SOME PROPERTIES OF TRACE CLASS ENTROPIC P-DETERMINANT OF POSITIVE OPERATORS IN HILBERT SPACES

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Abstract. Let H be a complex Hilbert space. For a given operator $P \geq 0$ with $P \in \mathcal{B}_1(H)$, the trace class associated to $\mathcal{B}(H)$ and $\operatorname{tr}(P) = 1$, we define the entropic P-*determinant* of the positive invertible operator A by

$$\eta_P(A) := \exp[-\operatorname{tr}(PA \ln A)].$$

In this paper we show among others that

$$\left[\frac{\operatorname{tr}(PA^2)}{\operatorname{tr}^2(PA)}\right]^{-\operatorname{tr}(PA)} \leq \frac{\eta_P(A)}{[\operatorname{tr}(PA)]^{-\operatorname{tr}(PA)}} \leq 1$$

and if there exists the constants $0 < m < M$ such that $m \leq A \leq M$, then

$$\left(\frac{m+M}{2\sqrt{mM}}\right)^{-2M} \leq \left(\frac{m+M}{2\sqrt{mM}}\right)^{-2\operatorname{tr}(PA)} \leq \left[\frac{\operatorname{tr}(PA^2)}{\operatorname{tr}^2(PA)}\right]^{-\operatorname{tr}(PA)}$$
$$\leq \frac{\eta_P(A)}{[\operatorname{tr}(PA)]^{-\operatorname{tr}(PA)}} \leq 1.$$

1. Introduction

In 1952, in the paper [8], B. Fuglede and R. V. Kadison introduced the determinant of a (invertible) operator and established its fundamental properties. The notion generalizes the usual determinant and can be considered for any operator in a finite von Neumann algebra (M, τ) with a faithful normal trace.

Let $T \in M$ be normal and $|T| := (T^*T)^{1/2}$ its modulus. By the spectral theorem one can represent T as an integral

$$T = \int_{\operatorname{Sp}(T)} \lambda dE(\lambda),$$

where $E(\lambda)$ is a projection valued measure and $\operatorname{Sp}(T)$ is the spectrum of T. The measure $\mu_T := \tau \circ E$ becomes a probability measure on the complex plane and has the support in the spectrum $\operatorname{Sp}(T)$.

For any $T \in M$ the Fuglede-Kadison determinant (*FK-determinant*) is defined by

$$\Delta_{FK}(T) := \exp\left(\int_0^\infty \ln t d\mu_{|T|}\right).$$

If T is invertible, then

$$\Delta_{FK}(T) := \exp(\tau(\ln(|T|))),$$

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where $\ln(|T|)$ is defined by the use of functional calculus.

Let $B(H)$ be the space of all bounded linear operators on a Hilbert space H, and I stands for the identity operator on H. An operator A in $B(H)$ is said to be positive (in symbol: $A \geq 0$) if $\langle Ax, x\rangle \geq 0$ for all $x \in H$. In particular, $A > 0$ means that A is positive and invertible. For a pair A, B of selfadjoint operators the order relation $A \geq B$ means as usual that $A - B$ is positive.

In 1998, Fujii et al. [9], [10], introduced the *normalized determinant* $\Delta_x(A)$ for positive invertible operators A on a Hilbert space H and a fixed unit vector $x \in H$, namely $\|x\| = 1$, defined by

$$\Delta_x(A) := \exp\langle \ln Ax, x\rangle$$

and discussed it as a continuous geometric mean and observed some inequalities around the determinant from this point of view. For some recent results, see [11].

We need now some preparations for trace of operators in Hilbert spaces.

Let $(H, \langle\cdot,\cdot\rangle)$ be a complex Hilbert space and $\{e_i\}_{i\in I}$ an *orthonormal basis* of H. We say that $A \in \mathcal{B}(H)$ is a *Hilbert-Schmidt operator* if

$$\sum_{i\in I} \|Ae_i\|^2 < \infty. \tag{1.1}$$

It is well know that, if $\{e_i\}_{i\in I}$ and $\{f_j\}_{j\in J}$ are orthonormal bases for H and $A \in \mathcal{B}(H)$ then

$$\sum_{i\in I} \|Ae_i\|^2 = \sum_{j\in I} \|Af_j\|^2 = \sum_{j\in I} \|A^*f_j\|^2 \tag{1.2}$$

showing that the definition (1.1) is independent of the orthonormal basis and A is a Hilbert-Schmidt operator iff A^* is a Hilbert-Schmidt operator.

Let $\mathcal{B}_2(H)$ the set of *Hilbert-Schmidt operators* in $\mathcal{B}(H)$. For $A \in \mathcal{B}_2(H)$ we define

$$\|A\|_2 := \left(\sum_{i\in I} \|Ae_i\|^2\right)^{1/2} \tag{1.3}$$

for $\{e_i\}_{i\in I}$ an orthonormal basis of H.

Using the triangle inequality in $l^2(I)$, one checks that $\mathcal{B}_2(H)$ is a *vector space* and that $\|\cdot\|_2$ is a norm on $\mathcal{B}_2(H)$, which is usually called in the literature as the *Hilbert-Schmidt norm.*

Denote *the modulus* of an operator $A \in \mathcal{B}(H)$ by $|A| := (A^*A)^{1/2}$.

Because $\||A|x\| = \|Ax\|$ for all $x \in H$, A is Hilbert-Schmidt iff $|A|$ is Hilbert-Schmidt and $\|A\|_2 = \||A|\|_2$. From (1.2) we have that if $A \in \mathcal{B}_2(H)$, then $A^* \in \mathcal{B}_2(H)$ and $\|A\|_2 = \|A^*\|_2$.

The following theorem collects some of the most important properties of Hilbert-Schmidt operators:

Theorem 1. *We have:*

(i) $(\mathcal{B}_2(H), \|\cdot\|_2)$ *is a Hilbert space with inner product*

$$\langle A, B\rangle_2 := \sum_{i\in I} \langle Ae_i, Be_i\rangle = \sum_{i\in I} \langle B^*Ae_i, e_i\rangle \tag{1.4}$$

and the definition does not depend on the choice of the orthonormal basis $\{e_i\}_{i\in I}$;

(ii) We have the inequalities

$$\|A\| \leq \|A\|_2 \tag{1.5}$$

for any $A \in \mathcal{B}_2(H)$ *and, if* $A \in \mathcal{B}_2(H)$ *and* $T \in \mathcal{B}(H)$, *then* AT, $TA \in \mathcal{B}_2(H)$ *with*

$$\|AT\|_2, \ \|TA\|_2 \leq \|T\| \|A\|_2 \tag{1.6}$$

(iii) $\mathcal{B}_2(H)$ *is an operator ideal in* $\mathcal{B}(H)$, *i.e.*

$$\mathcal{B}(H) \mathcal{B}_2(H) \mathcal{B}(H) \subseteq \mathcal{B}_2(H).$$

If $\{e_i\}_{i \in I}$ an orthonormal basis of H, we say that $A \in \mathcal{B}(H)$ is *trace class* if

$$\|A\|_1 := \sum_{i \in I} \langle |A| e_i, e_i \rangle < \infty. \tag{1.7}$$

The definition of $\|A\|_1$ does not depend on the choice of the orthonormal basis $\{e_i\}_{i \in I}$. We denote by $\mathcal{B}_1(H)$ the set of trace class operators in $\mathcal{B}(H)$.

The following proposition holds:

Proposition 1. *If* $A \in \mathcal{B}(H)$, *then the following are equivalent:*

(i) $A \in \mathcal{B}_1(H)$;

(ii) $|A|^{1/2} \in \mathcal{B}_2(H)$.

The following properties are also well known:

Theorem 2. *With the above notations:*

(i) We have

$$\|A\|_1 = \|A^*\|_1 \text{ and } \|A\|_2 \leq \|A\|_1 \tag{1.8}$$

for any $A \in \mathcal{B}_1(H)$;

(ii) $\mathcal{B}_1(H)$ *is an operator ideal in* $\mathcal{B}(H)$, *i.e.*

$$\mathcal{B}(H) \mathcal{B}_1(H) \mathcal{B}(H) \subseteq \mathcal{B}_1(H);$$

(iii) We have

$$\mathcal{B}_2(H) \mathcal{B}_2(H) = \mathcal{B}_1(H);$$

(iv) We have

$$\|A\|_1 = \sup \{ \langle A, B \rangle_2 \mid B \in \mathcal{B}_2(H), \ \|B\|_2 \leq 1 \};$$

(v) $(\mathcal{B}_1(H), \|\cdot\|_1)$ *is a Banach space.*

We define the *trace* of a trace class operator $A \in \mathcal{B}_1(H)$ to be

$$\operatorname{tr}(A) := \sum_{i \in I} \langle A e_i, e_i \rangle, \tag{1.9}$$

where $\{e_i\}_{i \in I}$ an orthonormal basis of H. Note that this coincides with the usual definition of the trace if H is finite-dimensional. We observe that the series (1.9) converges absolutely and it is independent from the choice of basis.

The following result collects some properties of the trace:

Theorem 3. *We have:*

(i) If $A \in \mathcal{B}_1(H)$ *then* $A^* \in \mathcal{B}_1(H)$ *and*

$$\operatorname{tr}(A^*) = \overline{\operatorname{tr}(A)}; \tag{1.10}$$

(ii) If $A\in\mathcal{B}_1(H)$ *and* $T\in\mathcal{B}(H)$, *then* $AT,\ TA\in\mathcal{B}_1(H)$,

$$\operatorname{tr}(AT)=\operatorname{tr}(TA) \text{ and } |\operatorname{tr}(AT)|\le\|A\|_1\|T\|; \tag{1.11}$$

(iii) $\operatorname{tr}(\cdot)$ *is a bounded linear functional on* $\mathcal{B}_1(H)$ *with* $\|\operatorname{tr}\|=1$;
(iv) If $A,\ B\in\mathcal{B}_2(H)$ *then* $AB,\ BA\in\mathcal{B}_1(H)$ *and* $\operatorname{tr}(AB)=\operatorname{tr}(BA)$.

Now, if we assume that $P\ge 0$ and $P\in\mathcal{B}_1(H)$, then for all $T\in\mathcal{B}(H)$, PT, $TP\in\mathcal{B}_1(H)$ and $\operatorname{tr}(PT)=\operatorname{tr}(TP)$. Also, since $P^{1/2}\in\mathcal{B}_2(H)$, $TP^{1/2}\in\mathcal{B}_2(H)$, hence $P^{1/2}TP^{1/2}$ and $TP^{1/2}P^{1/2}=TP\in\mathcal{B}_1(H)$ with $\operatorname{tr}\left(P^{1/2}TP^{1/2}\right)=\operatorname{tr}(TP)$. Therefore, if $P\ge 0$ and $P\in\mathcal{B}_1(H)$,

$$\operatorname{tr}(PT)=\operatorname{tr}(TP)=\operatorname{tr}\left(P^{1/2}TP^{1/2}\right)$$

for all $T\in\mathcal{B}(H)$.

If $T\ge 0$, then $P^{1/2}TP^{1/2}\ge 0$, which implies that $\operatorname{tr}(PT)\ge 0$ that shows that the functional $\mathcal{B}(H)\ni T\longmapsto\operatorname{tr}(PT)$ is linear and isotonic functional. Also, by (1.11), if $T_n\to T$ for $n\to\infty$ in $\mathcal{B}(H)$ then $\lim_{n\to\infty}\operatorname{tr}(PT_n)=\operatorname{tr}(PT)$, namely $\mathcal{B}(H)\ni T\longmapsto\operatorname{tr}(PT)$ is also continuous in the norm topology.

For a survey on recent trace inequalities see [6] and the references therein.

Now, for a given $P\ge 0$ with $P\in\mathcal{B}_1(H)$ and $\operatorname{tr}(P)=1$, we define the P-*determinant* of the positive invertible operator A by

$$\Delta_P(A):=\exp\operatorname{tr}(P\ln A)=\exp\operatorname{tr}((\ln A)P)=\exp\operatorname{tr}\left(P^{1/2}(\ln A)P^{1/2}\right). \tag{1.12}$$

Assume that $P\ge 0$ with $P\in\mathcal{B}_1(H)$ and $\operatorname{tr}(P)=1$. We observe that we have the following elementary properties [7]:

(i) *continuity*: the map $A\to\Delta_P(A)$ is norm continuous;
(ii) *power equality:* $\Delta_P(A^t)=\Delta_P(A)^t$ for all $t>0$;
(iii) *homogeneity*: $\Delta_P(tA)=t\Delta_P(A)$ and $\Delta_P(tI)=t$ for all $t>0$;
(iv) *monotonicity*: $0<A\le B$ implies $\Delta_P(A)\le\Delta_P(B)$.

In [7], we presented some fundamental properties of this determinant. Among others we showed that

$$1\le\frac{\operatorname{tr}(PA)}{\Delta_P(A)}\le\exp\left[\operatorname{tr}(PA)\operatorname{tr}\left(PA^{-1}\right)-1\right]$$

and

$$1\le\frac{\Delta_P(A)}{\left[\operatorname{tr}\left(PA^{-1}\right)\right]^{-1}}\le\exp\left[\operatorname{tr}\left(PA^{-1}\right)\operatorname{tr}(PA)-1\right],$$

for $A>0$ and $P\ge 0$ with $P\in\mathcal{B}_1(H)$ and $\operatorname{tr}(P)=1$.

For the entropy function $\eta(t)=-t\ln t$, $t>0$, the operator entropy has the following expression:

$$\eta(A)=-A\ln A$$

for positive A.

Now, for a given $P\ge 0$ with $P\in\mathcal{B}_1(H)$ and $\operatorname{tr}(P)=1$, we define the *entropic P-determinant* of the positive invertible operator A by

$$\eta_P(A):=\exp\left[-\operatorname{tr}(PA\ln A)\right]=\exp\left\{\operatorname{tr}\left[P\eta(A)\right]\right\}=\exp\left\{\operatorname{tr}\left[P^{1/2}\eta(A)P^{1/2}\right]\right\}.$$

Observe that the map $A \to \eta_P(A)$ is *norm continuous* and since

$$\begin{aligned}
&\exp\left(-\operatorname{tr}\left\{P\left[tA\ln\left(tA\right)\right]\right\}\right) \\
&= \exp\left(-\operatorname{tr}\left\{P\left[tA\left(\ln t + \ln A\right)\right]\right\}\right) = \exp\left(-\operatorname{tr}\left\{P\left(tA\ln t + tA\ln A\right)\right\}\right) \\
&= \exp\left(-t\ln t\operatorname{tr}\left(PA\right)\right)\exp\left(-t\operatorname{tr}\left(PA\ln A\right)\right) \\
&= \exp\ln\left(t^{-\operatorname{tr}(PA)t}\right)\left[\exp\left(-\operatorname{tr}\left(PA\ln A\right)\right)\right]^{-t},
\end{aligned}$$

hence

$$\eta_P(tA) = t^{-t\operatorname{tr}(PA)}\left[\eta_P(A)\right]^{-t} \tag{1.13}$$

for $t > 0$ and $A > 0$.

Observe also that

$$\eta_P(I) = 1 \text{ and } \eta_P(tI) = t^{-t} \tag{1.14}$$

for $t > 0$.

Motivated by the above results, in this paper we show among others that

$$\left[\frac{\operatorname{tr}\left(PA^2\right)}{\operatorname{tr}^2\left(PA\right)}\right]^{-\operatorname{tr}(PA)} \le \frac{\eta_P(A)}{\left[\operatorname{tr}\left(PA\right)\right]^{-\operatorname{tr}(PA)}} \le 1$$

and if there exists the constants $0 < m < M$ such that $m \le A \le M$, then

$$\begin{aligned}
\left(\frac{m+M}{2\sqrt{mM}}\right)^{-2M} &\le \left(\frac{m+M}{2\sqrt{mM}}\right)^{-2\operatorname{tr}(PA)} \le \left[\frac{\operatorname{tr}\left(PA^2\right)}{\operatorname{tr}^2\left(PA\right)}\right]^{-\operatorname{tr}(PA)} \\
&\le \frac{\eta_P(A)}{\left[\operatorname{tr}\left(PA\right)\right]^{-\operatorname{tr}(PA)}} \le 1.
\end{aligned}$$

2. Main Results

We have the following fundamental facts:

Proposition 2. *Let $P \ge 0$ with $P \in \mathcal{B}_1(H)$ and $\operatorname{tr}(P) = 1$. If A, $B > 0$, then we have the Ky Fan type inequality*

$$\eta_P((1-t)A + tB) \ge \left[\eta_P(A)\right]^{1-t}\left[\eta_P(B)\right]^t \tag{2.1}$$

for all $t \in [0,1]$.

Proof. Since entropy function $\eta(\cdot)$ is operator concave, then

$$\eta\left(\left(1-t\right)A + tB\right) \ge \left(1-t\right)\eta\left(A\right) + t\eta\left(B\right)$$

for all $t \in [0,1]$.

If we multiply both sides by $P^{1/2} \ge 0$, then we get

$$P^{1/2}\eta\left(\left(1-t\right)A + tB\right)P^{1/2} \ge \left(1-t\right)P^{1/2}\eta\left(A\right)P^{1/2} + tP^{1/2}\eta\left(B\right)P^{1/2}$$

and by taking the tr we derive

$$\operatorname{tr}\left[P\eta\left(\left(1-t\right)A + tB\right)\right] \ge \left(1-t\right)\operatorname{tr}\left[P\eta\left(A\right)\right] + t\operatorname{tr}\left[P\eta\left(B\right)\right]$$

for all $t \in [0,1]$.

If we take the exponential, then we derive that

$$\begin{aligned}\eta_P((1-t)A+tB) &= \exp\left(\operatorname{tr}\left[P\eta\left((1-t)A+tB\right)\right]\right)\\ &\geq \exp\left[(1-t)\operatorname{tr}\left[P\eta\left(A\right)\right]+t\operatorname{tr}\left[P\eta\left(B\right)\right]\right]\\ &= \left(\exp\left(\operatorname{tr}\left[P\eta\left(A\right)\right]\right)\right)^{1-t}\left(\exp\left(\operatorname{tr}\left[P\eta\left(B\right)\right]\right)\right)^{t}\\ &= \left[\eta_P\left(A\right)\right]^{1-t}\left[\eta_P\left(B\right)\right]^{t},\end{aligned}$$

which proves the desired inequality (2.1). □

We define the *logarithmic mean* of two positive numbers a, b by

$$L\left(a,b\right):=\begin{cases}\frac{b-a}{\ln b-\ln a} \text{ if } b\neq a,\\ \\ a \text{ if } b=a.\end{cases}$$

The following Hermite-Hadamard type integral inequalities hold:

Corollary 1. *With the assumptions of Proposition 2,*

$$\int_0^1 \eta_P((1-t)A+tB)dt \geq L\left(\eta_P\left(A\right),\eta_P\left(B\right)\right) \tag{2.2}$$

and

$$\eta_P\left(\frac{A+B}{2}\right) \geq \int_0^1 \left[\eta_P\left((1-t)A+tB\right)\right]^{1/2}\left[\eta_P\left(tA+(1-t)B\right)\right]^{1/2}dt. \tag{2.3}$$

Proof. If we take the integral over $t\in[0,1]$ in (2.5), then we get

$$\begin{aligned}\int_0^1 \eta_P((1-t)A+tB)dt &\geq \int_0^1 \left[\eta_P\left(A\right)\right]^{1-t}\left[\eta_P\left(B\right)\right]^{t}dt\\ &= L\left(\eta_P\left(A\right),\eta_P\left(B\right)\right)\end{aligned}$$

for all A, $B>0$, which proves (2.6).

We get from (2.5) for $t=1/2$ that

$$\eta_P\left(\frac{A+B}{2}\right)\geq\left[\eta_P\left(A\right)\right]^{1/2}\left[\eta_P\left(B\right)\right]^{1/2}.$$

If we replace A by $(1-t)A+tB$ and B by $tA+(1-t)B$ we obtain

$$\eta_P\left(\frac{A+B}{2}\right)\geq\left[\eta_P\left((1-t)A+tB\right)\right]^{1/2}\left[\eta_P\left(tA+(1-t)B\right)\right]^{1/2}.$$

By taking the integral, we derive the desired result (2.3). □

Theorem 4. *Let* $P\geq 0$ *with* $P\in\mathcal{B}_1(H)$ *and* $\operatorname{tr}(P)=1$. *If* $A>0$, *then*

$$\left[\frac{\operatorname{tr}\left(PA^2\right)}{\operatorname{tr}^2\left(PA\right)}\right]^{-\operatorname{tr}(PA)}\leq\frac{\eta_P(A)}{\left[\operatorname{tr}\left(PA\right)\right]^{-\operatorname{tr}(PA)}}\leq 1. \tag{2.4}$$

Proof. The entropy function $\eta(t)=-t\ln t$, $t>0$ is operator concave. By utilizing Jensen's trace inequality for concave function g on $(0,\infty)$, see [3], [4] or [6], we have for $B>0$, $P\geq 0$ with $P\in\mathcal{B}_1(H)$ and $\operatorname{tr}(P)=1$ that

$$\operatorname{tr}\left[Pg\left(B\right)\right]\leq g\left[\operatorname{tr}\left(PB\right)\right],$$

implying that

$$\eta_P(A) = \exp\left(\operatorname{tr}\left[P\eta\left(A\right)\right]\right) \leq \exp\left[\eta\left(\operatorname{tr}\left(PA\right)\right)\right]$$
$$= \exp\left(\ln\left[\operatorname{tr}\left(PA\right)\right]^{-\operatorname{tr}(PA)}\right) = \left[\operatorname{tr}\left(PA\right)\right]^{-\operatorname{tr}(PA)},$$

which proves the second inequality in (2.4).

Observe that if $Q := \frac{APA}{\operatorname{tr}(PA^2)}$, then $Q \geq 0$, $\operatorname{tr} Q = 1$ and for $B = A^{-1}$ we have

$$\begin{aligned}
\eta_Q(B) &= \eta_{\frac{APA}{\operatorname{tr}(PA^2)}}\left(A^{-1}\right) \\
&= \exp\left(\operatorname{tr}\left[-\frac{APA}{\operatorname{tr}\left(PA^2\right)}A^{-1}\ln\left(A^{-1}\right)\right]\right) \\
&= \exp\left(\operatorname{tr}\left[-\frac{AP}{\operatorname{tr}\left(PA^2\right)}\ln\left(A^{-1}\right)\right]\right) \\
&= \exp\left(\operatorname{tr}\left[\frac{AP}{\operatorname{tr}\left(PA^2\right)}\ln A\right]\right) \\
&= \exp\left(\frac{1}{\operatorname{tr}\left(PA^2\right)}\operatorname{tr}\left(AP\ln A\right)\right) = \exp\left(\frac{1}{\operatorname{tr}\left(PA^2\right)}\operatorname{tr}\left(P\left(\ln A\right)A\right)\right) \\
&= \left(\frac{1}{\operatorname{tr}\left(PA^2\right)}\operatorname{tr}\left(PA\ln A\right)\right) = \exp\left(\frac{-1}{\operatorname{tr}\left(PA^2\right)}\operatorname{tr}\left(-PA\ln A\right)\right) \\
&= \left[\exp\left(-\operatorname{tr}\left(PA\ln A\right)\right)\right]^{\frac{-1}{\operatorname{tr}(PA^2)}} = \left[\eta_P(A)\right]^{\frac{-1}{\operatorname{tr}(PA^2)}}
\end{aligned}$$

for $A > 0$, $P \geq 0$ with $P \in \mathcal{B}_1(H)$ and $\operatorname{tr}(P) = 1$, which gives that

$$\eta_P(A) = \left[\eta_{\frac{APA}{\operatorname{tr}(PA^2)}}\left(A^{-1}\right)\right]^{-\operatorname{tr}\left(PA^2\right)} \tag{2.5}$$

for $A > 0$, $P \geq 0$ with $P \in \mathcal{B}_1(H)$ and $\operatorname{tr}(P) = 1$.

Now, using the second inequality in (2.4) for $Q := \frac{APA}{\operatorname{tr}(PA^2)}$ and A^{-1} we have

$$\begin{aligned}
\eta_{\frac{APA}{\operatorname{tr}(PA^2)}}\left(A^{-1}\right) &\leq \left[\operatorname{tr}\left(\frac{APA}{\operatorname{tr}\left(PA^2\right)}A^{-1}\right)\right]^{-\operatorname{tr}\left(\frac{APA}{\operatorname{tr}(PA^2)}A^{-1}\right)} \\
&= \left[\operatorname{tr}\left(\frac{PA}{\operatorname{tr}\left(PA^2\right)}\right)\right]^{-\operatorname{tr}\left(\frac{PA}{\operatorname{tr}(PA^2)}\right)} = \left[\frac{\operatorname{tr}\left(PA\right)}{\operatorname{tr}\left(PA^2\right)}\right]^{-\frac{\operatorname{tr}(PA)}{\operatorname{tr}(PA^2)}}.
\end{aligned} \tag{2.6}$$

If we take the power $-\operatorname{tr}\left(PA^2\right) \leq 0$ in (2.6), then we get

$$\begin{aligned}
\left[\eta_{\frac{APA}{\operatorname{tr}(PA^2)}}\left(A^{-1}\right)\right]^{-\operatorname{tr}\left(PA^2\right)} &\geq \left[\frac{\operatorname{tr}\left(PA\right)}{\operatorname{tr}\left(PA^2\right)}\right]^{\operatorname{tr}(PA)} \\
&= \left[\frac{\operatorname{tr}\left(PA^2\right)}{\operatorname{tr}\left(PA\right)}\right]^{-\operatorname{tr}(PA)} \\
&= \left[\frac{\operatorname{tr}\left(PA^2\right)}{\operatorname{tr}^2\left(PA\right)}\right]^{-\operatorname{tr}(PA)}\left[\operatorname{tr}\left(PA\right)\right]^{-\operatorname{tr}(PA)}
\end{aligned}$$

and by (2.5) we derive the first inequality in (2.4). □

Corollary 2. *Let $P \geq 0$ with $P \in \mathcal{B}_1(H)$ and $\operatorname{tr}(P) = 1$. If there exists the constants $0 < m < M$ such that $m \leq A \leq M$, then*

$$\left(\frac{m+M}{2\sqrt{mM}}\right)^{-2M} \leq \left(\frac{m+M}{2\sqrt{mM}}\right)^{-2\operatorname{tr}(PA)} \leq \left[\frac{\operatorname{tr}\left(PA^2\right)}{\operatorname{tr}^2(PA)}\right]^{-\operatorname{tr}(PA)} \leq \frac{\eta_P(A)}{[\operatorname{tr}(PA)]^{-\operatorname{tr}(PA)}} \leq 1. \tag{2.7}$$

Proof. We use the following Kantorovich type inequality, see [5] or [6],

$$\frac{\operatorname{tr}\left(PA^2\right)}{\operatorname{tr}^2(PA)} \leq \left(\frac{m+M}{2\sqrt{mM}}\right)^2$$

that holds for A satisfying the condition $m \leq A \leq M$.

By employing the first inequality in (2.7) we derive the first part of (2.7). □

Theorem 5. *Let $P \geq 0$ with $P \in \mathcal{B}_1(H)$ and $\operatorname{tr}(P) = 1$. If $A > 0$, then*

$$\eta_P(A) \leq a^{-\operatorname{tr}(PA)} \exp\left[-\operatorname{tr}(PA) + a\right] \tag{2.8}$$

for all $a > 0$.

In particular, for $a = \operatorname{tr}(PA)$ we get the second inequality in (2.4), which is the best possible inequality from (2.8).

Proof. It is well know that, if f is differentiable convex on an interval I, then for all u, $v \in I$ we have

$$f(u) - f(v) \leq f'(u)(u - v). \tag{2.9}$$

Consider the convex function $f(t) = t \ln t$, $t > 0$. Since $f'(t) = \ln t + 1$, $t > 0$, hence by (2.9) we get

$$u \ln u - v \ln v \leq (\ln u + 1)(u - v) \tag{2.10}$$

namely

$$-v \ln v \leq -u \ln u + (\ln u + 1)(u - v)$$

giving that

$$-v \ln v \leq u - v - v \ln u$$

for u, $v > 0$.

If we take $u = a$ and use the functional calculus for $v = A > 0$, then we get

$$-A \ln A \leq a - A - A \ln a,$$

namely

$$-A \ln A \leq -\ln(ea) A + a \tag{2.11}$$

for all $a > 0$ and $A > 0$.

Now, if we multiply both sides of (2.11) with $P^{1/2} \geq 0$, then we get

$$-P^{1/2}(A \ln A) P^{1/2} \leq -\ln(ea) P^{1/2} A P^{1/2} + aP$$

and by taking the trace, we obtain

$$\begin{aligned} -\operatorname{tr}[P(A \ln A)] &\leq -\ln(ea)\operatorname{tr}(PA) + a \\ &= -\operatorname{tr}(PA) - \ln a \operatorname{tr}(PA) + a. \end{aligned}$$

Finally, if we take the exponential we derive

$$\eta_P(A) \leq a^{-\operatorname{tr}(PA)} \exp\left[-\operatorname{tr}(PA) + a\right].$$

For given $A>0$, $P\geq 0$ with $P\in\mathcal{B}_1(H)$ and $\operatorname{tr}(P)=1$, consider the function

$$f(t)=t^{-\operatorname{tr}(PA)}\exp\left[-\operatorname{tr}(PA)+t\right],\ t>0.$$

We have

$$\begin{aligned} f'(t) &= -\operatorname{tr}(PA)\,t^{-\operatorname{tr}(PA)-1}\exp\left[-\operatorname{tr}(PA)+t\right] \\ &\quad + t^{-\operatorname{tr}(PA)}\exp\left[-\operatorname{tr}(PA)+t\right] \\ &= \exp\left[-\operatorname{tr}(PA)+t\right]t^{-\operatorname{tr}(PA)-1}\left(t-\operatorname{tr}(PA)\right). \end{aligned}$$

We observe that the function f is decreasing on $(0,\operatorname{tr}(PA))$ and increasing on $(\operatorname{tr}(PA),\infty)$ showing that

$$\inf_{t\in(0,\infty)} f(t)=f(\operatorname{tr}(PA))=\operatorname{tr}(PA)^{-\operatorname{tr}(PA)}.$$

Therefore the best inequality we can get from (2.8) is for $a=\operatorname{tr}(PA)$. □

Remark 1. *For $a=1$ in (2.8) we get the inequality*

$$\eta_P(A)\leq\exp\left[1-\operatorname{tr}(PA)\right] \tag{2.12}$$

for $A>0$, $P\geq 0$ with $P\in\mathcal{B}_1(H)$ and $\operatorname{tr}(P)=1$.

Corollary 3. *With the assumptions of Theorem 5, we have*

$$\eta_P(A)\geq a^{\operatorname{tr}(PA)}\exp\left[\operatorname{tr}(PA)-a\operatorname{tr}\left(PA^2\right)\right]. \tag{2.13}$$

In particular, for $a=\frac{\operatorname{tr}(PA)}{\operatorname{tr}(PA^2)}$ we get the first inequality in (2.4), which is the best possible inequality from (2.13).

Proof. If we write the inequality (2.8) for A^{-1} and $\frac{APA}{\operatorname{tr}(PA^2)}$, then we get

$$\eta_P(A^{-1})\leq a^{-\operatorname{tr}\left(\frac{APA}{\operatorname{tr}(PA^2)}A^{-1}\right)}\exp\left[-\operatorname{tr}\left(\frac{APA}{\operatorname{tr}(PA^2)}A^{-1}\right)+a\right]$$

for all $a>0$.

This is equivalent to

$$\eta_P(A^{-1})\leq a^{-\frac{\operatorname{tr}(PA)}{\operatorname{tr}(PA^2)}}\exp\left[-\frac{\operatorname{tr}(PA)}{\operatorname{tr}(PA^2)}+a\right] \tag{2.14}$$

for all $a>0$.

Now, if we take the power $-\operatorname{tr}\left(PA^2\right)\leq 0$ in (2.14), then we get

$$\left[\eta_P(A^{-1})\right]^{-\operatorname{tr}\left(PA^2\right)}\geq a^{\operatorname{tr}(PA)}\exp\left[\operatorname{tr}(PA)-a\operatorname{tr}\left(PA^2\right)\right]$$

for all $a>0$ and by (2.5) we get (2.13).

For given $A>0$, $P\geq 0$ with $P\in\mathcal{B}_1(H)$ and $\operatorname{tr}(P)=1$, consider the function

$$g(t)=t^{\operatorname{tr}(PA)}\exp\left[\operatorname{tr}(PA)-t\operatorname{tr}\left(PA^2\right)\right],\ t>0.$$

We have

$$\begin{aligned} g'(t) &= \operatorname{tr}(PA)\,t^{\operatorname{tr}(PA)-1}\exp\left[\operatorname{tr}(PA)-t\operatorname{tr}\left(PA^2\right)\right] \\ &\quad - t^{\operatorname{tr}(PA)}\operatorname{tr}\left(PA^2\right)\exp\left[\operatorname{tr}(PA)-t\operatorname{tr}\left(PA^2\right)\right] \\ &= t^{\operatorname{tr}(PA)-1}\exp\left[\operatorname{tr}(PA)-t\operatorname{tr}\left(PA^2\right)\right]\left(\operatorname{tr}(PA)-t\operatorname{tr}\left(PA^2\right)\right). \end{aligned}$$

We observe that the function g is increasing on $\left(0, \frac{\operatorname{tr}(PA)}{\operatorname{tr}(PA^2)}\right)$ and decreasing on $\left(\frac{\operatorname{tr}(PA)}{\operatorname{tr}(PA^2)}, \infty\right)$ showing that

$$\sup_{t\in(0,\infty)} g(t) = g\left(\frac{\operatorname{tr}(PA)}{\operatorname{tr}(PA^2)}\right) = \left[\frac{\operatorname{tr}(PA)}{\operatorname{tr}(PA^2)}\right]^{\operatorname{tr}(PA)} = \left[\frac{\operatorname{tr}\left(PA^2\right)}{\operatorname{tr}(PA)}\right]^{-\operatorname{tr}(PA)}$$

and the statement is proved. □

Remark 2. *For $a = 1$ in (2.13) we get*

$$\eta_P(A) \geq \exp\left[\operatorname{tr}(PA) - \operatorname{tr}\left(PA^2\right)\right]. \tag{2.15}$$

for $A > 0$, $P \geq 0$ with $P \in \mathcal{B}_1(H)$ and $\operatorname{tr}(P) = 1$.

3. Related Results

In [4] we obtained, among others, the following reverse of Jensen's inequality:

Lemma 1. *Let A be a selfadjoint operator on the Hilbert space H and assume that $\operatorname{Sp}(A) \subseteq [m, M]$ for some scalars m, M with $m < M$. If f is a continuously differentiable convex function on $[m, M]$ and $Q \in \mathcal{B}_1(H) \setminus \{0\}$, $Q \geq 0$, then we have*

$$\begin{aligned}
0 &\leq \frac{\operatorname{tr}(Qf(A))}{\operatorname{tr}(Q)} - f\left(\frac{\operatorname{tr}(QA)}{\operatorname{tr}(Q)}\right) \\
&\leq \frac{\operatorname{tr}(Qf'(A)A)}{\operatorname{tr}(Q)} - \frac{\operatorname{tr}(QA)}{\operatorname{tr}(Q)}\frac{\operatorname{tr}(Qf'(A))}{\operatorname{tr}(Q)} \\
&\leq \begin{cases} \frac{1}{2}\left[f'(M) - f'(m)\right] \frac{\operatorname{tr}\left(Q\left|A - \frac{\operatorname{tr}(QA)}{\operatorname{tr}(Q)}1_H\right|\right)}{\operatorname{tr}(Q)} \\ \\ \frac{1}{2}(M-m)\frac{\operatorname{tr}\left(Q\left|f'(A) - \frac{\operatorname{tr}(Qf'(A))}{\operatorname{tr}(Q)}1_H\right|\right)}{\operatorname{tr}(Q)} \end{cases} \\
&\leq \begin{cases} \frac{1}{2}\left[f'(M) - f'(m)\right]\left[\frac{\operatorname{tr}\left(QA^2\right)}{\operatorname{tr}(Q)} - \left(\frac{\operatorname{tr}(QA)}{\operatorname{tr}(Q)}\right)^2\right]^{1/2} \\ \\ \frac{1}{2}(M-m)\left[\frac{\operatorname{tr}\left(Q\left[f'(A)\right]^2\right)}{\operatorname{tr}(Q)} - \left(\frac{\operatorname{tr}\left(Qf'(A)\right)}{\operatorname{tr}(Q)}\right)^2\right]^{1/2} \end{cases} \\
&\leq \frac{1}{4}\left[f'(M) - f'(m)\right](M-m).
\end{aligned} \tag{3.1}$$

By the use of this results we can obtain the following result:

Theorem 6. *Let $P \geq 0$ with $P \in \mathcal{B}_1(H)$ and $\operatorname{tr}(P) = 1$. If there exists the constants $0 < m < M$ such that $m \leq A \leq M$, then*

$$
\begin{aligned}
1 &\leq \frac{[\operatorname{tr}(PA)]^{-\operatorname{tr}(PA)}}{\eta_P(A)} \\
&\leq \begin{cases} \left(\frac{M}{m}\right)^{\frac{1}{2}\operatorname{tr}(P|A-\operatorname{tr}(PA)|)} \\ \exp\left[\frac{1}{2}(M-m)\operatorname{tr}(P|\ln A - \operatorname{tr}(P\ln A)|)\right] \end{cases} \\
&\leq \begin{cases} \left(\frac{M}{m}\right)^{\frac{1}{2}\left[\operatorname{tr}\left(PA^2\right)-(\operatorname{tr}(PA))^2\right]^{1/2}} \\ \exp\left\{\frac{1}{2}(M-m)\left[\operatorname{tr}\left(P[\ln(eA)]^2\right) - (\operatorname{tr}(P\ln(eA)))^2\right]^{1/2}\right\} \end{cases} \\
&\leq \left(\frac{M}{m}\right)^{\frac{1}{4}(M-m)}.
\end{aligned} \tag{3.2}
$$

Proof. If we write the inequality (3.1) for A satisfying the condition $0 < m \leq A \leq M$ and $Q = P$, then we get

$$
\begin{aligned}
0 &\leq \operatorname{tr}(PA\ln A) - \operatorname{tr}(PA)\ln(\operatorname{tr}(PA)) \\
&\leq \operatorname{tr}(P(\ln A + 1)A) - \operatorname{tr}(PA)\operatorname{tr}(P(\ln A + 1))
\end{aligned}
$$

$$
\begin{aligned}
&\leq \begin{cases} \frac{1}{2}\ln\left(\frac{M}{m}\right)\operatorname{tr}(P|A-\operatorname{tr}(PA)|) \\ \frac{1}{2}(M-m)\operatorname{tr}(P|\ln A + 1 - \operatorname{tr}(P(\ln A+1))|) \end{cases} \\
&\leq \begin{cases} \frac{1}{2}\ln\left(\frac{M}{m}\right)\left[\operatorname{tr}\left(PA^2\right) - (\operatorname{tr}(PA))^2\right]^{1/2} \\ \frac{1}{2}(M-m)\left[\operatorname{tr}\left(P[\ln(eA)]^2\right) - (\operatorname{tr}(P\ln(eA)))^2\right]^{1/2} \end{cases} \\
&\leq \frac{1}{4}\ln\left(\frac{M}{m}\right)(M-m),
\end{aligned}
$$

namely

$$
\begin{aligned}
0 &\leq \operatorname{tr}(PA\ln A) - \operatorname{tr}(PA)\ln(\operatorname{tr}(PA)) \\
&\leq \begin{cases} \frac{1}{2}\ln\left(\frac{M}{m}\right)\operatorname{tr}(P|A-\operatorname{tr}(PA)|) \\ \frac{1}{2}(M-m)\operatorname{tr}(P|\ln A - \operatorname{tr}(P\ln A)|) \end{cases} \\
&\leq \begin{cases} \frac{1}{2}\ln\left(\frac{M}{m}\right)\left[\operatorname{tr}\left(PA^2\right) - (\operatorname{tr}(PA))^2\right]^{1/2} \\ \frac{1}{2}(M-m)\left[\operatorname{tr}\left(P[\ln(eA)]^2\right) - (\operatorname{tr}(P\ln(eA)))^2\right]^{1/2} \end{cases} \\
&\leq \frac{1}{4}\ln\left(\frac{M}{m}\right)(M-m).
\end{aligned}
$$

If we take the exponential in this inequality, we get

$$1 \leq \frac{\exp\left[-\operatorname{tr}(PA)\ln(\operatorname{tr}(PA))\right]}{\exp\left[-\operatorname{tr}(PA\ln A)\right]}$$
$$\leq \begin{cases} \exp\left[\frac{1}{2}\ln\left(\frac{M}{m}\right)\operatorname{tr}\left(P\left|A-\operatorname{tr}(PA)\right|\right)\right] \\ \\ \exp\left[\frac{1}{2}(M-m)\operatorname{tr}\left(P\left|\ln A-\operatorname{tr}(P\ln A)\right|\right)\right] \end{cases}$$
$$\leq \begin{cases} \exp\left[\frac{1}{2}\ln\left(\frac{M}{m}\right)\left[\operatorname{tr}\left(PA^2\right)-\left(\operatorname{tr}(PA)\right)^2\right]^{1/2}\right] \\ \\ \exp\left\{\frac{1}{2}(M-m)\left[\operatorname{tr}\left(P\left[\ln(eA)\right]^2\right)-\left(\operatorname{tr}\left(P\ln(eA)\right)\right)^2\right]^{1/2}\right\} \end{cases}$$
$$\leq \exp\left[\frac{1}{4}\ln\left(\frac{M}{m}\right)(M-m)\right]$$

and the inequality (3.1) is proved. □

Corollary 4. *With the assumptions of Theorem 6, we have*

$$(3.3)\quad 1 \geq \frac{\left[\frac{\operatorname{tr}(PA)}{\operatorname{tr}(PA^2)}\right]^{\operatorname{tr}(PA)}}{\eta_P(A)}$$
$$\geq \begin{cases} \left(\frac{M}{m}\right)^{-\frac{1}{2}\operatorname{tr}\left(APA\left|A^{-1}-\operatorname{tr}\left(\frac{PA}{\operatorname{tr}(PA^2)}\right)\right|\right)} \\ \\ \exp\left[-\frac{1}{2}\left(m^{-1}-M^{-1}\right)\operatorname{tr}\left(APA\left|\ln A^{-1}-\operatorname{tr}\left(\frac{APA}{\operatorname{tr}(PA^2)}\ln A^{-1}\right)\right|\right)\right] \end{cases}$$

$$\geq \begin{cases} \left(\frac{M}{m}\right)^{-\frac{1}{2}\left[\operatorname{tr}\left(PA^2\right)-\left(\operatorname{tr}(PA)\right)^2\right]^{1/2}} \\ \\ \exp\left\{-\frac{1}{2}\left(m^{-1}-M^{-1}\right)\right. \\ \left.\times\left[\operatorname{tr}\left(PA^2\right)\operatorname{tr}\left(APA\left[\ln\left(eA^{-1}\right)\right]^2\right)-\left(\operatorname{tr}\left(APA\ln\left(eA^{-1}\right)\right)\right)^2\right]^{1/2}\right\} \end{cases}$$
$$\geq \left(\frac{M}{m}\right)^{-\frac{1}{4mM}(M-m)\operatorname{tr}\left(PA^2\right)} \geq \left(\frac{M}{m}\right)^{-\frac{M}{4m}(M-m)}.$$

Proof. If $0 < m \leq A \leq M$, then $0 < M^{-1} \leq A^{-1} \leq m^{-1}$. Now, using (3.2) for $Q := \frac{APA}{\operatorname{tr}(PA^2)}$ and A^{-1} we have

$$1 \leq \frac{\left[\operatorname{tr}\left(\frac{APA}{\operatorname{tr}(PA^2)}A^{-1}\right)\right]^{-\operatorname{tr}\left(\frac{APA}{\operatorname{tr}(PA^2)}A^{-1}\right)}}{\eta_{\frac{APA}{\operatorname{tr}(PA^2)}}\left(A^{-1}\right)}$$
$$\leq \begin{cases} \left(\frac{M}{m}\right)^{\frac{1}{2}\operatorname{tr}\left(\frac{APA}{\operatorname{tr}(PA^2)}\left|A^{-1}-\operatorname{tr}\left(\frac{APA}{\operatorname{tr}(PA^2)}A^{-1}\right)\right|\right)} \\ \\ \exp\left[\frac{1}{2}\left(m^{-1}-M^{-1}\right)\operatorname{tr}\left(\frac{APA}{\operatorname{tr}(PA^2)}\left|\ln A^{-1}-\operatorname{tr}\left(\frac{APA}{\operatorname{tr}(PA^2)}\ln A^{-1}\right)\right|\right)\right] \end{cases}$$

$$\leq \begin{cases} \left(\frac{M}{m}\right)^{\frac{1}{2}\left[\operatorname{tr}\left(\frac{APA}{\operatorname{tr}(PA^2)}A^{-2}\right)-\left(\operatorname{tr}\left(\frac{APA}{\operatorname{tr}(PA^2)}A^{-1}\right)\right)^2\right]^{1/2}} \\ \exp\left\{\frac{1}{2}\left(m^{-1}-M^{-1}\right)\right. \\ \left.\times\left[\operatorname{tr}\left(\frac{APA}{\operatorname{tr}(PA^2)}\left[\ln\left(eA^{-1}\right)\right]^2\right)-\left(\operatorname{tr}\left(\frac{APA}{\operatorname{tr}(PA^2)}\ln\left(eA^{-1}\right)\right)\right)^2\right]^{1/2}\right\} \end{cases}$$

$$\leq \left(\frac{M}{m}\right)^{\frac{1}{4}\left(m^{-1}-M^{-1}\right)},$$

namely

$$1 \leq \frac{\left[\frac{\operatorname{tr}(PA)}{\operatorname{tr}(PA^2)}\right]^{-\frac{\operatorname{tr}(PA)}{\operatorname{tr}(PA^2)}}}{\eta_{\frac{APA}{\operatorname{tr}(PA^2)}}\left(A^{-1}\right)} \tag{3.4}$$

$$\leq \begin{cases} \left(\frac{M}{m}\right)^{\frac{1}{2}\operatorname{tr}\left(\frac{APA}{\operatorname{tr}(PA^2)}\left|A^{-1}-\operatorname{tr}\left(\frac{PA}{\operatorname{tr}(PA^2)}\right)\right|\right)} \\ \exp\left[\frac{1}{2}\left(m^{-1}-M^{-1}\right)\operatorname{tr}\left(\frac{APA}{\operatorname{tr}(PA^2)}\left|\ln A^{-1}-\operatorname{tr}\left(\frac{PA^2}{\operatorname{tr}(PA^2)}\ln A^{-1}\right)\right|\right)\right] \end{cases}$$

$$\leq \begin{cases} \left(\frac{M}{m}\right)^{\frac{1}{2}\left[\frac{1}{\operatorname{tr}(PA^2)}-\left(\frac{\operatorname{tr}(PA)}{\operatorname{tr}(PA^2)}\right)^2\right]^{1/2}} \\ \exp\left\{\frac{1}{2}\left(m^{-1}-M^{-1}\right)\right. \\ \left.\times\left[\operatorname{tr}\left(\frac{APA}{\operatorname{tr}(PA^2)}\left[\ln\left(eA^{-1}\right)\right]^2\right)-\left(\operatorname{tr}\left(\frac{APA}{\operatorname{tr}(PA^2)}\ln\left(eA^{-1}\right)\right)\right)^2\right]^{1/2}\right\} \end{cases}$$

$$\leq \left(\frac{M}{m}\right)^{\frac{1}{4mM}(M-m)}.$$

Now, if we take the power $-\operatorname{tr}\left(PA^2\right) \leq 0$ in (2.14), then we get

$$1 \geq \frac{\left[\frac{\operatorname{tr}(PA)}{\operatorname{tr}(PA^2)}\right]^{\operatorname{tr}(PA)}}{\left[\eta_{\frac{APA}{\operatorname{tr}(PA^2)}}\left(A^{-1}\right)\right]^{-\operatorname{tr}(PA^2)}}$$

$$\geq \begin{cases} \left(\frac{M}{m}\right)^{-\frac{1}{2}\operatorname{tr}\left(APA\left|A^{-1}-\operatorname{tr}\left(\frac{PA}{\operatorname{tr}(PA^2)}\right)\right|\right)} \\ \exp\left[-\frac{1}{2}\left(m^{-1}-M^{-1}\right)\operatorname{tr}\left(APA\left|\ln A^{-1}-\operatorname{tr}\left(\frac{APA}{\operatorname{tr}(PA^2)}\ln A^{-1}\right)\right|\right)\right] \end{cases}$$

$$\geq \begin{cases} \left(\frac{M}{m}\right)^{-\frac{1}{2}\left[\operatorname{tr}\left(PA^2\right)-(\operatorname{tr}(PA))^2\right]^{1/2}} \\ \exp\left\{-\frac{1}{2}\left(m^{-1}-M^{-1}\right)\right. \\ \left.\times\left[\operatorname{tr}\left(PA^2\right)\operatorname{tr}\left(APA\left[\ln\left(eA^{-1}\right)\right]^2\right)-\left(\operatorname{tr}\left(APA\ln\left(eA^{-1}\right)\right)\right)^2\right]^{1/2}\right\} \end{cases}$$

$$\geq \left(\frac{M}{m}\right)^{-\frac{1}{4mM}(M-m)\operatorname{tr}\left(PA^2\right)}.$$

This proves the desired result (3.3). □

References

[1] S. S. Dragomir, Hermite–Hadamard's type inequalities for operator convex functions, *Applied Mathematics and Computation*, **218** (2011), Issue 3, pp. 766–772.

[2] S. S. Dragomir, Some Grüss type inequalities for trace of operators in Hilbert spaces, *Oper. Matrices*, **10** (2016), no. 4, 923–943. Preprint *RGMIA Res. Rep. Coll.* **17** (2014), Art. 114. [https://rgmia.org/papers/v17/v17a114.pdf].

[3] S. S. Dragomir, Some trace inequalities for convex functions of selfadjoint operators in Hilbert spaces, *Korean J. Math.*, 24 (2016), no. 2, 273–296. Preprint *RGMIA Res. Rep. Coll.*, **17** (2014), Art. 115. [https://rgmia.org/papers/v17/v17a115.pdf]

[4] S. S. Dragomir, Jensen's type trace inequalities for convex functions of selfadjoint operators in Hilbert spaces, *Facta Univ. Ser. Math. Inform.*, **31** (2016), no. 5, 981–998. Preprint *RGMIA Res. Rep. Coll.*, **17** (2014), Art. 116. [https://rgmia.org/papers/v17/v17a116.pdf].

[5] S. S. Dragomir, Trace inequalities of Cassels and Grüss type for operators in Hilbert spaces, *Acta Univ. Sapientiae Math.*, 9 (2017), no. 1, 74–93. Preprint *RGMIA Res. Rep. Coll.*, **17** (2014), Art. 121. [https://rgmia.org/papers/v17/v17a121.pdf].

[6] S. S. Dragomir, Trace inequalities for operators in Hilbert spaces: a survey of recent results, *Aust. J. Math. Anal. Appl.* Vol. **19** (2022), No. 1, Art. 1, 202 pp. [Online https://ajmaa.org/searchroot/files/pdf/v19n1/v19i1p1.pdf].

[7] S. S. Dragomir, Some properties of trace class P-determinant of positive operators in Hilbert spaces, Preprint *RGMIA Res. Rep. Coll.* **25** (2022), Art. 15, 14 pp. [Online https://rgmia.org/papers/v25/v25a16.pdf].

[8] B. Fuglede and R. V. Kadison, Determinant theory in finite factors, *Ann. of Math.* (2) **55** (1952), 520–530.

[9] J. I. Fujii and Y. Seo, Determinant for positive operators, *Sci. Math.,* **1** (1998), 153–156.

[10] J. I. Fujii, S. Izumino and Y. Seo, Determinant for positive operators and Specht's Theorem, *Sci. Math.*, **1** (1998), 307–310.

[11] S. Hiramatsu and Y. Seo, Determinant for positive operators and Oppenheim's inequality, *J. Math. Inequal.*, Volume **15** (2021), Number 4, 1637–1645

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