b-x)^{\alpha}\right)\Bigg\} =: \delta_{12}(f,x), \tag{56}$$

iii)

$$|{}_1A_n(f,x)-f(x)| \leq (0.0868)\cdot$$

$$\left\{\sum_{j=1}^{N-1}\frac{\left|f^{(j)}(x)\right|}{j!}\left\{\frac{1}{n^{\beta j}}+(b-a)^j\frac{2}{\pi^2\left(n^{1-\beta}-2\right)}\right\}+\right.$$

$$\frac{1}{\Gamma(\alpha+1)}\left\{\frac{\left(\omega_1\left(D_{x-}^{\alpha}f,\frac{1}{n^{\beta}}\right)_{[a,x]}+\omega_1\left(D_{*x}^{\alpha}f,\frac{1}{n^{\beta}}\right)_{[x,b]}\right)}{n^{\alpha\beta}}+\right.$$

$$\frac{2}{\pi^2\left(n^{1-\beta}-2\right)}\left(\left\|D_{x-}^{\alpha}f\right\|_{\infty,[a,x]}(x-a)^{\alpha}+\left\|D_{*x}^{\alpha}f\right\|_{\infty,[x,b]}(b-x)^{\alpha}\right)\Bigg\}\Bigg\} =: \delta_{13}(f,x), \tag{57}$$

$\forall\ x\in[a,b]$,

and

iv)

$$\|{}_1A_nf-f\|_{\infty} \leq (0.0868)\cdot$$

$$\left\{\sum_{j=1}^{N-1}\frac{\left\|f^{(j)}\right\|_{\infty}}{j!}\left\{\frac{1}{n^{\beta j}}+(b-a)^j\frac{2}{\pi^2\left(n^{1-\beta}-2\right)}\right\}+\right.$$

$$\frac{1}{\Gamma(\alpha+1)}\left\{\frac{\left(\sup_{x\in[a,b]}\omega_1\left(D_{x-}^{\alpha}f,\frac{1}{n^{\beta}}\right)_{[a,x]}+\sup_{x\in[a,b]}\omega_1\left(D_{*x}^{\alpha}f,\frac{1}{n^{\beta}}\right)_{[x,b]}\right)}{n^{\alpha\beta}}+\right.$$

$$\frac{2}{\pi^2\left(n^{1-\beta}-2\right)}(b-a)^{\alpha}\left(\sup_{x\in[a,b]}\left\|D_{x-}^{\alpha}f\right\|_{\infty,[a,x]}+\sup_{x\in[a,b]}\left\|D_{*x}^{\alpha}f\right\|_{\infty,[x,b]}\right)\Bigg\}\Bigg\} =: \delta_{14}(f). \tag{58}$$

Above, when $N=1$ the sum $\sum_{j=1}^{N-1}\cdot=0$.

As we see here we obtain the real valued fractionally type pointwise and uniform convergence with rates of ${}_1A_n\to I$ the unit operator, as $n\to\infty$.

3.2 About the algebraic activation function neural networks

Here see also [21].

We consider the generator algebraic function

$$\varphi(x)=\frac{x}{\sqrt[2m]{1+x^{2m}}},\quad m\in\mathbb{N},\ x\in\mathbb{R}, \tag{59}$$

which is a sigmoidal type of function and is a strictly increasing function.

We see that $\varphi(-x)=-\varphi(x)$ with $\varphi(0)=0$. We get that

$$\varphi'(x)=\frac{1}{\left(1+x^{2m}\right)^{\frac{2m+1}{2m}}}>0,\ \ \forall\ x\in\mathbb{R}, \tag{60}$$

proving φ as strictly increasing over $\mathbb{R},\varphi'(x)=\varphi'(-x)$. We easily find that $\lim_{x\to+\infty}\varphi(x)=1$, $\varphi(+\infty)=1$, and $\lim_{x\to-\infty}\varphi(x)=-1$, $\varphi(-\infty)=-1$.

We consider the activation function

$$\psi_2(x)=\frac{1}{4}\left[\varphi(x+1)-\varphi(x-1)\right]. \tag{61}$$

Clearly it is $\psi_2(x)=\psi_2(-x)$, $\forall\ x\in\mathbb{R}$, so that ψ_2 is an even function and symmetric with respect to the y-axis. Clealry $\psi_2(x)>0$, $\forall\ x\in\mathbb{R}$.

Also it is

$$\psi_2(0)=\frac{1}{2\sqrt[2m]{2}}. \tag{62}$$

By [21], we have that $\psi_2'(x)<0$ for $x>0$. That is ψ_2 is strictly decreasing over $(0,+\infty)$.

Clearly, ψ_2 is strictly increasing over $(-\infty,0)$ and $\psi_2'(0)=0$.

Furthermore we obtain that

$$\lim_{x\to+\infty}\psi_2(x)=\frac{1}{4}\left[\varphi(+\infty)-\varphi(+\infty)\right]=0, \tag{63}$$

and

$$\lim_{x\to-\infty}\psi_2(x)=\frac{1}{4}\left[\varphi(-\infty)-\varphi(-\infty)\right]=0. \tag{64}$$

That is the x-axis is the horizontal asymptote of ψ_2.

Conclusion, ψ_2 is a bell shape symmetric function with maximum

$$\psi_2(0)=\frac{1}{2\sqrt[2m]{2}},\quad m\in\mathbb{N}. \tag{65}$$

We need

Theorem 51 *([21]) We have that*

$$\sum_{i=-\infty}^{\infty}\psi_2(x-i)=1,\ \ \forall\ x\in\mathbb{R}. \tag{66}$$

Theorem 52 *([21]) It holds*

$$\int_{-\infty}^{\infty} \psi_2(x)\, dx = 1. \tag{67}$$

Theorem 53 *([21]) Let $0 < \alpha < 1$, and $n \in \mathbb{N}$ with $n^{1-\alpha} > 2$. It holds*

$$\sum_{\begin{cases} k = -\infty \\ : |nx - k| \geq n^{1-\alpha} \end{cases}}^{\infty} \psi_2(nx - k) < \frac{1}{4m\left(n^{1-\alpha} - 2\right)^{2m}} =: c_2(\alpha, n), \quad m \in \mathbb{N}. \tag{68}$$

Theorem 54 *([21]) Let $[a, b] \subset \mathbb{R}$ and $n \in \mathbb{N}$ so that $\lceil na \rceil \leq \lfloor nb \rfloor$. It holds*

$$\frac{1}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \psi_2(nx - k)} < 2\left(\sqrt[2m]{1 + 4^m}\right) =: \alpha_2, \tag{69}$$

$\forall\, x \in [a, b]$, $m \in \mathbb{N}$.

We make

Remark 55 *([21]) (1) We have that*

$$\lim_{n \to \infty} \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \psi_2(nx - k) \neq 1, \quad \text{for at least some } x \in [a, b]. \tag{70}$$

(2) In general it holds that

$$\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \psi_2(nx - k) \leq 1. \tag{71}$$

We mention

Definition 56 *([21]) Let $f \in C([a, b])$ and $n \in \mathbb{N} : \lceil na \rceil \leq \lfloor nb \rfloor$. We introduce and define the real positive valued linear neural network operator*

$${}_2A_n(f, x) := \frac{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} f\left(\frac{k}{n}\right) \psi_2(nx - k)}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \psi_2(nx - k)}, \quad x \in [a, b]. \tag{72}$$

Clearly here ${}_2A_n(f, x) \in C([a, b])$.

We mention here about the pointwise and uniform convergence of ${}_2A_n(f, x)$ to $f(x)$ with rates.

Theorem 57 *([21]) Let $f \in C([a,b])$, $0<\alpha<1$, $n \in \mathbb{N} : n^{1-\alpha} > 2$, $x \in [a,b]$, $m \in \mathbb{N}$. Then*

i)

$$\left|{}_2A_n(f,x) - f(x)\right| \leq \left(\sqrt[2m]{1+4^m}\right)\left[2\omega_1\left(f, \frac{1}{n^\alpha}\right) + \frac{\|f\|_\infty}{m\left(n^{1-\alpha}-2\right)^{2m}}\right] =: \rho_2(f), \tag{73}$$

and

ii)

$$\left\|{}_2A_n(f) - f\right\|_\infty \leq \rho_2(f). \tag{74}$$

We get that $\lim_{n\to\infty} {}_2A_n(f) = f$, pointwise and uniformly.

In the next we mention the high order neural network real valued approximation result by using the smoothness of f.

Theorem 58 *([21]) Let $f \in C^N([a,b])$, $n, N, m \in \mathbb{N}$, $0<\alpha<1$, $x \in [a,b]$ and $n^{1-\alpha} > 2$. Then*

i)

$$\left|{}_2A_n(f,x) - f(x)\right| \leq \left(\sqrt[2m]{1+4^m}\right)\left\{\sum_{j=1}^{N} \frac{\left\|f^{(j)}(x)\right\|}{j!}\left[\frac{2}{n^{\alpha j}} + \frac{(b-a)^j}{2m\left(n^{1-\alpha}-2\right)^{2m}}\right] + \right. \tag{75}$$

$$\left.\left[\omega_1\left(f^{(N)}, \frac{1}{n^\alpha}\right)\frac{2}{n^{\alpha N} N!} + \frac{\left\|f^{(N)}\right\|_\infty (b-a)^N}{N! m\left(n^{1-\alpha}-2\right)^{2m}}\right]\right\} =: \gamma_{21}(f,x),$$

ii) assume further $f^{(j)}(x_0) = 0$, $j = 1, ..., N$, for some $x_0 \in [a,b]$, it holds

$$\left|{}_2A_n(f,x_0) - f(x_0)\right| \leq \left(\sqrt[2m]{1+4^m}\right) \cdot \tag{76}$$

$$\left[\omega_1\left(f^{(N)}, \frac{1}{n^\alpha}\right)\frac{2}{n^{\alpha N} N!} + \frac{\left\|f^{(N)}\right\|_\infty (b-a)^N}{N! m\left(n^{1-\alpha}-2\right)^{2m}}\right] =: \gamma_{22}(f),$$

and

iii)

$$\left\|{}_2A_n(f) - f\right\|_\infty \leq \left(\sqrt[2m]{1+4^m}\right)\left\{\sum_{j=1}^{N} \frac{\left\|f^{(j)}\right\|_\infty}{j!}\left[\frac{2}{n^{\alpha j}} + \frac{(b-a)^j}{2m\left(n^{1-\alpha}-2\right)^{2m}}\right] + \right.$$

$$\left.\left[\omega_1\left(f^{(N)}, \frac{1}{n^\alpha}\right)\frac{2}{n^{\alpha N} N!} + \frac{\left\|f^{(N)}\right\|_\infty (b-a)^N}{N! m\left(n^{1-\alpha}-2\right)^{2m}}\right]\right\} =: \gamma_{23}(f). \tag{77}$$

We derive that $\lim_{n\to\infty} {}_2A_n(f) = f$, pointwise and uniformly.

The corresponding fractional approximation result follows:

Theorem 59 *([21]) Let* $\alpha > 0$, $N = \lceil \alpha \rceil$, $\alpha \notin \mathbb{N}$, $f \in C^N([a,b])$, $0 < \beta < 1$, $m \in \mathbb{N}$, $x \in [a,b]$, $n \in \mathbb{N} : n^{1-\beta} > 2$. *Then*

i)

$$\left| {}_2A_n(f,x) - \sum_{j=1}^{N-1} \frac{f^{(j)}(x)}{j!}\ {}_2A_n\left((\cdot - x)^j\right)(x) - f(x)\right| \leq$$

$$\frac{2\left(\sqrt[2m]{1+4^m}\right)}{\Gamma(\alpha+1)}\left\{\frac{\left(\omega_1\left(D_{x-}^{\alpha}f, \frac{1}{n^\beta}\right)_{[a,x]} + \omega_1\left(D_{*x}^{\alpha}f, \frac{1}{n^\beta}\right)_{[x,b]}\right)}{n^{\alpha\beta}} + \right.$$

$$\left.\frac{1}{4m\left(n^{1-\beta}-2\right)^{2m}}\left(\left\|D_{x-}^{\alpha}f\right\|_{\infty,[a,x]}(x-a)^{\alpha} + \left\|D_{*x}^{\alpha}f\right\|_{\infty,[x,b]}(b-x)^{\alpha}\right)\right\} =: \delta_{21}(f,x), \tag{78}$$

ii) if $f^{(j)}(x) = 0$, *for* $j = 1, ..., N-1$, *we have*

$$\left|{}_2A_n(f,x) - f(x)\right| \leq \frac{2\left(\sqrt[2m]{1+4^m}\right)}{\Gamma(\alpha+1)}$$

$$\left\{\frac{\left(\omega_1\left(D_{x-}^{\alpha}f, \frac{1}{n^\beta}\right)_{[a,x]} + \omega_1\left(D_{*x}^{\alpha}f, \frac{1}{n^\beta}\right)_{[x,b]}\right)}{n^{\alpha\beta}} + \right.$$

$$\left.\frac{1}{4m\left(n^{1-\beta}-2\right)^{2m}}\left(\left\|D_{x-}^{\alpha}f\right\|_{\infty,[a,x]}(x-a)^{\alpha} + \left\|D_{*x}^{\alpha}f\right\|_{\infty,[x,b]}(b-x)^{\alpha}\right)\right\} =: \delta_{22}(f,x), \tag{79}$$

iii)

$$\left|{}_2A_n(f,x) - f(x)\right| \leq 2\left(\sqrt[2m]{1+4^m}\right) \cdot$$

$$\left\{\sum_{j=1}^{N-1} \frac{\left\|f^{(j)}(x)\right\|}{j!}\left\{\frac{1}{n^{\beta j}} + \frac{(b-a)^j}{4m\left(n^{1-\beta}-2\right)^{2m}}\right\} + \right.$$

$$\frac{1}{\Gamma(\alpha+1)}\left\{\frac{\left(\omega_1\left(D_{x-}^{\alpha}f, \frac{1}{n^\beta}\right)_{[a,x]} + \omega_1\left(D_{*x}^{\alpha}f, \frac{1}{n^\beta}\right)_{[x,b]}\right)}{n^{\alpha\beta}} + \right.$$

$$\left.\left.\frac{1}{4m\left(n^{1-\beta}-2\right)^{2m}}\left(\left\|D_{x-}^{\alpha}f\right\|_{\infty,[a,x]}(x-a)^{\alpha} + \left\|D_{*x}^{\alpha}f\right\|_{\infty,[x,b]}(b-x)^{\alpha}\right)\right\}\right\} =: \delta_{23}(f,x), \tag{80}$$

$\forall\ x \in [a,b]$,

and

iv)

$$\left\|{}_2A_nf - f\right\|_{\infty} \leq 2\left(\sqrt[2m]{1+4^m}\right) \cdot$$

$$\left\{\sum_{j=1}^{N-1}\frac{\left\|f^{(j)}\right\|_{\infty}}{j!}\left\{\frac{1}{n^{\beta j}}+\frac{(b-a)^{j}}{4m\left(n^{1-\beta}-2\right)^{2m}}\right\}+\right.$$

$$\frac{1}{\Gamma(\alpha+1)}\left\{\frac{\left(\sup_{x\in[a,b]}\omega_{1}\left(D_{x-}^{\alpha}f,\frac{1}{n^{\beta}}\right)_{[a,x]}+\sup_{x\in[a,b]}\omega_{1}\left(D_{*x}^{\alpha}f,\frac{1}{n^{\beta}}\right)_{[x,b]}\right)}{n^{\alpha\beta}}+\right.$$

$$\left.\left.\frac{(b-a)^{\alpha}}{4m\left(n^{1-\beta}-2\right)^{2m}}\left(\sup_{x\in[a,b]}\left\|D_{x-}^{\alpha}f\right\|_{\infty,[a,x]}+\sup_{x\in[a,b]}\left\|D_{*x}^{\alpha}f\right\|_{\infty,[x,b]}\right)\right\}\right\}=:\delta_{24}(f). \tag{81}$$

Above, when $N=1$ the sum $\sum_{j=1}^{N-1}\cdot=0$.

As we see here we obtain the real valued fractionally type pointwise and uniform convergence with rates of ${}_2A_n\rightarrow I$ the unit operator, as $n\rightarrow\infty$.

3.3 About the Gudermannian activation function neural networks

See also [22], [41].

Here we consider $gd(x)$ the Gudermannian function [22], which is a sigmoid function, as a generator function:

$$\sigma(x)=2\arctan\left(\tanh\left(\frac{x}{2}\right)\right)=\int_{0}^{x}\frac{dt}{\cosh t}=:gd(x),\ x\in\mathbb{R}. \tag{82}$$

Let the normalized generator sigmoid function

$$f(x):=\frac{4}{\pi}\sigma(x)=\frac{4}{\pi}\int_{0}^{x}\frac{dt}{\cosh t}=\frac{8}{\pi}\int_{0}^{x}\frac{1}{e^{t}+e^{-t}}dt,\quad x\in\mathbb{R}. \tag{83}$$

Here

$$f'(x)=\frac{4}{\pi\cosh x}>0,\quad\forall\ x\in\mathbb{R},$$

hence f is strictly increasing on $\mathbb{R}$.

Notice that $\tanh(-x)=-\tanh x$ and $\arctan(-x)=-\arctan x$, $x\in\mathbb{R}$.

So, here the neural network activation function will be:

$$\psi_{3}(x)=\frac{1}{4}\left[f(x+1)-f(x-1)\right],\ x\in\mathbb{R}. \tag{84}$$

By [22], we get that

$$\psi_{3}(x)=\psi_{3}(-x),\quad\forall\ x\in\mathbb{R}, \tag{85}$$

i.e. it is even and symmetric with respect to the y-axis. Here we have $f(+\infty)=1$, $f(-\infty)=-1$ and $f(0)=0$. Clearly it is

$$f(-x)=-f(x),\ \ \forall\ x\in\mathbb{R}, \tag{86}$$

an odd function, symmetric with respect to the origin. Since $x+1>x-1$, and $f(x+1)>f(x-1)$, we obtain $\psi_3(x)>0$, $\forall\ x\in\mathbb{R}$.

By [22], we have that

$$\psi_3(0)=\frac{2}{\pi}gd(1)\cong 0.551. \tag{87}$$

By [22] ψ_3 is strictly decreasing on $(0,+\infty)$, and strictly increasing on $(-\infty,0)$, and $\psi_3'(0)=0$.

Also we have that

$$\lim_{x\to+\infty}\psi_3(x)=\lim_{x\to-\infty}\psi_3(x)=0, \tag{88}$$

that is the x-axis is the horizontal asymptote for ψ_3.

Conclusion, ψ_3 is a bell shaped symmetric function with maximum $\psi_3(0)\cong 0.551$.

We mention

Theorem 60 *([22]) It holds that*

$$\sum_{i=-\infty}^{\infty}\psi_3(x-i)=1,\ \ \forall\ x\in\mathbb{R}. \tag{89}$$

Theorem 61 *([22]) We have that*

$$\int_{-\infty}^{\infty}\psi_3(x)\,dx=1. \tag{90}$$

So $\psi_3(x)$ is a density function.

Theorem 62 *([22]) Let $0<\alpha<1$, and $n\in\mathbb{N}$ with $n^{1-\alpha}>2$. It holds*

$$\sum_{\begin{cases} k=-\infty \\ :|nx-k|\geq n^{1-\alpha}\end{cases}}^{\infty}\psi_3(nx-k)<\frac{4}{\pi e^{(n^{1-\alpha}-2)}}=\frac{4e^2}{\pi e^{n^{1-\alpha}}}=:c_3(\alpha,n). \tag{91}$$

Theorem 63 *([22]) Let $[a,b]\subset\mathbb{R}$ and $n\in\mathbb{N}$ so that $\lceil na\rceil\leq\lfloor nb\rfloor$. It holds*

$$\frac{1}{\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}\psi_3(nx-k)}<2.412=:\alpha_3, \tag{92}$$

$\forall\ x\in[a,b]$.

We make

Remark 64 *([22]) (1) We have that*

$$\lim_{n\to\infty} \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \psi_3 (nx - k) \neq 1, \quad \text{for at least some } x \in [a,b]. \tag{93}$$

(2) In general it holds

$$\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \psi_3 (nx - k) \leq 1. \tag{94}$$

Definition 65 *([22]) Let $f \in C([a,b])$ and $n \in \mathbb{N} : \lceil na \rceil \leq \lfloor nb \rfloor$. We define the real positive valued linear neural network operator*

$${}_3A_n (f, x) := \frac{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} f\left(\frac{k}{n}\right) \psi_3 (nx - k)}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \psi_3 (nx - k)}, \quad x \in [a,b]. \tag{95}$$

Clearly here ${}_3A_n (f, x) \in C([a,b])$.

We mention here about the pointwise and uniform convergence of ${}_3A_n (f, x)$ to $f(x)$ with rates.

Theorem 66 *([22]) Let $f \in C([a,b])$, $0 < \alpha < 1$, $n \in \mathbb{N} : n^{1-\alpha} > 2$, $x \in [a,b]$. Then*

i)

$$\left|{}_3A_n (f, x) - f(x)\right| \leq 2.412 \left[\omega_1 \left(f, \frac{1}{n^\alpha}\right) + \frac{8 \|f\|_\infty}{\pi e^{(n^{1-\alpha}-2)}}\right] =: \rho_3 (f), \tag{96}$$

and

ii)

$$\left\|{}_3A_n (f) - f\right\|_\infty \leq \rho_3 (f). \tag{97}$$

We get that $\lim_{n\to\infty} {}_3A_n (f) = f$, pointwise and uniformly.

In the next we mention the high order neural network real valued approximation result by using the smoothness of f.

Theorem 67 *([22]) Let $f \in C^N([a,b])$, $n, N \in \mathbb{N}$, $0 < \alpha < 1$, $x \in [a,b]$ and $n^{1-\alpha} > 2$. Then*

i)

$$\left|{}_3A_n (f, x) - f(x)\right| \leq 2.412 \left\{ \sum_{j=1}^{N} \frac{\left|f^{(j)}(x)\right|}{j!} \left[\frac{1}{n^{\alpha j}} + \frac{4 (b-a)^j}{\pi e^{(n^{1-\alpha}-2)}}\right] + \right. \tag{98}$$

$$\left[\omega_1\left(f^{(N)},\frac{1}{n^{\alpha}}\right)\frac{1}{n^{\alpha N}N!}+\frac{8\left\|f^{(N)}\right\|_{\infty}(b-a)^N}{N!\pi e^{(n^{1-\alpha}-2)}}\right]\Bigg\}=:\gamma_{31}(f,x),$$

ii) assume further $f^{(j)}(x_0)=0$, $j=1,...,N$, *for some* $x_0\in[a,b]$, *it holds*

$$\left|{}_3A_n(f,x_0)-f(x_0)\right|\leq 2.412\cdot$$

$$\left\{\omega_1\left(f^{(N)},\frac{1}{n^{\alpha}}\right)\frac{1}{n^{\alpha N}N!}+\frac{8\left\|f^{(N)}\right\|_{\infty}(b-a)^N}{N!\pi e^{(n^{1-\alpha}-2)}}\right\}=:\gamma_{32}(f),\tag{99}$$

and

iii)

$$\left\|{}_3A_n(f)-f\right\|_{\infty}\leq 2.412\left\{\sum_{j=1}^{N}\frac{\left\|f^{(j)}\right\|_{\infty}}{j!}\left[\frac{1}{n^{\alpha j}}+\frac{4(b-a)^j}{\pi e^{(n^{1-\alpha}-2)}}\right]+\right.$$

$$\left.\left[\omega_1\left(f^{(N)},\frac{1}{n^{\alpha}}\right)\frac{1}{n^{\alpha N}N!}+\frac{8\left\|f^{(N)}\right\|_{\infty}(b-a)^N}{N!\pi e^{(n^{1-\alpha}-2)}}\right]\right\}=:\gamma_{33}(f).\tag{100}$$

Again we obtain $\lim_{n\to\infty}{}_3A_n(f)=f$, *pointwise and uniformly.*

We also mention the following real valued fractional approximation result by neural networks.

Theorem 68 *([22]) Let* $\alpha>0$, $N=\lceil\alpha\rceil$, $\alpha\notin\mathbb{N}$, $f\in C^N([a,b])$, $0<\beta<1$, $x\in[a,b]$, $n\in\mathbb{N}:n^{1-\beta}>2$. *Then*

i)

$$\left|{}_3A_n(f,x)-\sum_{j=1}^{N-1}\frac{f^{(j)}(x)}{j!}\,{}_3A_n\left((\cdot-x)^j\right)(x)-f(x)\right|\leq$$

$$\frac{2.412}{\Gamma(\alpha+1)}\left\{\frac{\left(\omega_1\left(D_{x-}^{\alpha}f,\frac{1}{n^{\beta}}\right)_{[a,x]}+\omega_1\left(D_{*x}^{\alpha}f,\frac{1}{n^{\beta}}\right)_{[x,b]}\right)}{n^{\alpha\beta}}+\right.$$

$$\left.\frac{4}{\pi e^{(n^{1-\alpha}-2)}}\left(\left\|D_{x-}^{\alpha}f\right\|_{\infty,[a,x]}(x-a)^{\alpha}+\left\|D_{*x}^{\alpha}f\right\|_{\infty,[x,b]}(b-x)^{\alpha}\right)\right\}=:\delta_{31}(f,x),\tag{101}$$

ii) if $f^{(j)}(x)=0$, *for* $j=1,...,N-1$, *we have*

$$\left|{}_3A_n(f,x)-f(x)\right|\leq\frac{2.412}{\Gamma(\alpha+1)}$$

$$\left\{\frac{\left(\omega_1\left(D_{x-}^{\alpha}f,\frac{1}{n^{\beta}}\right)_{[a,x]}+\omega_1\left(D_{*x}^{\alpha}f,\frac{1}{n^{\beta}}\right)_{[x,b]}\right)}{n^{\alpha\beta}}+\right.$$

$$\frac{4}{\pi^2\left(n^{1-\beta}-2\right)}\left(\left\|D_{x-}^{\alpha}f\right\|_{\infty,[a,x]}(x-a)^{\alpha}+\left\|D_{*x}^{\alpha}f\right\|_{\infty,[x,b]}(b-x)^{\alpha}\right)\Bigg\} =: \delta_{32}(f,x), \tag{102}$$

iii)

$$\left|{}_3A_n(f,x)-f(x)\right| \leq 2.412\cdot$$

$$\left\{\sum_{j=1}^{N-1}\frac{\left|f^{(j)}(x)\right|}{j!}\left\{\frac{1}{n^{\beta j}}+(b-a)^j\frac{4}{\pi e^{\left(n^{1-\alpha}-2\right)}}\right\}+\right.$$

$$\frac{1}{\Gamma(\alpha+1)}\left\{\frac{\left(\omega_1\left(D_{x-}^{\alpha}f,\frac{1}{n^{\beta}}\right)_{[a,x]}+\omega_1\left(D_{*x}^{\alpha}f,\frac{1}{n^{\beta}}\right)_{[x,b]}\right)}{n^{\alpha\beta}}+\right.$$

$$\left.\left.\frac{4}{\pi e^{\left(n^{1-\alpha}-2\right)}}\left(\left\|D_{x-}^{\alpha}f\right\|_{\infty,[a,x]}(x-a)^{\alpha}+\left\|D_{*x}^{\alpha}f\right\|_{\infty,[x,b]}(b-x)^{\alpha}\right)\right\}\right\} =: \delta_{33}(f,x), \tag{103}$$

$\forall\ x\in[a,b]$,

and

iv)

$$\left\|{}_3A_nf-f\right\|_{\infty} \leq 2.412\cdot$$

$$\left\{\sum_{j=1}^{N-1}\frac{\left\|f^{(j)}\right\|_{\infty}}{j!}\left\{\frac{1}{n^{\beta j}}+(b-a)^j\frac{4}{\pi e^{\left(n^{1-\alpha}-2\right)}}\right\}+\right.$$

$$\frac{1}{\Gamma(\alpha+1)}\left\{\frac{\left(\sup_{x\in[a,b]}\omega_1\left(D_{x-}^{\alpha}f,\frac{1}{n^{\beta}}\right)_{[a,x]}+\sup_{x\in[a,b]}\omega_1\left(D_{*x}^{\alpha}f,\frac{1}{n^{\beta}}\right)_{[x,b]}\right)}{n^{\alpha\beta}}+\right.$$

$$\left.\left.\frac{4}{\pi e^{\left(n^{1-\alpha}-2\right)}}(b-a)^{\alpha}\left(\sup_{x\in[a,b]}\left\|D_{x-}^{\alpha}f\right\|_{\infty,[a,x]}+\sup_{x\in[a,b]}\left\|D_{*x}^{\alpha}f\right\|_{\infty,[x,b]}\right)\right\}\right\} =: \delta_{34}(f). \tag{104}$$

Above, when $N=1$ the sum $\sum_{j=1}^{N-1}\cdot=0$.

As we see here we obtain real valued fractionally type pointwise and uniform convergence with rates of ${}_3A_n\to I$ the unit operator, as $n\to\infty$.

3.4 About the generalized symmetrical activation function neural networks

Here we consider the generalized symmetrical sigmoid function ([23], [30])

$$f_1(x)=\frac{x}{\left(1+|x|^{\mu}\right)^{\frac{1}{\mu}}},\quad \mu>0,\ x\in\mathbb{R}. \tag{105}$$

This has applications in immunology and protection from disease together with probability theory. It is also called a symmetrical protection curve.

The parameter μ is a shape parameter controling how fast the curve approaches the asymptotes for a given slope at the inflection point. When $\mu = 1$ f_1 is the absolute sigmoid function, and when $\mu = 2$, f_1 is the square root sigmoid function. When $\mu = 1.5$ the function approximates the arctangent function, when $\mu = 2.9$ it approximates the logistic function, and when $\mu = 3.4$ it approximates the error function. Parameter μ is estimated in the likelihood maximization ([30]). For more see [30].

Next we study the particular generator sigmoid function

$$f_2(x) = \frac{x}{\left(1 + |x|^{\lambda}\right)^{\frac{1}{\lambda}}}, \quad \lambda \text{ is an odd number, } x \in \mathbb{R}. \tag{106}$$

We have that $f_2(0) = 0$, and

$$f_2(-x) = -f_2(x), \tag{107}$$

so f_2 is symmetric with respect to zero.

When $x \geq 0$, we get that ([23])

$$f_2'(x) = \frac{1}{\left(1 + x^{\lambda}\right)^{\frac{\lambda+1}{\lambda}}} > 0, \tag{108}$$

that is f_2 is strictly increasing on $[0, +\infty)$ and f_2 is strictly increasing on $(-\infty, 0]$. Hence f_2 is strictly increasing on $\mathbb{R}$.

We also have $f_2(+\infty) = f_2(-\infty) = 1$.

Let us consider the activation function ([23]):

$$\psi_4(x) = \frac{1}{4}\left[f_2(x+1) - f_2(x-1)\right] =$$

$$\frac{1}{4}\left[\frac{(x+1)}{\left(1 + |x+1|^{\lambda}\right)^{\frac{1}{\lambda}}} - \frac{(x-1)}{\left(1 + |x-1|^{\lambda}\right)^{\frac{1}{\lambda}}}\right]. \tag{109}$$

Clearly it holds ([23])

$$\psi_4(x) = \psi_4(-x), \quad \forall\, x \in \mathbb{R}. \tag{110}$$

and

$$\psi_4(0) = \frac{1}{2\sqrt[\lambda]{2}}, \tag{111}$$

and $\psi_4(x) > 0$, $\forall\, x \in \mathbb{R}$.

Following [23], we have that ψ_4 is strictly decreasing over $[0, +\infty)$, and ψ_4 is strictly increasing on $(-\infty, 0]$, by ψ_4-symmetry with respect to y-axis, and $\psi_4'(0) = 0$.

Clearly it is

$$\lim_{x \to +\infty} \psi_4(x) = \lim_{x \to -\infty} \psi_4(x) = 0, \tag{112}$$

therefore the x-axis is the horizontal asymptote of $\psi_4(x)$.

The value

$$\psi_4(0) = \frac{1}{2\sqrt[\lambda]{2}}, \ \lambda \text{ is an odd number}, \tag{113}$$

is the maximum of ψ_4, which is a bell shaped function.

We need

Theorem 69 *([23]) It holds*

$$\sum_{i=-\infty}^{\infty} \psi_4(x - i) = 1, \ \ \forall \ x \in \mathbb{R}. \tag{114}$$

Theorem 70 *([23]) We have that*

$$\int_{-\infty}^{\infty} \psi_4(x) \, dx = 1. \tag{115}$$

So that $\psi_4(x)$ is a density function on $\mathbb{R}$.

We need

Theorem 71 *([23]) Let $0 < \alpha < 1$, and $n \in \mathbb{N}$ with $n^{1-\alpha} > 2$. It holds*

$$\sum_{\begin{cases} j = -\infty \\ : |nx - j| \geq n^{1-\alpha} \end{cases}}^{\infty} \psi_4(nx - j) < \frac{1}{2\lambda \left(n^{1-\alpha} - 2\right)^{\lambda}} =: c_4(\alpha, n), \tag{116}$$

where $\lambda \in \mathbb{N}$ is an odd number.

Theorem 72 *([23]) Let $[a, b] \subset \mathbb{R}$ and $n \in \mathbb{N}$ so that $\lceil na \rceil \leq \lfloor nb \rfloor$. Then*

$$\frac{1}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \psi_4(|nx - k|)} < 2\sqrt[\lambda]{1 + 2^{\lambda}} =: \alpha_4, \tag{117}$$

where λ is an odd number, $\forall \ x \in [a, b]$.

We make

Remark 73 *([23]) (1) We have that*

$$\lim_{n\to\infty}\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}\psi_4(nx-k)\neq 1,\quad \textit{for at least some } x\in[a,b]. \tag{118}$$

(2) In general it holds

$$\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}\psi_4(nx-k)\leq 1. \tag{119}$$

We mention

Definition 74 *([23]) Let $f\in C([a,b])$ and $n\in\mathbb{N}:\lceil na\rceil\leq\lfloor nb\rfloor$. We define the real positive valued linear neural network operator*

$${}_4A_n(f,x):=\frac{\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}f\left(\frac{k}{n}\right)\psi_4(nx-k)}{\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}\psi_4(nx-k)},\quad x\in[a,b]. \tag{120}$$

Clearly here ${}_4A_n(f,x)\in C([a,b])$.

We mention results about the pointwise and uniform convergence of ${}_4A_n(f,x)$ to $f(x)$ with rates.

Theorem 75 *([23]) Let $f\in C([a,b])$, $0<\alpha<1$, $n\in\mathbb{N}:n^{1-\alpha}>2$, $x\in[a,b]$, $\lambda\in\mathbb{N}$ is odd. Then*

i)

$$|{}_4A_n(f,x)-f(x)|\leq 2\sqrt[\lambda]{1+2^{\lambda}}\left[\omega_1\left(f,\frac{1}{n^{\alpha}}\right)+\frac{\|f\|_{\infty}}{\lambda\left(n^{1-\alpha}-2\right)^{\lambda}}\right]=:\rho_4(f), \tag{121}$$

and

ii)

$$\|{}_4A_n(f)-f\|_{\infty}\leq\rho_4(f). \tag{122}$$

We get that $\lim_{n\to\infty}{}_4A_n(f)=f$, pointwise and uniformly.

In the next we mention about the high order neural network real valued approximation result by using the smoothness of f.

Theorem 76 *([23]) Let $f\in C^N([a,b])$, $n,N\in\mathbb{N}$, λ is odd, $0<\alpha<1$, $x\in[a,b]$ and $n^{1-\alpha}>2$. Then* *i)*

$$\left|{}_4A_n(f,x)-f(x)\right| \leq \sqrt[\lambda]{1+2^{\lambda}}\left\{\sum_{j=1}^{N}\frac{\left|f^{(j)}(x)\right|}{j!}\left[\frac{2}{n^{\alpha j}}+\frac{(b-a)^{j}}{\lambda\left(n^{1-\alpha}-2\right)^{\lambda}}\right]+\right.$$

$$\left.\left[\omega_1\left(f^{(N)},\frac{1}{n^{\alpha}}\right)\frac{2}{n^{\alpha N}N!}+\frac{2\left\|f^{(N)}\right\|_{\infty}(b-a)^{N}}{N!\lambda\left(n^{1-\alpha}-2\right)^{\lambda}}\right]\right\} =: \gamma_{41}(f,x), \tag{123}$$

ii) assume further $f^{(j)}(x_0)=0$, $j=1,...,N$, *for some* $x_0\in[a,b]$, *it holds*

$$\left|{}_4A_n(f,x_0)-f(x_0)\right| \leq \sqrt[\lambda]{1+2^{\lambda}}\cdot$$

$$\left[\omega_1\left(f^{(N)},\frac{1}{n^{\alpha}}\right)\frac{2}{n^{\alpha N}N!}+\frac{2\left\|f^{(N)}\right\|_{\infty}(b-a)^{N}}{N!\lambda\left(n^{1-\alpha}-2\right)^{\lambda}}\right] =: \gamma_{42}(f), \tag{124}$$

and

iii)

$$\left\|{}_4A_n(f)-f\right\|_{\infty} \leq \sqrt[\lambda]{1+2^{\lambda}}\left\{\sum_{j=1}^{N}\frac{\left\|f^{(j)}\right\|_{\infty}}{j!}\left[\frac{2}{n^{\alpha j}}+\frac{(b-a)^{j}}{\lambda\left(n^{1-\alpha}-2\right)^{\lambda}}\right]+\right.$$

$$\left.\left[\omega_1\left(f^{(N)},\frac{1}{n^{\alpha}}\right)\frac{2}{n^{\alpha N}N!}+\frac{2\left\|f^{(N)}\right\|_{\infty}(b-a)^{N}}{N!\lambda\left(n^{1-\alpha}-2\right)^{\lambda}}\right]\right\} =: \gamma_{43}(f). \tag{125}$$

We derive that $\lim_{n\to\infty} {}_4A_n(f)=f$, *pointwise and uniformly.*

Next we mention the corresponding real valued fractional approximation result by neural networks.

Theorem 77 *([23]) Let* $\alpha>0$, $N=\lceil\alpha\rceil$, $\alpha\notin\mathbb{N}$, $f\in C^N([a,b])$, $0<\beta<1$, λ *is odd,* $x\in[a,b]$, $n\in\mathbb{N}: n^{1-\beta}>2$. *Then*

i)

$$\left|{}_4A_n(f,x)-\sum_{j=1}^{N-1}\frac{f^{(j)}(x)}{j!}\,{}_4A_n\left((\cdot-x)^{j}\right)(x)-f(x)\right| \leq$$

$$\frac{2\sqrt[\lambda]{1+2^{\lambda}}}{\Gamma(\alpha+1)}\left\{\frac{\left(\omega_1\left(D_{x-}^{\alpha}f,\frac{1}{n^{\beta}}\right)_{[a,x]}+\omega_1\left(D_{*x}^{\alpha}f,\frac{1}{n^{\beta}}\right)_{[x,b]}\right)}{n^{\alpha\beta}}+\right.$$

$$\left.\frac{1}{2\lambda\left(n^{1-\beta}-2\right)^{\lambda}}\left(\left\|D_{x-}^{\alpha}f\right\|_{\infty,[a,x]}(x-a)^{\alpha}+\left\|D_{*x}^{\alpha}f\right\|_{\infty,[x,b]}(b-x)^{\alpha}\right)\right\} =: \delta_{41}(f,x), \tag{126}$$

ii) if $f^{(j)}(x)=0$*, for* $j=1,...,N-1$*, we have*

$$\left|{}_4A_n(f,x)-f(x)\right| \leq \frac{2\sqrt[\lambda]{1+2^{\lambda}}}{\Gamma(\alpha+1)}$$

$$\left\{\frac{\left(\omega_1\left(D_{x-}^{\alpha}f,\frac{1}{n^{\beta}}\right)_{[a,x]}+\omega_1\left(D_{*x}^{\alpha}f,\frac{1}{n^{\beta}}\right)_{[x,b]}\right)}{n^{\alpha\beta}}+\right.$$

$$\left.\frac{1}{2\lambda\left(n^{1-\beta}-2\right)^{\lambda}}\left(\left\|D_{x-}^{\alpha}f\right\|_{\infty,[a,x]}(x-a)^{\alpha}+\left\|D_{*x}^{\alpha}f\right\|_{\infty,[x,b]}(b-x)^{\alpha}\right)\right\}=:\delta_{42}(f,x), \tag{127}$$

iii)

$$\left|{}_4A_n(f,x)-f(x)\right| \leq 2\sqrt[\lambda]{1+2^{\lambda}}\cdot$$

$$\left\{\sum_{j=1}^{N-1}\frac{\left\|f^{(j)}(x)\right\|}{j!}\left\{\frac{1}{n^{\beta j}}+\frac{(b-a)^{j}}{2\lambda\left(n^{1-\beta}-2\right)^{\lambda}}\right\}+\right.$$

$$\frac{1}{\Gamma(\alpha+1)}\left\{\frac{\left(\omega_1\left(D_{x-}^{\alpha}f,\frac{1}{n^{\beta}}\right)_{[a,x]}+\omega_1\left(D_{*x}^{\alpha}f,\frac{1}{n^{\beta}}\right)_{[x,b]}\right)}{n^{\alpha\beta}}+\right.$$

$$\left.\left.\frac{1}{2\lambda\left(n^{1-\beta}-2\right)^{\lambda}}\left(\left\|D_{x-}^{\alpha}f\right\|_{\infty,[a,x]}(x-a)^{\alpha}+\left\|D_{*x}^{\alpha}f\right\|_{\infty,[x,b]}(b-x)^{\alpha}\right)\right\}\right\}=:\delta_{43}(f,x), \tag{128}$$

$\forall\ x\in[a,b]$,

and

iv)

$$\left\|{}_4A_nf-f\right\|_{\infty} \leq 2\sqrt[\lambda]{1+2^{\lambda}}\cdot$$

$$\left\{\sum_{j=1}^{N-1}\frac{\left\|f^{(j)}\right\|_{\infty}}{j!}\left\{\frac{1}{n^{\beta j}}+\frac{(b-a)^{j}}{2\lambda\left(n^{1-\beta}-2\right)^{\lambda}}\right\}+\right.$$

$$\frac{1}{\Gamma(\alpha+1)}\left\{\frac{\left(\sup_{x\in[a,b]}\omega_1\left(D_{x-}^{\alpha}f,\frac{1}{n^{\beta}}\right)_{[a,x]}+\sup_{x\in[a,b]}\omega_1\left(D_{*x}^{\alpha}f,\frac{1}{n^{\beta}}\right)_{[x,b]}\right)}{n^{\alpha\beta}}+\right.$$

$$\left.\left.\frac{(b-a)^{\alpha}}{2\lambda\left(n^{1-\beta}-2\right)^{\lambda}}\left(\sup_{x\in[a,b]}\left\|D_{x-}^{\alpha}f\right\|_{\infty,[a,x]}+\sup_{x\in[a,b]}\left\|D_{*x}^{\alpha}f\right\|_{\infty,[x,b]}\right)\right\}\right\}=:\delta_{44}(f). \tag{129}$$

Above, when $N=1$ *the sum* $\sum_{j=1}^{N-1}\cdot=0$.

As we see here we obtain real valued fractionally type pointwise and uniform convergence with rates of ${}_4A_n\to I$ *the unit operator, as* $n\to\infty$.

For further related results see [24]-[27].

4 Main Results: Approximation by Fuzzy Quasi-Interpolation Neural Network Operators with respect to arctangent, algebraic, Gudermannian and generalized symmetrical activation functions

Let $f \in C_{\mathcal{F}}([a,b])$ (fuzzy continuous functions on $[a,b] \subset \mathbb{R}$), $n \in \mathbb{N}$. We define the following Fuzzy Quasi-Interpolation Neural Network operators

$$ {}_jA_n^{\mathcal{F}}(f,x) = \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor *} f\left(\frac{k}{n}\right) \odot \frac{\psi_j(nx-k)}{\sum\limits_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \psi_j(nx-k)}, \tag{130} $$

$\forall\, x \in [a,b]$, see also (49), (72), (95) and (120); $j = 1,2,3,4$.

The fuzzy sums in (130) are finite.

Let $r \in [0,1]$, we observe that

$$ {}_j\left[A_n^{\mathcal{F}}(f,x)\right]^r = \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \left[f\left(\frac{k}{n}\right)\right]^r \left(\frac{\psi_j(nx-k)}{\sum\limits_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \psi_j(nx-k)}\right) = $$

$$ \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \left[f_-^{(r)}\left(\frac{k}{n}\right), f_+^{(r)}\left(\frac{k}{n}\right)\right] \left(\frac{\psi_j(nx-k)}{\sum\limits_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \psi_j(nx-k)}\right) = $$

$$ \left[\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} f_-^{(r)}\left(\frac{k}{n}\right) \left(\frac{\psi_j(nx-k)}{\sum\limits_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \psi_j(nx-k)}\right), \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} f_+^{(r)}\left(\frac{k}{n}\right) \left(\frac{\psi_j(nx-k)}{\sum\limits_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \psi_j(nx-k)}\right)\right] \tag{131} $$

$$ = \left[{}_jA_n\left(f_-^{(r)}, x\right),\ {}_jA_n\left(f_+^{(r)}, x\right)\right]. \tag{132} $$

We have proved that

$$ \left({}_jA_n^{\mathcal{F}}(f,x)\right)_{\pm}^{(r)} = {}_jA_n\left(f_{\pm}^{(r)}, x\right), \quad j = 1,2,3,4, \tag{133} $$

respectively, $\forall\, r \in [0,1]$, $\forall\, x \in [a,b]$.

Therefore we get

$$ D\left({}_jA_n^{\mathcal{F}}(f,x), f(x)\right) = $$

$$\sup_{r\in[0,1]} \max\left\{\left|{}_{j}A_{n}\left(f_{-}^{(r)},x\right)-f_{-}^{(r)}(x)\right|,\left|{}_{j}A_{n}\left(f_{+}^{(r)},x\right)-f_{+}^{(r)}(x)\right|\right\}, \tag{134}$$

$\forall\ x\in[a,b]$; $j=1,2,3,4$.

We present our first fuzzy neural network approximation result.

Theorem 78 *Let $f\in C_{\mathcal{F}}([a,b])$, $0<\alpha<1$, $x\in[a,b]$, $n\in\mathbb{N}$ with $n^{1-\alpha}>2$; $j=1,2,3,4$. Then*

1)

$$D\left({}_{j}A_{n}^{\mathcal{F}}(f,x),f(x)\right)\leq\alpha_{j}\left[\omega_{1}^{(\mathcal{F})}\left(f,\frac{1}{n^{\alpha}}\right)+2c_{j}(\alpha,n)D^{*}(f,\widetilde{o})\right]=:\ {}_{j}T_{1n}, \tag{135}$$

and

2)

$$D^{*}\left({}_{j}A_{n}^{\mathcal{F}}(f),f\right)\leq\ {}_{j}T_{1n}. \tag{136}$$

We notice that $\lim_{n\to\infty}\left({}_{j}A_{n}^{\mathcal{F}}(f)\right)(x)\overset{D}{\to}f(x)$, $\lim_{n\to\infty}{}_{j}A_{n}^{\mathcal{F}}(f)\overset{D^{*}}{\to}f$, pointwise and uniformly.

Above α_{j} are as in (46), (69), (92), (117), and $c_{j}(\alpha,n)$ are as in (45), (68), (91), (116); $j=1,2,3,4$, respectively.

Proof. We have that $f_{\pm}^{(r)}\in C([a,b])$, $\forall\ r\in[0,1]$. Hence by (50), (73), (96) and (121), we obtain $(j=1,2,3,4)$

$$\left|{}_{j}A_{n}\left(f_{\pm}^{(r)},x\right)-f_{\pm}^{(r)}(x)\right|\leq\alpha_{j}\left[\omega_{1}\left(f_{\pm}^{(r)},\frac{1}{n^{\alpha}}\right)+2c_{j}(\alpha,n)\left\|f_{\pm}^{(r)}\right\|_{\infty}\right] \tag{137}$$

(by Proposition 7 and $\left\|f_{\pm}^{(r)}\right\|_{\infty}\leq D^{*}(f,\widetilde{o})$)

$$\leq\alpha_{j}\left[\omega_{1}^{(\mathcal{F})}\left(f,\frac{1}{n^{\alpha}}\right)+2c_{j}(\alpha,n)D^{*}(f,\widetilde{o})\right]. \tag{138}$$

Taking into account (134) the theorem is proved. ■

We also give

Theorem 79 *Let $f\in C_{\mathcal{F}}^{N}([a,b])$, $N\in\mathbb{N}$, $0<\alpha<1$, $x\in[a,b]$, $n\in\mathbb{N}$ with $n^{1-\alpha}>2$; $j=1,2,3,4$. Then*

1)

$$D\left({}_{j}A_{n}^{\mathcal{F}}(f,x),f(x)\right)\leq\alpha_{j}$$

$$\left\{\sum_{j_{*}=1}^{N}\frac{D\left(f^{(j_{*})}(x),\widetilde{o}\right)}{j_{*}!}\left[\frac{1}{n^{\alpha j_{*}}}+c_{j}(\alpha,n)(b-a)^{j_{*}}\right]+\right.$$

$$\left.\left[\omega_{1}^{(\mathcal{F})}\left(f^{(N)},\frac{1}{n^{\alpha}}\right)\frac{1}{n^{\alpha N}N!}+2c_{j}(\alpha,n)D^{*}\left(f^{(N)},\widetilde{o}\right)\frac{(b-a)^{N}}{N!}\right]\right\}, \tag{139}$$

2) assume further that $f^{(j_*)}(x_0) = \widetilde{o}$, $j_* = 1, ..., N$, *for some* $x_0 \in [a,b]$, *it holds*

$$D\left({}_jA_n^{\mathcal{F}}(f, x_0), f(x_0)\right) \leq$$

$$\left[\omega_1^{(\mathcal{F})}\left(f^{(N)}, \frac{1}{n^\alpha}\right)\frac{1}{n^{\alpha N}N!} + 2c_j(\alpha, n) D^*\left(f^{(N)}, \widetilde{o}\right)\frac{(b-a)^N}{N!}\right], \tag{140}$$

notice here the extremely high rate of convergence $n^{-(N+1)\alpha}$,

3)

$$D^*\left({}_jA_n^{\mathcal{F}}(f), f\right) \leq \alpha_j$$

$$\left\{\sum_{j_*=1}^{N}\frac{D^*\left(f^{(j_*)}, \widetilde{o}\right)}{j_*!}\left[\frac{1}{n^{\alpha j_*}} + c_j(\alpha, n)(b-a)^{j_*}\right] + \right.$$

$$\left.\left[\omega_1^{(\mathcal{F})}\left(f^{(N)}, \frac{1}{n^\alpha}\right)\frac{1}{n^{\alpha N}N!} + 2c_j(\alpha, n) D^*\left(f^{(N)}, \widetilde{o}\right)\frac{(b-a)^N}{N!}\right]\right\}. \tag{141}$$

Proof. Since $f \in C_{\mathcal{F}}^N([a,b])$, $N \geq 1$, we have that $f_{\pm}^{(r)} \in C^N([a,b])$, $\forall$ $r \in [0,1]$; $j = 1,2,3,4$. Using (52), (75), (98), (123), we get

$$\left|{}_jA_n\left(f_{\pm}^{(r)}, x\right) - f_{\pm}^{(r)}(x)\right| \leq \alpha_j \tag{74}$$

$$\left\{\sum_{j_*=1}^{N}\frac{\left|\left(f_{\pm}^{(r)}\right)^{(j_*)}(x)\right|}{j_*!}\left[\frac{1}{n^{\alpha j_*}} + c_j(\alpha, n)(b-a)^{j_*}\right] + \right.$$

$$\left.\left[\omega_1\left(\left(f_{\pm}^{(r)}\right)^{(N)}, \frac{1}{n^\alpha}\right)\frac{1}{n^{\alpha N}N!} + 2c_j(\alpha, n)\left\|\left(f_{\pm}^{(r)}\right)^{(N)}\right\|_\infty\frac{(b-a)^N}{N!}\right]\right\} \tag{142}$$

(by Remark 12)

$$= \alpha_j\left\{\sum_{j_*=1}^{N}\frac{\left|\left(f^{(j_*)}\right)_{\pm}^{(r)}(x)\right|}{j_*!}\left[\frac{1}{n^{\alpha j_*}} + c_j(\alpha, n)(b-a)^{j_*}\right] + \right.$$

$$\left.\left[\omega_1\left(\left(f^{(N)}\right)_{\pm}^{(r)}, \frac{1}{n^\alpha}\right)\frac{1}{n^{\alpha N}N!} + 2c_j(\alpha, n)\left\|\left(f^{(N)}\right)_{\pm}^{(r)}\right\|_\infty\frac{(b-a)^N}{N!}\right]\right\} \leq$$

$$\alpha_j\left\{\sum_{j_*=1}^{N}\frac{D\left(f^{(j_*)}(x), \widetilde{o}\right)}{j_*!}\left[\frac{1}{n^{\alpha j_*}} + c_j(\alpha, n)(b-a)^{j_*}\right] + \right.$$

$$\left.\left[\omega_1^{(\mathcal{F})}\left(f^{(N)}, \frac{1}{n^\alpha}\right)\frac{1}{n^{\alpha N}N!} + 2c_j(\alpha, n) D^*\left(f^{(N)}, \widetilde{o}\right)\frac{(b-a)^N}{N!}\right]\right\}, \tag{143}$$

by Proposition 7, $\left\| \left(f^{(N)}\right)_{\pm}^{(r)} \right\|_{\infty} \leq D^{*}\left(f^{(N)}, \widetilde{o}\right)$ and apply (134).

The theorem is proved. ■

Next we present

Theorem 80 *Let $\alpha > 0$, $N = \lceil \alpha \rceil$, $\alpha \notin \mathbb{N}$, $f \in C_{\mathcal{F}}^{N}([a,b])$, $0 < \beta < 1$, $x \in [a,b]$, $n \in \mathbb{N}$, $n^{1-\beta} > 2$; $j = 1,2,3,4$. Then*

i)

$$D\left({}_{j}A_{n}^{\mathcal{F}}(f,x), f(x)\right) \leq$$

$$\alpha_{j}\left\{\sum_{j_{*}=1}^{N-1} \frac{D\left(f^{(j_{*})}(x), \widetilde{o}\right)}{j_{*}!}\left[\frac{1}{n^{\beta j_{*}}} + c_{j}(\beta,n)(b-a)^{j_{*}}\right] + \right.$$

$$\frac{1}{\Gamma(\alpha+1)}\left\{\frac{\left[\omega_{1}^{(\mathcal{F})}\left(\left(D_{x-}^{\alpha\mathcal{F}}f\right), \frac{1}{n^{\beta}}\right)_{[a,x]} + \omega_{1}^{(\mathcal{F})}\left(\left(D_{*x}^{\alpha\mathcal{F}}f\right), \frac{1}{n^{\beta}}\right)_{[x,b]}\right]}{n^{\alpha\beta}} + \right. \tag{144}$$

$$\left.\left. c_{j}(\beta,n)\left[D^{*}\left(\left(D_{x-}^{\alpha\mathcal{F}}f\right), \widetilde{o}\right)_{[a,x]}(x-a)^{\alpha} + D^{*}\left(\left(D_{*x}^{\alpha\mathcal{F}}f\right), \widetilde{o}\right)_{[x,b]}(b-x)^{\alpha}\right]\right\}\right\},$$

ii) if $f^{(j)}(x_{0}) = 0$, $j = 1,...,N-1$, for some $x_{0} \in [a,b]$, we have

$$D\left({}_{j}A_{n}^{\mathcal{F}}(f,x_{0}), f(x_{0})\right) \leq$$

$$\frac{\alpha_{j}}{\Gamma(\alpha+1)}\left\{\frac{\left[\omega_{1}^{(\mathcal{F})}\left(\left(D_{x_{0}-}^{\alpha\mathcal{F}}f\right), \frac{1}{n^{\beta}}\right)_{[a,x_{0}]} + \omega_{1}^{(\mathcal{F})}\left(\left(D_{*x_{0}}^{\alpha\mathcal{F}}f\right), \frac{1}{n^{\beta}}\right)_{[x_{0},b]}\right]}{n^{\alpha\beta}} + \right. \tag{145}$$

$$\left. c_{j}(\beta,n)\left[D^{*}\left(\left(D_{x_{0}-}^{\alpha\mathcal{F}}f\right), \widetilde{o}\right)_{[a,x_{0}]}(x_{0}-a)^{\alpha} + D^{*}\left(\left(D_{*x_{0}}^{\alpha\mathcal{F}}f\right), \widetilde{o}\right)_{[x_{0},b]}(b-x_{0})^{\alpha}\right]\right\},$$

when $\alpha > 1$ notice here the extremely high rate of convergence at $n^{-(\alpha+1)\beta}$, and

iii)

$$D^{*}\left({}_{j}A_{n}^{\mathcal{F}}(f), f\right) \leq$$

$$\alpha_{j}\left\{\sum_{j_{*}=1}^{N-1} \frac{D^{*}\left(f^{(j_{*})}, \widetilde{o}\right)}{j_{*}!}\left[\frac{1}{n^{\beta j_{*}}} + c_{j}(\beta,n)(b-a)^{j_{*}}\right] + \right.$$

$$\frac{1}{\Gamma(\alpha+1)}\left\{\frac{\left[\sup_{x\in[a,b]}\omega_{1}^{(\mathcal{F})}\left(\left(D_{x-}^{\alpha\mathcal{F}}f\right), \frac{1}{n^{\beta}}\right)_{[a,x]} + \sup_{x\in[a,b]}\omega_{1}^{(\mathcal{F})}\left(\left(D_{*x}^{\alpha\mathcal{F}}f\right), \frac{1}{n^{\beta}}\right)_{[x,b]}\right]}{n^{\alpha\beta}} + \right. \tag{146}$$

$$\left.\left. c_{j}(\beta,n)(b-a)^{\alpha}\left[\sup_{x\in[a,b]}D^{*}\left(\left(D_{x-}^{\alpha\mathcal{F}}f\right), \widetilde{o}\right)_{[a,x]} + \sup_{x\in[a,b]}D^{*}\left(\left(D_{*x}^{\alpha\mathcal{F}}f\right), \widetilde{o}\right)_{[x,b]}\right]\right\}\right\},$$

above, when $N=1$ the sum $\sum_{j=1}^{N-1} \cdot = 0$.

As we see here we obtain fractionally the fuzzy pointwise and uniform convergence with rates of ${}_jA_n^{\mathcal{F}} \to I$ the unit operator, as $n \to \infty$.

Proof. Here $f_{\pm}^{(r)} \in C^N([a,b])$, $\forall\ r \in [0,1]$, and $D_{x-}^{\alpha\mathcal{F}} f$, $D_{*x}^{\alpha\mathcal{F}} f$ are fuzzy continuous over $[a,b]$, $\forall\ x \in [a,b]$, so that $\left(D_{x-}^{\alpha\mathcal{F}} f\right)_{\pm}^{(r)}$, $\left(D_{*x}^{\alpha\mathcal{F}} f\right)_{\pm}^{(r)} \in C([a,b])$, $\forall$ $r \in [0,1]$, $\forall\ x \in [a,b]$.

We observe by (57), (80), (103), (128), $\forall\ x \in [a,b]$, that (respectively in $\pm$)

$$\left| {}_jA_n\left(f_{\pm}^{(r)}, x\right) - f_{\pm}^{(r)}(x)\right| \leq \alpha_j \cdot$$

$$\left\{ \sum_{j_*=1}^{N-1} \frac{\left|\left(f_{\pm}^{(r)}\right)^{(j_*)}(x)\right|}{j_*!} \left\{\frac{1}{n^{\beta j_*}} + c_j(\beta,n)(b-a)^{j_*}\right\} + \right. \tag{147}$$

$$\frac{1}{\Gamma(\alpha+1)} \left\{ \frac{\left(\omega_1\left(D_{x-}^{\alpha}\left(f_{\pm}^{(r)}\right), \frac{1}{n^{\beta}}\right)_{[a,x]} + \omega_1\left(D_{*x}^{\alpha}\left(f_{\pm}^{(r)}\right), \frac{1}{n^{\beta}}\right)_{[x,b]}\right)}{n^{\alpha\beta}} + \right.$$

$$\left.\left. c_j(\beta,n)\left(\left\|D_{x-}^{\alpha}\left(f_{\pm}^{(r)}\right)\right\|_{\infty,[a,x]}(x-a)^{\alpha} + \left\|D_{*x}^{\alpha}\left(f_{\pm}^{(r)}\right)\right\|_{\infty,[x,b]}(b-x)^{\alpha}\right)\right\}\right\} =$$

(by Remark 12, (30), (33))

$$\alpha_j \left\{ \sum_{j_*=1}^{N-1} \frac{\left|\left(f^{(j_*)}(x)\right)_{\pm}^{(r)}\right|}{j_*!} \left\{\frac{1}{n^{\beta j_*}} + c_j(\beta,n)(b-a)^{j_*}\right\} + \right.$$

$$\frac{1}{\Gamma(\alpha+1)} \left\{ \frac{\left(\omega_1\left(\left(D_{x-}^{\alpha\mathcal{F}} f\right)_{\pm}^{(r)}, \frac{1}{n^{\beta}}\right)_{[a,x]} + \omega_1\left(\left(D_{*x}^{\alpha\mathcal{F}} f\right)_{\pm}^{(r)}, \frac{1}{n^{\beta}}\right)_{[x,b]}\right)}{n^{\alpha\beta}} + \right. \tag{148}$$

$$\left.\left. c_j(\beta,n)\left(\left\|\left(D_{x-}^{\alpha\mathcal{F}} f\right)_{\pm}^{(r)}\right\|_{\infty,[a,x]}(x-a)^{\alpha} + \left\|\left(D_{*x}^{\alpha\mathcal{F}} f\right)_{\pm}^{(r)}\right\|_{\infty,[x,b]}(b-x)^{\alpha}\right)\right\}\right\} \leq$$

$$\alpha_j \left\{ \sum_{j_*=1}^{N-1} \frac{D\left(f^{(j_*)}(x), \widetilde{o}\right)}{j_*!} \left\{\frac{1}{n^{\beta j_*}} + c_j(\beta,n)(b-a)^{j_*}\right\} + \right.$$

$$\frac{1}{\Gamma(\alpha+1)} \left\{ \frac{\left[\omega_1^{(\mathcal{F})}\left(\left(D_{x-}^{\alpha\mathcal{F}} f\right), \frac{1}{n^{\beta}}\right)_{[a,x]} + \omega_1^{(\mathcal{F})}\left(\left(D_{*x}^{\alpha\mathcal{F}} f\right), \frac{1}{n^{\beta}}\right)_{[x,b]}\right]}{n^{\alpha\beta}} + \right. \tag{149}$$

$$\left.\left. c_j(\beta,n)\left[D^*\left(\left(D_{x-}^{\alpha\mathcal{F}} f\right), \widetilde{o}\right)_{[a,x]}(x-a)^{\alpha} + D^*\left(\left(D_{*x}^{\alpha\mathcal{F}} f\right), \widetilde{o}\right)_{[x,b]}(b-x)^{\alpha}\right]\right\}\right\},$$

along with (134) proving all inequalities of theorem.

Here we notice that

$$\left(D_{x-}^{\alpha\mathcal{F}} f\right)_{\pm}^{(r)}(t)=\left(D_{x-}^{\alpha}\left(f_{\pm}^{(r)}\right)\right)(t)$$

$$=\frac{(-1)^{N}}{\Gamma(N-\alpha)}\int_{t}^{x}(s-t)^{N-\alpha-1}\left(f_{\pm}^{(r)}\right)^{(N)}(s)\,ds,$$

where $a\leq t\leq x$.

Hence

$$\left|\left(D_{x-}^{\alpha\mathcal{F}} f\right)_{\pm}^{(r)}(t)\right|\leq\frac{1}{\Gamma(N-\alpha)}\int_{t}^{x}(s-t)^{N-\alpha-1}\left|\left(f_{\pm}^{(r)}\right)^{(N)}(s)\right|ds$$

$$\leq\frac{\left\|\left(f^{(N)}\right)_{\pm}^{(r)}\right\|_{\infty}}{\Gamma(N-\alpha+1)}(b-a)^{N-\alpha}\leq\frac{D^{*}\left(f^{(N)},\widetilde{o}\right)}{\Gamma(N-\alpha+1)}(b-a)^{N-\alpha}.$$

So we have

$$\left|\left(D_{x-}^{\alpha\mathcal{F}} f\right)_{\pm}^{(r)}(t)\right|\leq\frac{D^{*}\left(f^{(N)},\widetilde{o}\right)}{\Gamma(N-\alpha+1)}(b-a)^{N-\alpha},$$

all $a\leq t\leq x$.

And it holds

$$\left\|\left(D_{x-}^{\alpha\mathcal{F}} f\right)_{\pm}^{(r)}\right\|_{\infty,[a,x]}\leq\frac{D^{*}\left(f^{(N)},\widetilde{o}\right)}{\Gamma(N-\alpha+1)}(b-a)^{N-\alpha},\tag{150}$$

that is

$$D^{*}\left(\left(D_{x-}^{\alpha\mathcal{F}} f\right),\widetilde{o}\right)_{[a,x]}\leq\frac{D^{*}\left(f^{(N)},\widetilde{o}\right)}{\Gamma(N-\alpha+1)}(b-a)^{N-\alpha},$$

and

$$\sup_{x\in[a,b]}D^{*}\left(\left(D_{x-}^{\alpha\mathcal{F}} f\right),\widetilde{o}\right)_{[a,x]}\leq\frac{D^{*}\left(f^{(N)},\widetilde{o}\right)}{\Gamma(N-\alpha+1)}(b-a)^{N-\alpha}<\infty.\tag{151}$$

Similarly we have

$$\left(D_{*x}^{\alpha\mathcal{F}} f\right)_{\pm}^{(r)}(t)=\left(D_{*x}^{\alpha}\left(f_{\pm}^{(r)}\right)\right)(t)$$

$$=\frac{1}{\Gamma(N-\alpha)}\int_{x}^{t}(t-s)^{N-\alpha-1}\left(f_{\pm}^{(r)}\right)^{(N)}(s)\,ds,$$

where $x\leq t\leq b$.

Hence

$$\left|\left(D_{*x}^{\alpha\mathcal{F}} f\right)_{\pm}^{(r)}(t)\right|\leq\frac{1}{\Gamma(N-\alpha)}\int_{x}^{t}(t-s)^{N-\alpha-1}\left|\left(f^{(N)}\right)_{\pm}^{(r)}(s)\right|ds\leq$$

$$\frac{\left\|\left(f^{(N)}\right)_{\pm}^{(r)}\right\|_{\infty}}{\Gamma(N-\alpha+1)}(b-a)^{N-\alpha}\leq\frac{D^{*}\left(f^{(N)},\widetilde{o}\right)}{\Gamma(N-\alpha+1)}(b-a)^{N-\alpha},$$

$x \leq t \leq b$.

So we have

$$\left\| \left(D_{*x}^{\alpha \mathcal{F}} f \right)_{\pm}^{(r)} \right\|_{\infty, [x,b]} \leq \frac{D^{*}\left(f^{(N)}, \widetilde{o} \right)}{\Gamma \left(N - \alpha + 1 \right)} \left(b - a \right)^{N - \alpha}, \tag{152}$$

that is

$$D^{*}\left(\left(D_{*x}^{\alpha \mathcal{F}} f \right), \widetilde{o} \right)_{[x,b]} \leq \frac{D^{*}\left(f^{(N)}, \widetilde{o} \right)}{\Gamma \left(N - \alpha + 1 \right)} \left(b - a \right)^{N - \alpha},$$

and

$$\sup_{x \in [a,b]} D^{*}\left(\left(D_{*x}^{\alpha \mathcal{F}} f \right), \widetilde{o} \right)_{[x,b]} \leq \frac{D^{*}\left(f^{(N)}, \widetilde{o} \right)}{\Gamma \left(N - \alpha + 1 \right)} \left(b - a \right)^{N - \alpha} < +\infty. \tag{153}$$

Furthermore we notice

$$\omega_{1}^{(\mathcal{F})} \left(\left(D_{x-}^{\alpha \mathcal{F}} f \right), \frac{1}{n^{\beta}} \right)_{[a,x]} = \sup_{\substack{s,t \in [a,x] \\ |s-t| \leq \frac{1}{n^{\beta}}}} D\left(\left(D_{x-}^{\alpha \mathcal{F}} f \right)(s), \left(D_{x-}^{\alpha \mathcal{F}} f \right)(t) \right) \leq$$

$$\sup_{\substack{s,t \in [a,x] \\ |s-t| \leq \frac{1}{n^{\beta}}}} \left\{ D\left(\left(D_{x-}^{\alpha \mathcal{F}} f \right)(s), \widetilde{o} \right) + D\left(\left(D_{x-}^{\alpha \mathcal{F}} f \right)(t), \widetilde{o} \right) \right\} \leq 2D^{*}\left(\left(D_{x-}^{\alpha \mathcal{F}} f \right), \widetilde{o} \right)_{[a,x]}$$

$$\leq \frac{2D^{*}\left(f^{(N)}, \widetilde{o} \right)}{\Gamma \left(N - \alpha + 1 \right)} \left(b - a \right)^{N - \alpha}.$$

Therefore it holds

$$\sup_{x \in [a,b]} \omega_{1}^{(\mathcal{F})} \left(\left(D_{x-}^{\alpha \mathcal{F}} f \right), \frac{1}{n^{\beta}} \right)_{[a,x]} \leq \frac{2D^{*}\left(f^{(N)}, \widetilde{o} \right)}{\Gamma \left(N - \alpha + 1 \right)} \left(b - a \right)^{N - \alpha} < +\infty. \tag{154}$$

Similarly we observe

$$\omega_{1}^{(\mathcal{F})} \left(\left(D_{*x}^{\alpha \mathcal{F}} f \right), \frac{1}{n^{\beta}} \right)_{[x,b]} = \sup_{\substack{s,t \in [x,b] \\ |s-t| \leq \frac{1}{n^{\beta}}}} D\left(\left(D_{*x}^{\alpha \mathcal{F}} f \right)(s), \left(D_{*x}^{\alpha \mathcal{F}} f \right)(t) \right) \leq$$

$$2D^{*}\left(\left(D_{*x}^{\alpha \mathcal{F}} f \right), \widetilde{o} \right)_{[x,b]} \leq \frac{2D^{*}\left(f^{(N)}, \widetilde{o} \right)}{\Gamma \left(N - \alpha + 1 \right)} \left(b - a \right)^{N - \alpha}.$$

Consequently it holds

$$\sup_{x \in [a,b]} \omega_{1}^{(\mathcal{F})} \left(\left(D_{*x}^{\alpha \mathcal{F}} f \right), \frac{1}{n^{\beta}} \right)_{[x,b]} \leq \frac{2D^{*}\left(f^{(N)}, \widetilde{o} \right)}{\Gamma \left(N - \alpha + 1 \right)} \left(b - a \right)^{N - \alpha} < +\infty. \tag{155}$$

So everything in the statements of the theorem makes sense.

The proof of the theorem is now completed. ■

Corollary 81 *(to Theorem 80, $N=1$ case) Let $0<\alpha,\beta<1$, $f\in C_{\mathcal{F}}^{1}([a,b])$, $n\in\mathbb{N}$, $n^{1-\beta}>2$; $j=1,2,3,4$. Then*

$$D^{*}\left({}_{j}A_{n}^{\mathcal{F}}(f),f\right)\leq$$

$$\frac{\alpha_{j}}{\Gamma(\alpha+1)}\left\{\frac{\left[\sup_{x\in[a,b]}\omega_{1}^{(\mathcal{F})}\left(\left(D_{x-}^{\alpha\mathcal{F}}f\right),\frac{1}{n^{\beta}}\right)_{[a,x]}+\sup_{x\in[a,b]}\omega_{1}^{(\mathcal{F})}\left(\left(D_{*x}^{\alpha\mathcal{F}}f\right),\frac{1}{n^{\beta}}\right)_{[x,b]}\right]}{n^{\alpha\beta}}+\right.$$

$$\left.c_{j}(\beta,n)(b-a)^{\alpha}\left[\sup_{x\in[a,b]}D^{*}\left(\left(D_{x-}^{\alpha\mathcal{F}}f\right),\widetilde{o}\right)_{[a,x]}+\sup_{x\in[a,b]}D^{*}\left(\left(D_{*x}^{\alpha\mathcal{F}}f\right),\widetilde{o}\right)_{[x,b]}\right]\right\}. \tag{156}$$

Proof. By (146). ■

Finally we specialize to $\alpha=\frac{1}{2}$.

Corollary 82 *(to Theorem 80) Let $0<\beta<1$, $f\in C_{\mathcal{F}}^{1}([a,b])$, $n\in\mathbb{N}$, $n^{1-\beta}>2$; $j=1,2,3,4$. Then*

$$D^{*}\left({}_{j}A_{n}^{\mathcal{F}}(f),f\right)\leq$$

$$\frac{2\alpha_{j}}{\sqrt{\pi}}\left\{\frac{\left[\sup_{x\in[a,b]}\omega_{1}^{(\mathcal{F})}\left(\left(D_{x-}^{\frac{1}{2}\mathcal{F}}f\right),\frac{1}{n^{\beta}}\right)_{[a,x]}+\sup_{x\in[a,b]}\omega_{1}^{(\mathcal{F})}\left(\left(D_{*x}^{\frac{1}{2}\mathcal{F}}f\right),\frac{1}{n^{\beta}}\right)_{[x,b]}\right]}{n^{\frac{\beta}{2}}}+\right.$$

$$\left.c_{j}(\beta,n)\sqrt{b-a}\left[\sup_{x\in[a,b]}D^{*}\left(\left(D_{x-}^{\frac{1}{2}\mathcal{F}}f\right),\widetilde{o}\right)_{[a,x]}+\sup_{x\in[a,b]}D^{*}\left(\left(D_{*x}^{\frac{1}{2}\mathcal{F}}f\right),\widetilde{o}\right)_{[x,b]}\right]\right\}. \tag{157}$$

Proof. By (156). ■

Conclusion 83 *We have extended to the fuzzy setting all the main approximation theorems of section 3.*

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